Moving Frames and their Applications

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History of Moving Frames

Classical contributions:

M. Bartels ($\sim$1800), J. Serret, J. Frénet, G. Darboux, É. Cotton,

Élie Cartan

Modern developments: (1970's)

S.S. Chern, M. Green, P. Griffiths, G. Jensen, . . .

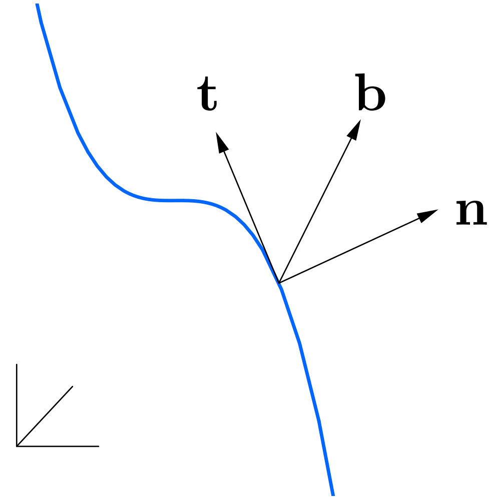
The equivariant approach: (1997 –)

PJO, M. Fels, G. Marí–Beffa, I. Kogan, J. Cheh, J. Pohjanpelto, P. Kim, M. Boutin, D. Lewis, E. Mansfield, E. Hubert, O. Morozov, R. McLenaghan, R. Smirnov, J. Yue, A. Nikitin, J. Patera, F. Valiquette, R. Thompson, . . .

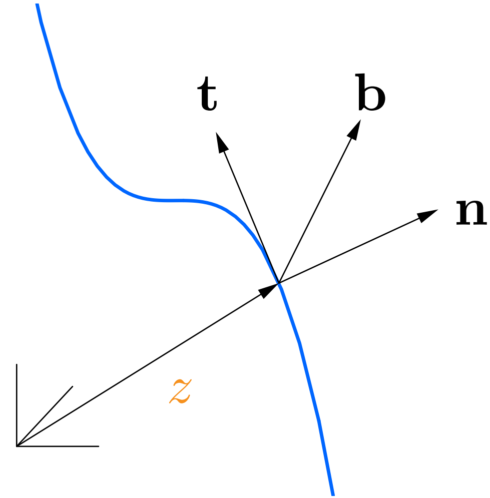
Moving Frame — Space Curves

tangent	normal	binormal
$\mathbf{t} = \dfrac{dz}{ds}$	$\mathbf{n} = \dfrac{z_{ss}}{\Vert z_{ss} \Vert}$	$\mathbf{b} = \mathbf{t} \times \mathbf{n}$

s — arc length



Moving Frame — Space Curves



tangent $\quad$ normal $\quad$ binormal

$$\mathbf{t} = \frac{dz}{ds} \qquad \mathbf{n} = \frac{z_{ss}}{\| \, z_{ss} \, \|} \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n}$$

s — arc length

z — point on the curve

Frénet–Serret equations

$$\frac{d\mathbf{t}}{ds} = \kappa \, \mathbf{n} \qquad \frac{d\mathbf{n}}{ds} = -\kappa \, \mathbf{t} + \tau \, \mathbf{b} \qquad \frac{d\mathbf{b}}{ds} = -\tau \, \mathbf{n}$$

κ — curvature $\qquad$ τ — torsion

"I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear."

"Nevertheless, I must admit I found the book, like most of Cartan's papers, hard reading."

— Hermann Weyl

"Cartan on groups and differential geometry"
Bull. Amer. Math. Soc. **44** (1938) 598–601

Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry groups and groupoids
- Differential invariants
- Rigidity
- Joint invariants and semi-differential invariants
- Invariant differential forms and tensors
- Identities and syzygies
- Classical invariant theory

- Computer vision
  — object recognition & symmetry detection
- Invariant numerical methods
- Invariant variational problems
- Invariant submanifold flows
- Poisson geometry & solitons
- Killing tensors in relativity
- Invariants of Lie algebras in quantum mechanics
- Lie pseudo-groups

The Basic Equivalence Problem

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group

Equivalence:

Determine when two p-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \subset M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Find all symmetries,

i.e., self-equivalences or *self-congruences*:

$$N = g \cdot N$$

Classical Geometry — *F. Klein*

- **Euclidean group:** $G = \begin{cases} \mathrm{SE}(m) = \mathrm{SO}(m) \ltimes \mathbb{R}^m \\ \mathrm{E}(m) = \mathrm{O}(m) \ltimes \mathbb{R}^m \end{cases}$

  $\boxed{z \longmapsto A \cdot z + b}$ $\quad A \in \mathrm{SO}(m)$ or $\mathrm{O}(m)$, $\quad b \in \mathbb{R}^m$, $\quad z \in \mathbb{R}^m$

  $\Rightarrow$ isometries: rotations, translations , (reflections)

- **Equi-affine group:** $G = \mathrm{SA}(m) = \mathrm{SL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{SL}(m)$ — volume-preserving

- **Affine group:** $G = \mathrm{A}(m) = \mathrm{GL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{GL}(m)$

- **Projective group:** $G = \mathrm{PSL}(m+1)$

  acting on $\mathbb{R}^m \subset \mathbb{RP}^m$

$\Longrightarrow$ Applications in computer vision

Tennis, Anyone?

Moving Frames

Definition.

A moving frame is a G-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Isotropy & Freeness

Isotropy subgroup: $\quad G_z = \{\, g \mid g \cdot z = z \,\} \quad$ for $\; z \in M$

free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity

$$\Longrightarrow \quad G_z = \{e\} \text{ for all } z \in M$$

locally free — the orbits all have the same dimension as G

$$\Longrightarrow \quad G_z \subset G \text{ is discrete for all } z \in M$$

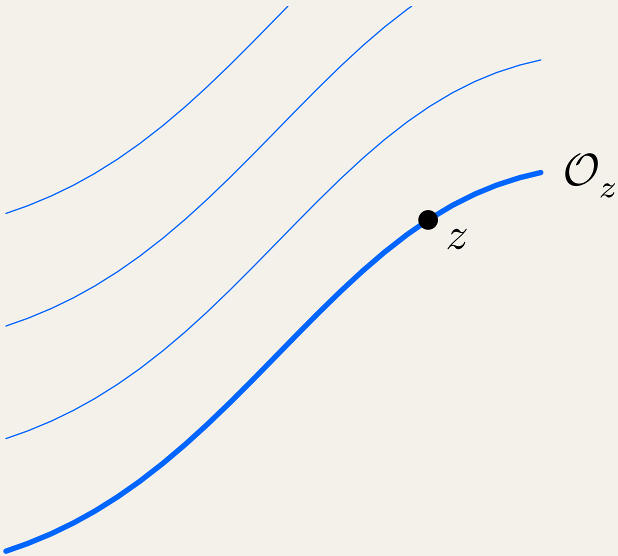
regular — the orbits form a regular foliation

$$\not\approx \quad \text{irrational flow on the torus}$$

effective — the only group element which fixes *every* point in M is the identity: $g \cdot z = z$ for all $z \in M$ iff $g = e$:

