# *Moving Frames and Differential Invariants of Lie Pseudo–Groups*

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*Sur la théorie, si importante sans doute, mais pour nous si obscure, des ≪groupes de Lie infinis≫, nous ne savons rien que ce qui trouve dans les mémoires de Cartan, première exploration à travers une jungle presque impénétrable; mais celle-ci menace de se refermer sur les sentiers déjà tracés, si l'on ne procède bientôt à un indispensable travail de défrichement.*

— André Weil, 1947

# Pseudo-groups in Action

- Lie — Medolaghi — Vessiot
- Cartan . . . Guillemin, Sternberg
- Kuranishi, Spencer, Goldschmidt, Kumpera, . . .
- Relativity
- Noether's Second Theorem
- Gauge theory and field theories
  Maxwell, Yang–Mills, conformal, string, . . .
- Fluid Mechanics, Metereology
  Euler, Navier–Stokes,
  boundary layer, quasi-geostropic , . . .
- Linear and linearizable PDEs
- Solitons (in $2 + 1$ dimensions)
  K–P, Davey-Stewartson, . . .
- Kac–Moody
- *Lie groups*!

# What's New?

Direct constructive algorithms for:

- Invariant Maurer–Cartan forms
- Structure equations
- Moving frames
- Differential invariants
- Invariant differential operators
- Basis Theorem
- Syzygies and recurrence formulae
- Further applications

$\Longrightarrow$ Symmetry groups of differential equations $\Longrightarrow$ Vessiot group splitting
$\Longrightarrow$ Gauge theories
$\Longrightarrow$ Calculus of variations

# Symmetry Groups — Review

System of differential equations:

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, 2, \ldots, k$$

Prolonged vector field:

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=0}^{n} \varphi_\alpha^J \frac{\partial}{\partial u_J^\alpha}$$

where

$$\varphi_\alpha^J = D_J \left( \varphi^\alpha - \sum_{i=1}^{p} u_