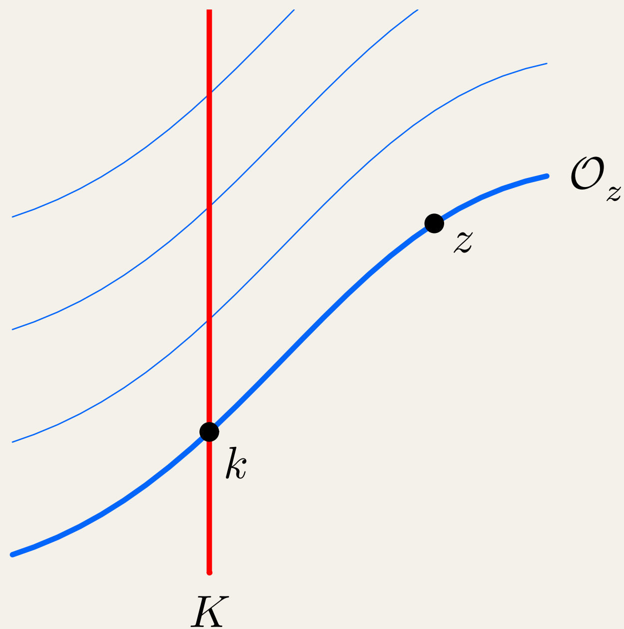
$$G_M^* = \bigcap_{z \in M} G_z = \{e\}$$

Geometric Construction



Normalization = choice of cross-section to the group orbits

Geometric Construction



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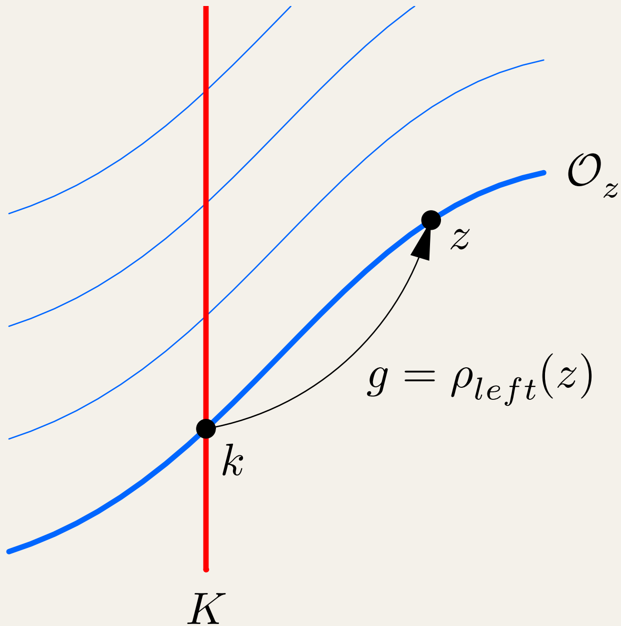
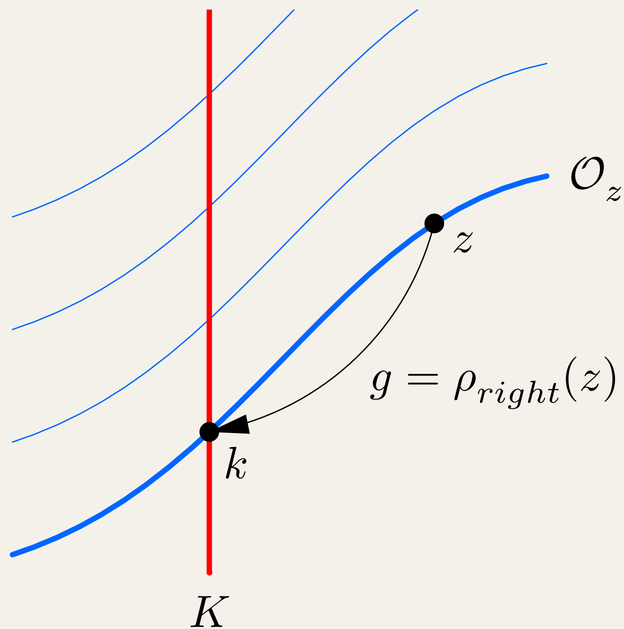


Normalization = choice of cross-section to the group orbits

Geometric Construction



Normalization = choice of cross-section to the group orbits

K — cross-section to the group orbits

$\mathcal{O}_z$ — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the intersection

- k is the *canonical form* of z
- the (nonconstant) coordinates of k are the fundamental invariants

$g \in G$ — *unique* group element mapping k to z

$\Longrightarrow$ freeness

Then $\rho_{left}(z) = g$ is a left moving frame: $\rho_{left}(h \cdot z) = h \cdot \rho_{left}(z)$

$$k = \rho_{left}(z)^{-1} \cdot z = \rho_{right}(z) \cdot z$$

Algebraic Construction

$$r = \dim G \ \leq \ m = \dim M$$

Coordinate cross-section

$$K = \{ z_1 = c_1, \ \ldots \ , z_r = c_r \}$$

left	right
$w(g, z) = g^{-1} \cdot z$	$w(g, z) = g \cdot z$

$g = (g_1, \ldots, g_r)$ — group parameters

$z = (z_1, \ldots, z_m)$ — coordinates on M

Choose $r = \dim G$ components to *normalize*:

$$w_1(g, z) = c_1 \qquad \ldots \qquad w_r(g, z) = c_r \qquad\qquad (*)$$

Solve $(*)$ for the group parameters $g = (g_1, \ldots, g_r)$

$\Longrightarrow$ Implicit Function Theorem

The solution

$$g = \rho(z)$$

is a (local) moving frame.

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

$\Longrightarrow$ These are the coordinates of the canonical form $k \in K$.

Completeness of Invariants

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \; \ldots \; , I_{m-r}(z))$$

Invariantization

Definition. The invariantization of a function $F : M \to \mathbb{R}$ with respect to a right moving frame $g = \rho(z)$ is the the invariant function $I = \iota(F)$ defined by

$$I(z) = F(\rho(z) \cdot z).$$

$$\iota(z_1) = c_1, \quad \ldots \quad \iota(z_r) = c_r, \qquad \iota(z_{r+1}) = I_1(z), \quad \ldots \quad \iota(z_m) = I_{m-r}(z).$$

cross-section variables — fundamental invariants

“phantom invariants”

Invariantization respects all algebraic operations:

$$\boxed{\iota\,[\,F(z_1, \ldots, z_m)\,] \;=\; F(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z))}$$

Invariantization amounts to restricting F to the cross-section

$$I \,|\, K = F \,|\, K$$

and then requiring that $I = \iota(F)$ be constant along the orbits.

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$.

Invariantization defines a canonical projection

$$\iota \;:\; \text{functions} \quad \longmapsto \quad \text{invariants}$$

The Replacement Theorem

If $I(z_1, \ldots, z_m)$ is any invariant, then

$$\boxed{\iota\,[\,I(z_1, \ldots, z_m)\,] \;=\; I(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z))}$$

This "Rewrite Rule" trivially proves that any invariant can be easily expressed (rewritten) in terms of the fundamental invariants!

The Rotation Group

$$G = \mathrm{SO}(2) \qquad \text{acting on} \qquad \mathbb{R}^2$$

$$z = (x, u) \longmapsto g \cdot z = (\, x \cos \phi - u \sin \phi \,,\; x \sin \phi + u \cos \phi \,)$$

$$\Longrightarrow \quad \text{Free on } M = \mathbb{R}^2 \setminus \{0\}$$

Left moving frame:

$$w(g, z) = g^{-1} \cdot z = (y, v)$$

$$y = x \cos \phi + u \sin \phi \qquad v = -\, x \sin \phi + u \cos \phi$$

Cross-section:

$$K = \{\, u = 0,\; x > 0 \,\}$$

Normalization equation:

$$v = -x \sin \phi + u \cos \phi = 0$$

Left moving frame:

$$\phi = \tan^{-1} \frac{u}{x} \quad \Longrightarrow \quad \phi = \rho(x, u) \in \mathrm{SO}(2)$$

Fundamental invariant

$$r = \iota(x) = \sqrt{x^2 + u^2}$$

Invariantization

$$\iota[\, F(x, u) \,] = F(r, 0)$$

Prolongation

- The moving frame construction requires freeness of the group action
- Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!
- Freeness typically fails because the dimension of the underlying manifold is not large enough, i.e., $m < r = \dim G$.
- Thus, to make the action free, we must increase the dimension of the space via some natural prolongation procedure.

An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

- Prolonging to derivatives (jet space)

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- Prolonging to "multi-space"

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$\Longrightarrow$ joint or semi-differential invariants

$\Longrightarrow$ invariant numerical approximations

Euclidean Plane Curves

Special Euclidean group: $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

acts on $M = \mathbb{R}^2$ via rigid motions: $w = R\,z + b$

To obtain the classical (left) moving frame we invert the group transformations:

$$\left.\begin{aligned} y &= \quad \cos\phi\,(x-a) + \sin\phi\,(u-b) \\ v &= -\sin\phi\,(x-a) + \cos\phi\,(u-b) \end{aligned}\right\} \qquad w = R^{-1}(z-b)$$

Assume for simplicity the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

$\Longrightarrow$ extensions to parametrized curves are straightforward

Prolong the action to J^n via implicit differentiation:

$$
\begin{aligned}
y &= \quad \cos\phi\,(x-a) + \sin\phi\,(u-b) \\
v &= -\sin\phi\,(x-a) + \cos\phi\,(u-b) \\
v_y &= \frac{-\sin\phi + u_x\cos\phi}{\cos\phi + u_x\sin\phi} \\
v_{yy} &= \frac{u_{xx}}{(\cos\phi + u_x\sin\phi)^3} \\
v_{yyy} &= \frac{(\cos\phi + u_x\sin\phi)u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi + u_x\sin\phi)^5} \\
&\vdots
\end{aligned}
$$

Normalization: $\qquad r = \dim G = 3$

$$y = \cos\phi\,(x - a) + \sin\phi\,(u - b) = 0$$

$$v = -\sin\phi\,(x - a) + \cos\phi\,(u - b) = 0$$

$$v_y = \frac{-\sin\phi + u_x \cos\phi}{\cos\phi + u_x \sin\phi} = 0$$

$$v_{yy} = \frac{u_{xx}}{(\cos\phi + u_x \sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi + u_x \sin\phi)u_{xxx} - 3u_{xx}^2 \sin\phi}{(\cos\phi + u_x \sin\phi)^5}$$

$$\vdots$$

Solve for the group parameters:

$$y = \quad \cos\phi\,(x - a) + \sin\phi\,(u - b) = 0$$

$$v = -\sin\phi\,(x - a) + \cos\phi\,(u - b) = 0$$

$$v_y = \frac{-\sin\phi \; + u_x \cos\phi}{\cos\phi \; + u_x \sin\phi} = 0$$

$\Longrightarrow$ Left moving frame $\qquad \rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$a = x \qquad b = u \qquad \phi \; = \tan^{-1} u_x$$

$$a = x \qquad b = u \qquad \phi = \tan^{-1} u_x$$

Differential invariants — invariantization

$$v_{yy} = \frac{u_{xx}}{(\cos\phi + u_x \sin\phi)^3} \longmapsto \iota(u_{xx}) = \frac{u_{xx}}{(1+u_x^2)^{3/2}} = \kappa$$

$$v_{yyy} = \cdots \qquad \longmapsto \iota(u_{xxx}) = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3} = \frac{d\kappa}{ds}$$

$$v_{yyyy} = \cdots \qquad \longmapsto \iota(u_{xxxx}) = \cdots = \frac{d^2\kappa}{ds^2} - 3\kappa^3$$

$\Longrightarrow$ recurrence formulae

Invariant one-form — arc length

$$dy = (\cos\phi + u_x \sin\phi)\, dx \qquad \longmapsto \qquad \iota(dx) = \sqrt{1+u_x^2}\; dx = ds$$

Dual invariant differential operator

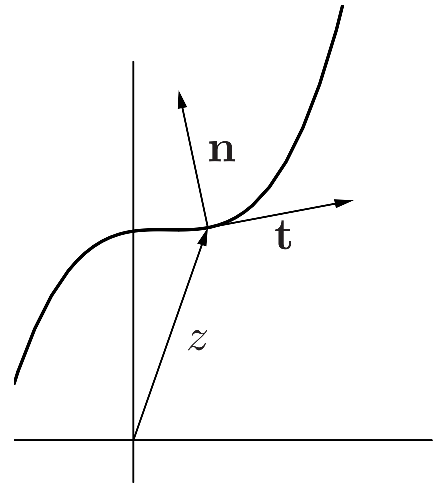
— arc length derivative

$$\frac{d}{dy} = \frac{1}{\cos\phi + u_x \sin\phi}\,\frac{d}{dx} \qquad \longmapsto \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa, \qquad \frac{d\kappa}{ds}, \qquad \frac{d^2\kappa}{ds^2}, \qquad \cdots$$

The Classical Picture:



Moving frame $\qquad \rho : \ (x, u, u_x) \ \longmapsto \ (R, \mathbf{a}) \in \mathrm{SE}(2)$

$$R = \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\,\mathbf{t},\ \mathbf{n}\,) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} = \begin{pmatrix} x_s \\ y_s \end{pmatrix}, \qquad \mathbf{n} = \mathbf{t}^{\perp} = \begin{pmatrix} -\,y_s \\ x_s \end{pmatrix}.$$

Frenet equations = Pulled-back Maurer–Cartan forms:

$$\frac{d\mathbf{x}}{ds} = \mathbf{t}, \qquad \frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -\,\kappa\,\mathbf{t}.$$

Equi-affine Plane Curves $\quad G = \mathrm{SA}(2)$

$$z \longmapsto A\,z + \mathbf{b} \qquad A \in \mathrm{SL}(2), \qquad \mathbf{b} \in \mathbb{R}^2$$

Invert for left moving frame:

$$\left.\begin{aligned} y &= \delta\,(x - a) - \beta\,(u - b) \\ v &= -\,\gamma\,(x - a) + \alpha\,(u - b) \end{aligned}\right\} \qquad w = A^{-1}(z - b)$$

$$\alpha\,\delta - \beta\,\gamma = 1$$

Prolong to J^3 via implicit differentiation

$$dy = (\delta - \beta\,u_x)\;dx \qquad D_y = \frac{1}{\delta - \beta\,u_x}\,D_x$$

Prolongation:

$$y = \delta\,(x - a) - \beta\,(u - b)$$

$$v = -\,\gamma\,(x - a) + \alpha\,(u - b)$$

$$v_y = -\,\frac{\gamma - \alpha\,u_x}{\delta - \beta\,u_x}$$

$$v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta\,u_x)^3}$$

$$v_{yyy} = -\,\frac{(\delta - \beta\,u_x)\,u_{xxx} + 3\,\beta\,u_{xx}^2}{(\delta - \beta\,u_x)^5}$$

$$v_{yyyy} = -\,\frac{u_{xxxx}(\delta - \beta\,u_x)^2 + 10\,\beta\,(\delta - \beta\,u_x)\,u_{xx}\,u_{xxx} + 15\,\beta^2\,u_{xx}^3}{(\delta - \beta\,u_x)^7}$$

$$v_{yyyyy} = \;\ldots$$

Normalization: $\qquad r = \dim G = 5$

$$y = \delta\,(x - a) - \beta\,(u - b) = 0$$

$$v = -\,\gamma\,(x - a) + \alpha\,(u - b) = 0$$

$$v_y = -\,\frac{\gamma - \alpha\,u_x}{\delta - \beta\,u_x} = 0$$

$$v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta\,u_x)^3} = 1$$

$$v_{yyy} = -\,\frac{(\delta - \beta\,u_x)\,u_{xxx} + 3\,\beta\,u_{xx}^2}{(\delta - \beta\,u_x)^5} = 0$$

$$v_{yyyy} = -\,\frac{u_{xxxx}(\delta - \beta\,u_x)^2 + 10\,\beta\,(\delta - \beta\,u_x)\,u_{xx}\,u_{xxx} + 15\,\beta^2\,u_{xx}^3}{(\delta - \beta\,u_x)^7}$$

$$v_{yyyyy} = \ \cdots$$

Equi-affine Moving Frame

$$\rho : \ (x, u, u_x, u_{xx}, u_{xxx}) \ \longmapsto \ (A, \mathbf{b}) \in \mathrm{SA}(2)$$

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Nondegeneracy condition: $u_{xx} \neq 0.$

Equi-affine arc length

$$dy = (\delta - \beta\, u_x)\; dx \qquad \longmapsto \qquad ds = \iota(dx) = \sqrt[3]{u_{xx}}\; dx$$

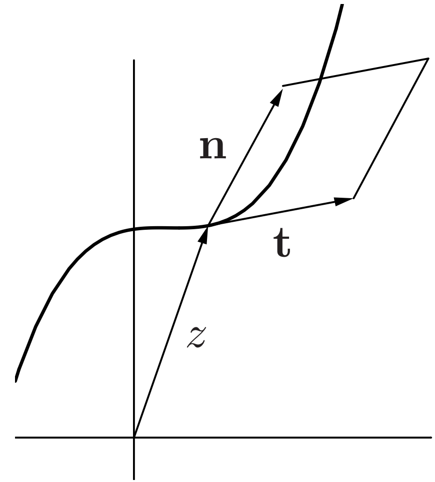
Equi-affine curvature

$$v_{yyyy} \qquad \longmapsto \qquad \iota(u_{xxxx}) \;=\; \frac{5\, u_{xx} u_{xxxx} - 3\, u_{xxx}^2}{9\, u_{xx}^{8/3}} = \kappa$$

$$v_{yyyyy} \qquad \longmapsto \qquad \iota(u_{xxxxx}) = \;\cdots\; = \frac{d\kappa}{ds}$$

$$v_{yyyyyy} \qquad \longmapsto \qquad \iota(u_{xxxxxx}) = \;\cdots\; = \frac{d^2\kappa}{ds^2} - 5\kappa^2$$

The Classical Picture:



$$A = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \end{pmatrix} = (\, \mathbf{t},\ \mathbf{n}\,) \qquad \mathbf{b} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{t} = \frac{dz}{ds}, \qquad\qquad \mathbf{n} = \frac{d^2 z}{ds^2}.$$

Frenet equations = Pulled-back Maurer–Cartan forms:

$$\frac{dz}{ds} = \mathbf{t}, \qquad \frac{d\mathbf{t}}{ds} = \mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = \kappa\, \mathbf{t}.$$

The Recurrence Formula

★ While invariantization respects all algebraic operations it *does not* commute with differentiation!

For *any* function or differential form Ω:

$$\boxed{\boxed{d\,\iota(\Omega) = \iota(d\,\Omega) + \sum_{k=1}^{r} \nu^k \wedge \iota\,[\mathbf{v}_k(\Omega)]}}$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — basis for $\mathfrak{g}$ — infinitesimal generators

$\nu^1, \ldots, \nu^r$ — dual invariantized Maurer–Cartan forms

★ The ν^k are uniquely determined by the recurrence formulae for the phantom differential invariants

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{k=1}^{r} \nu^k \wedge \iota\,[\mathbf{v}_k(\Omega)]$$

★★★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant differential forms follow from this universal recurrence formula by letting Ω range over the basic functions and differential forms!

★★★ Therefore, the entire structure of the differential invariant algebra and invariant variational bicomplex can be completely determined using only linear differential algebra; this does not require explicit formulas for the moving frame, the differential invariants, the invariant differential forms, or the group transformations!

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(G)$ is generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim N$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

$\Longrightarrow$ *Lie, Tresse, Ovsiannikov, Kumpera*

$\Longrightarrow$ Moving frames provides a constructive proof.

The Differential Invariant Algebra

Thus, remarkably, the structure of $\mathcal{I}(G)$ can be determined without knowing the explicit formulae for either the moving frame, or the differential invariants, or the invariant differential operators!

The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

Theorem. If G acts transitively on M, or if the infinitesimal generator coefficients depend rationally in the coordinates, then all recurrence formulae are rational in the basic differential invariants and so $\mathcal{I}(G)$ is a rational, non-commutative differential algebra.

Curves

Theorem. Let G be an ordinary* Lie group acting on the m-dimensional manifold M. Then, locally, there exist $m-1$ generating differential invariants $\kappa_1, \ldots, \kappa_{m-1}$. Every other differential invariant can be written as a function of the generating differential invariants and their derivatives with respect to the G-invariant arc length element ds.

* ordinary = transitive + no pseudo-stabilization.

Minimal Generating Invariants

A set of differential invariants is a generating system if all other differential invariants can be written in terms of them and their invariant derivatives.

Euclidean space curves $C \subset \mathbb{R}^3$:

- curvature κ and torsion τ

Equi–affine space curves $C \subset \mathbb{R}^3$:

- affine curvature κ and torsion τ

Euclidean surfaces $S \subset \mathbb{R}^3$:

- mean curvature H

⋆ Gauss curvature $K = \Phi(\mathcal{D}^{(4)} H)$.

Equi–affine surfaces $S \subset \mathbb{R}^3$:

- Pick invariant P.

Curves

Theorem. Let G be an ordinary* Lie group acting on the m-dimensional manifold M. Then, locally, there exist $m-1$ generating differential invariants $\kappa_1, \ldots, \kappa_{m-1}$. Every other differential invariant can be written as a function of the generating differential invariants and their derivatives with respect to the G-invariant arc length element ds.

* ordinary = transitive + no pseudo-stabilization.

$\Longrightarrow \quad m = 3 \quad$ — curvature κ & torsion τ

Euclidean Surfaces

Theorem.

The algebra of Euclidean differential invariants for a non-degenerate surface is generated by the mean curvature through invariant differentiation.

Euclidean Surfaces

Theorem. The algebra of Euclidean differential invariants for a non-degenerate surface is generated by the mean curvature through invariant differentiation.

$$K = \Phi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots\ )$$

Euclidean Proof

Commutation relation:

$$[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = Z_2 \mathcal{D}_1 - Z_1 \mathcal{D}_2,$$

Commutator invariants:

$$Z_1 = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad Z_2 = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

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Codazzi relation:

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Z_1) Z_1 - (\mathcal{D}_2 + Z_2) Z_2$$

Euclidean Proof

Commutation relation:

$$[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = Z_2 \mathcal{D}_1 - Z_1 \mathcal{D}_2,$$

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Codazzi relation:

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Z_1) Z_1 - (\mathcal{D}_2 + Z_2) Z_2$$

$\Longrightarrow$ Gauss' Theorema Egregium

(*Guggenheimer*)

To determine the commutator invariants:

$$\mathcal{D}_1\mathcal{D}_2H - \mathcal{D}_2\mathcal{D}_1H = Z_2\,\mathcal{D}_1H - Z_1\,\mathcal{D}_2H$$
$$\mathcal{D}_1\mathcal{D}_2\mathcal{D}_JH - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_JH = Z_2\,\mathcal{D}_1\mathcal{D}_JH - Z_1\,\mathcal{D}_2\mathcal{D}_JH \qquad (*)$$

Nondegenerate surface:

$$\det\begin{pmatrix} \mathcal{D}_1H & \mathcal{D}_2H \\ \mathcal{D}_1\mathcal{D}_JH & \mathcal{D}_2\mathcal{D}_JH \end{pmatrix} \neq 0,$$

Solve $(*)$ for Z_1, Z_2 in terms of derivatives of H.

Q.E.D.

★ A surface is mean curvature degenerate if

$$\mathcal{D}_jH = F_j(H) \quad \text{for} \quad j = 1, 2.$$

Totally umbilic and constant mean curvature surfaces, including minimal surfaces are degenerate. Geometry?

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants

$\Longrightarrow$ invariant numerical approximations

Joint Invariants

A joint invariant is an invariant of the k-fold Cartesian product action of G on $M \times \cdots \times M$:

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) \;=\; I(z_1, \ldots, z_k)}$$

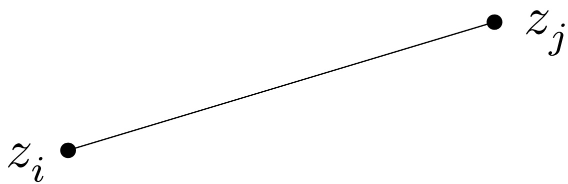
A joint differential invariant or semi-differential invariant is an invariant depending on the derivatives at several points $z_1, \ldots, z_k \in N$ on the submanifold:

$$\boxed{I(g \cdot z_1^{(n)}, \ldots, g \cdot z_k^{(n)}) \;=\; I(z_1^{(n)}, \ldots, z_k^{(n)})}$$

Joint Euclidean Invariants

Theorem. Every joint Euclidean invariant is a function of the interpoint distances

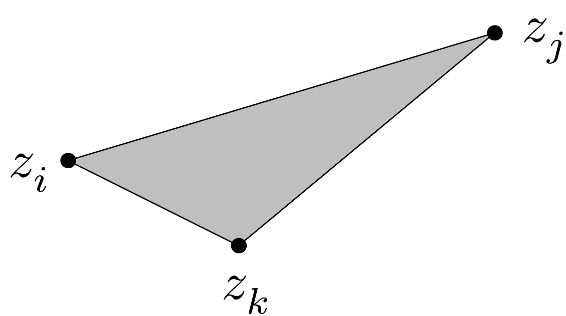
$$d(z_i, z_j) = \| \, z_i - z_j \, \|$$



Joint Equi–Affine Invariants

Theorem. Every planar joint equi–affine invariant is a function of the triangular areas

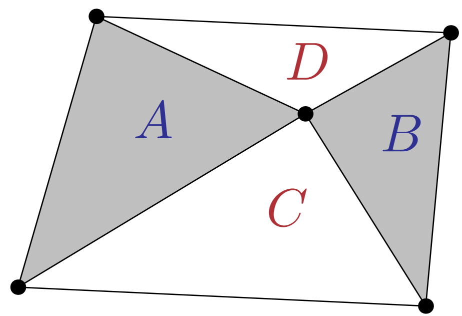
$$[\, i \ j \ k \,] = \tfrac{1}{2}\,(z_i - z_j) \wedge (z_i - z_k)$$



Joint Projective Invariants

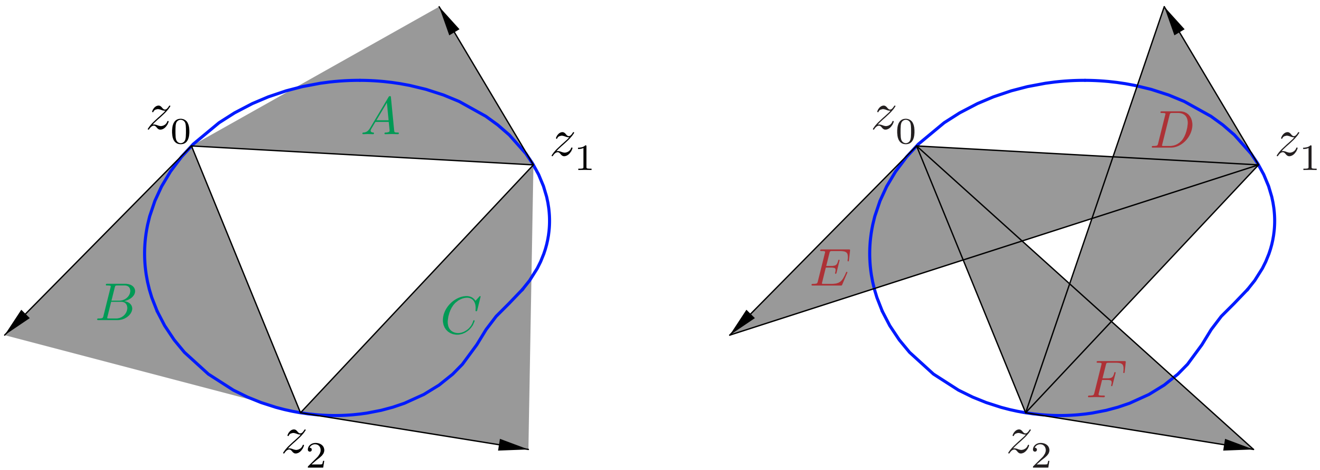
Theorem. Every planar joint projective invariant is a function of the planar cross-ratios

$$[\, z_i, z_j, z_k, z_l, z_m \,] = \frac{A\,B}{C\,D}$$



Projective joint differential invariant:

— tangent triangle ratio



$$\frac{A\,B\,C}{D\,E\,F}$$

- Prolonging to derivatives (jet space)

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$\Longrightarrow$ joint or semi-differential invariants

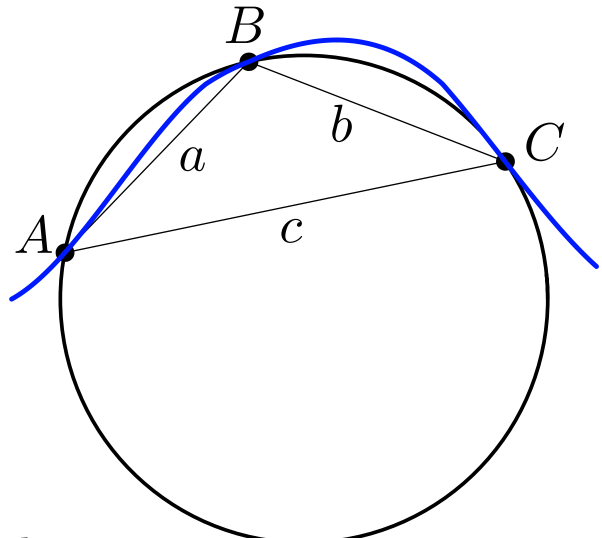
$\Longrightarrow$ invariant numerical approximations

Symmetry–Preserving Numerical Methods

- Invariant numerical approximations to differential invariants.
- Invariantization of numerical integration methods.

$\Longrightarrow$ Structure-preserving algorithms

Numerical approximation to curvature



Heron's formula

$$\widetilde{\kappa}(A,B,C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

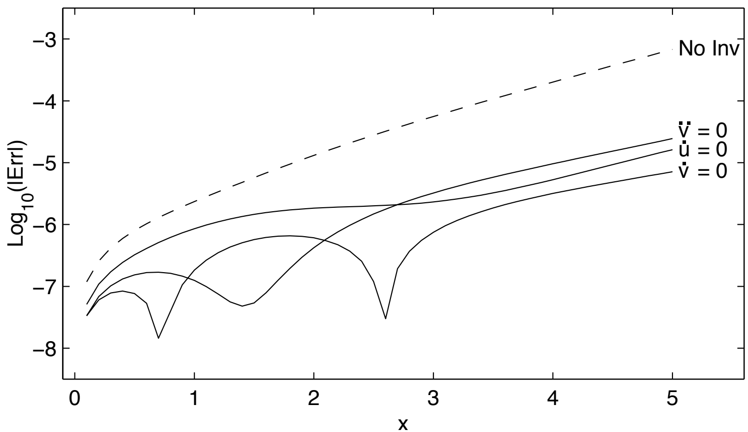
$$s = \frac{a+b+c}{2} \qquad \text{— semi-perimeter}$$

Invariantization of Numerical Schemes

$\Longrightarrow$ Pilwon Kim

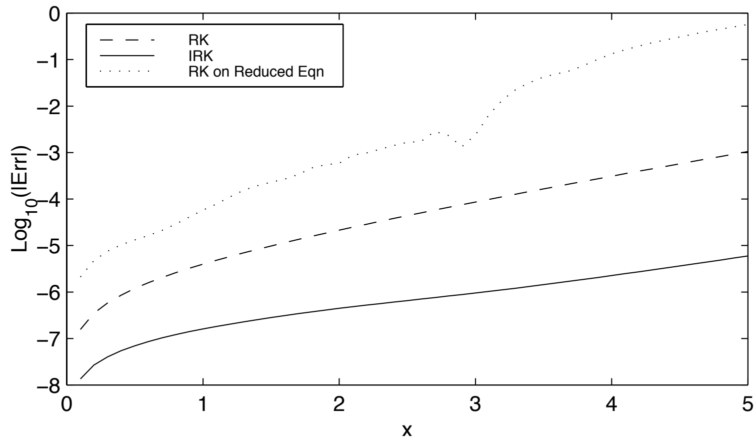
Suppose we are given a numerical scheme for integrating a differential equation, e.g., a Runge–Kutta Method for ordinary differential equations, or the Crank–Nicolson method for parabolic partial differential equations.

If G is a symmetry group of the differential equation, then one can use an appropriately chosen moving frame to invariantize the numerical scheme, leading to an invariant numerical scheme that preserves the symmetry group. In challenging regimes, the resulting invariantized numerical scheme can, with an inspired choice of moving frame, perform significantly better than its progenitor.



Invariant Runge–Kutta schemes

$$u_{xx} + x\,u_x - (x+1)u = \sin x, \qquad u(0) = u_x(0) = 1.$$

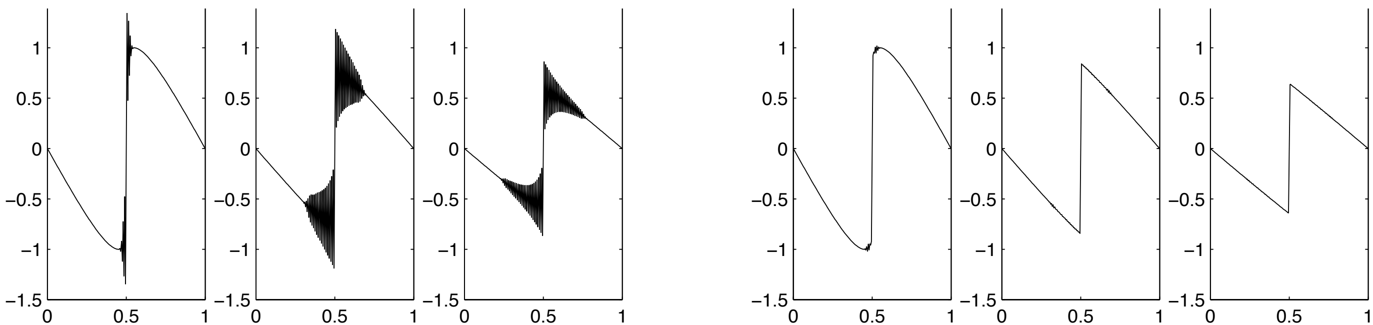


Comparison of symmetry reduction and invariantization for

$$u_{xx} + x\,u_x - (x+1)u = \sin x, \qquad u(0) = u_x(0) = 1.$$

Invariantization of Crank–Nicolson for Burgers' Equation

$$u_t = \varepsilon\, u_{xx} + u\, u_x$$



The Calculus of Variations

$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x}$ — variational problem

$L(x, u^{(n)})$ — Lagrangian

To construct the Euler-Lagrange equations: $\mathbf{E}(L) = 0$

- Take the first variation:

$$\delta(L\, d\mathbf{x}) = \sum_{\alpha, J} \frac{\partial L}{\partial u^\alpha_J}\, \delta u^\alpha_J\, d\mathbf{x}$$

- Integrate by parts:

$$\delta(L\, d\mathbf{x}) = \sum_{\alpha, J} \frac{\partial L}{\partial u^\alpha_J}\, D_J(\delta u^\alpha)\, d\mathbf{x}$$

$$\equiv \sum_{\alpha, J} (-D)^J \frac{\partial L}{\partial u^\alpha_J}\, \delta u^\alpha\, d\mathbf{x} = \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\, \delta u^\alpha\, d\mathbf{x}$$

Invariant Variational Problems

According to Lie, any G–invariant variational problem can be written in terms of the differential invariants:

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}}$$

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

If the variational problem is G-invariant, so

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}$$

then its Euler–Lagrange equations admit G as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$\mathbf{E}(L) \simeq F(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ ) = 0$$

Main Problem:

Construct F directly from P.

(*P. Griffiths, I. Anderson*)

Planar Euclidean group $\quad G = \mathrm{SE}(2)$

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}} \qquad \text{— curvature (differential invariant)}$$

$$ds = \sqrt{1+u_x^2}\, dx \qquad \text{— arc length}$$

$$\mathcal{D} = \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}} \frac{d}{dx} \qquad \text{— arc length derivative}$$

Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) \simeq F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

Euclidean Curve Examples

Minimal curves (geodesics):

$$\mathcal{I}[\,u\,] = \int ds = \int \sqrt{1 + u_x^2}\ dx$$

$$\mathbf{E}(L) = -\,\kappa = 0$$

$\Longrightarrow$ straight lines

The Elastica (Euler):

$$\mathcal{I}[\,u\,] = \int \tfrac{1}{2}\,\kappa^2\,ds = \int \frac{u_{xx}^2\,dx}{(1 + u_x^2)^{5/2}}$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

To construct the invariant Euler-Lagrange equations:

Take the first variation:

$$\delta(P\, ds) = \sum_j \frac{\partial P}{\partial \kappa_j}\, \delta\kappa_j\, ds + P\, \delta(ds)$$

Invariant variation of curvature:

$$\delta\kappa = \mathcal{A}_\kappa(\delta u) \qquad\qquad \mathcal{A}_\kappa = \mathcal{D}^2 + \kappa^2$$

Invariant variation of arc length:

$$\delta(ds) = \mathcal{B}(\delta u)\, ds \qquad\qquad \mathcal{B} = -\,\kappa$$

$\Longrightarrow$ moving frame recurrence formulae

Integrate by parts:

$$\delta(P\,ds) \equiv [\,\mathcal{E}(P)\,\mathcal{A}(\delta u) - \mathcal{H}(P)\,\mathcal{B}(\delta u)\,]\,ds$$
$$\equiv [\,\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P)\,]\,\delta u\,ds = \mathbf{E}(L)\,\delta u\,ds$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j}\,(-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

Euclidean–invariant Euler-Lagrange formula

$$\mathbf{E}(L) = \mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P) = (\mathcal{D}^2 + \kappa^2)\,\mathcal{E}(P) + \kappa\,\mathcal{H}(P) = 0.$$

The Elastica:

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\,ds \qquad P = \tfrac{1}{2}\,\kappa^2$$

$$\mathcal{E}(P) = \kappa \qquad\qquad \mathcal{H}(P) = -\,P = -\,\tfrac{1}{2}\,\kappa^2$$

$$\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\,\kappa + \kappa\,(\,-\,\tfrac{1}{2}\,\kappa^2\,) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

The shape of a Möbius strip

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The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1,2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping[3] and paper crumpling[4,5]. This could give new insight into energy localization phenomena in unstretchable sheets[6], which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures[7–9].

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher[10]. In engineering, pulley belts are often used in the form of Möbius strips to wear 'both' sides equally. At a much smaller scale, Möbius strips have recently been formed in ribbon-shaped $NbSe_3$ crystals under certain growth conditions involving a large temperature gradient[7,8].

Figure 1 Photo of a paper Möbius strip of aspect ratio 2π. The strip adopts a characteristic shape. Inextensibility of the material causes the surface to be developable. Its straight generators are drawn and the colouring varies according to the bending energy density.

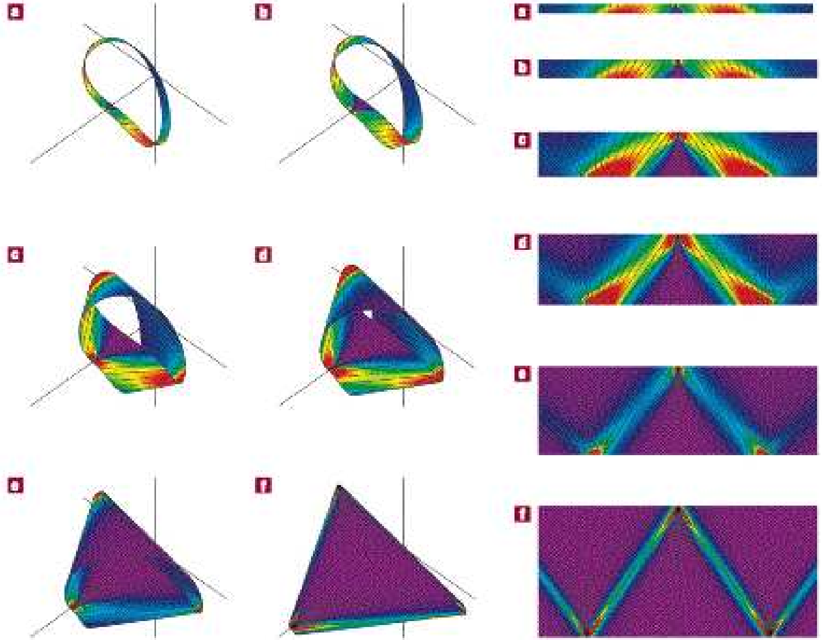


Figure 2 Computed Möbius strips. The left panel shows their three-dimensional shapes for $w = 0.1$ (**a**), 0.2 (**b**), 0.5 (**c**), 0.8 (**d**), 1.0 (**e**) and 1.5 (**f**), and the right panel the corresponding developments on the plane. The colouring changes according to the local bending energy density, from violet for regions of low bending to red for regions of high bending (scales are individually adjusted). Solution **c** may be compared with the paper model in Fig. 1 on which the generator field and density colouring have been printed.

Evolution of Invariants and Signatures

G — Lie group acting on $\mathbb{R}^2$

$C(t)$ — parametrized family of plane curves

G–invariant curve flow:

$$\frac{dC}{dt} = \mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$$

- I, J — differential invariants
- $\mathbf{t}$ — "unit tangent"
- $\mathbf{n}$ — "unit normal"
- The tangential component $I\,\mathbf{t}$ only affects the underlying parametrization of the curve. Thus, we can set I to be anything we like without affecting the curve evolution.

Normal Curve Flows

$$C_t = J\,\mathbf{n}$$

Examples — Euclidean–invariant curve flows

- $C_t = \mathbf{n}$ — geometric optics or grassfire flow;
- $C_t = \kappa\,\mathbf{n}$ — curve shortening flow;
- $C_t = \kappa^{1/3}\,\mathbf{n}$ — equi-affine invariant curve shortening flow: $$C_t = \mathbf{n}_{\text{equi-affine}}\,;$$
- $C_t = \kappa_s\,\mathbf{n}$ — modified Korteweg–deVries flow;
- $C_t = \kappa_{ss}\,\mathbf{n}$ — thermal grooving of metals.

Intrinsic Curve Flows

Theorem. The curve flow generated by

$$\mathbf{v} = I\,\mathbf{t} + J\,\mathbf{n}$$

preserves arc length if and only if

$$\mathcal{B}(J) + \mathcal{D}\,I = 0.$$

$\mathcal{D}$ — invariant arc length derivative

$\mathcal{B}$ — invariant arc length variation

$$\delta(ds) = \mathcal{B}(\delta u)\,ds$$

Normal Evolution of Differential Invariants

Theorem. Under a normal flow $C_t = J\,\mathbf{n}$,

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J), \qquad \frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J).$$

Invariant variations:

$$\delta\kappa = \mathcal{A}_\kappa(\delta u), \qquad \delta\kappa_s = \mathcal{A}_{\kappa_s}(\delta u).$$

$\mathcal{A}_\kappa = \mathcal{A}$ — invariant variation of curvature;

$\mathcal{A}_{\kappa_s} = \mathcal{D}\,\mathcal{A} + \kappa\,\kappa_s$ — invariant variation of κ_s.

Euclidean–invariant Curve Evolution

Normal flow: $C_t = J\,\mathbf{n}$

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J) = (\mathcal{D}^2 + \kappa^2)\, J,$$

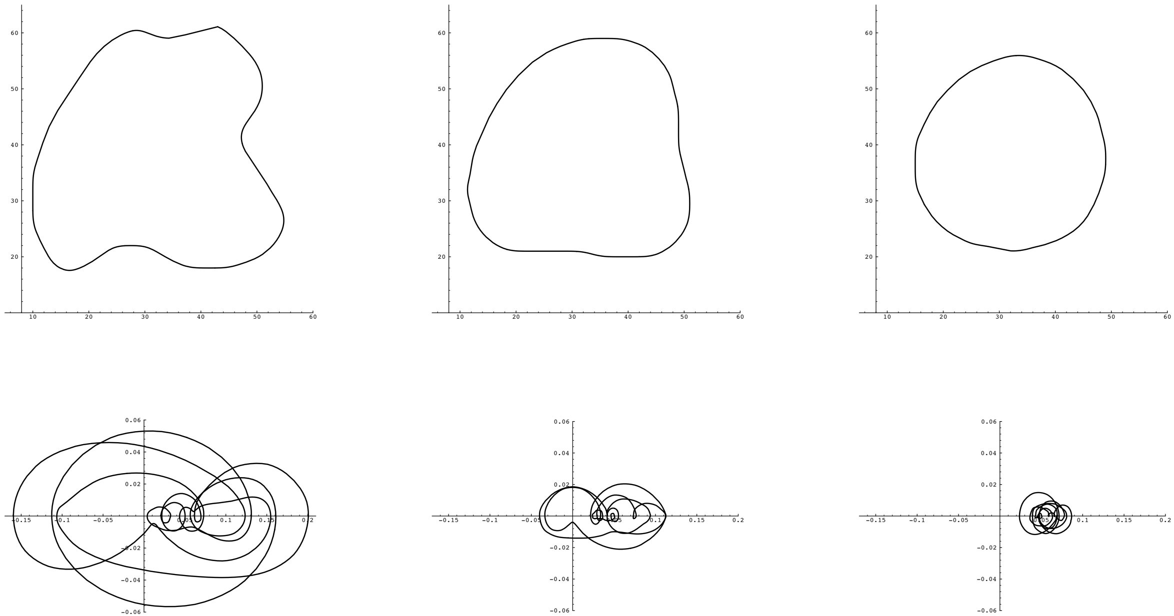
$$\frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J) = (\mathcal{D}^3 + \kappa^2 \mathcal{D} + 3\,\kappa\,\kappa_s)\, J.$$

Warning: For non-intrinsic flows, ∂_t and ∂_s *do not commute*!

Theorem. Under the curve shortening flow $C_t = -\kappa\,\mathbf{n}$, the signature curve $\kappa_s = H(t,\kappa)$ evolves according to the parabolic equation

$$\frac{\partial H}{\partial t} = H^2\, H_{\kappa\kappa} - \kappa^3 H_\kappa + 4\,\kappa^2 H$$

Smoothed Ventricle Signature



Intrinsic Evolution of Differential Invariants

Theorem.

Under an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \qquad \text{where} \qquad \mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} \tag{$*$}$$

In surprisingly many situations, (*) is a well-known integrable evolution equation, and $\mathcal{R}$ is its recursion operator!

$\Longrightarrow$ Hasimoto

$\Longrightarrow$ Langer, Singer, Perline

$\Longrightarrow$ Marí–Beffa, Sanders, Wang

$\Longrightarrow$ Qu, Chou, Anco, and many more ...

Euclidean plane curves

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$\mathcal{A} = \mathcal{D}^2 + \kappa^2 \qquad\qquad \mathcal{B} = -\kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

$$\boxed{\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\kappa^2 \kappa_s}$$

$\Longrightarrow$ modified Korteweg-deVries equation

Equi-affine plane curves

$$G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$$

$$\mathcal{A} = \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2$$

$$\mathcal{B} = \tfrac{1}{3}\,\mathcal{D}^2 - \tfrac{2}{9}\,\kappa$$

$$\begin{aligned}\mathcal{R} &= \mathcal{A} - \kappa_s\mathcal{D}^{-1}\mathcal{B} \\ &= \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{4}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2 + \tfrac{2}{9}\,\kappa_s\mathcal{D}^{-1}\cdot\kappa\end{aligned}$$

$$\boxed{\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + \tfrac{5}{3}\,\kappa\,\kappa_{sss} + \tfrac{5}{3}\,\kappa_s\kappa_{ss} + \tfrac{5}{9}\,\kappa^2\kappa_s}$$

$\Longrightarrow$ Sawada–Kotera equation

Recursion operator: $\widehat{\mathcal{R}} = \mathcal{R}\cdot\left(\mathcal{D}^2 + \tfrac{1}{3}\,\kappa + \tfrac{1}{3}\,\kappa_s\mathcal{D}^{-1}\right)$

Euclidean space curves

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & -2\tau D_s - \tau_s \\ \frac{2\tau}{\kappa} D_s^2 + \frac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \frac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} & \frac{1}{\kappa} D_s^3 - \frac{\kappa_s}{\kappa^2} D_s^2 + \frac{\kappa^2 - \tau^2}{\kappa} D_s + \frac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} \kappa & 0 \end{pmatrix}$$

$$\mathcal{R} = \mathcal{A} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B}$$

$$\boxed{\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix}}$$

$\Longrightarrow$ vortex filament flow (Hasimoto)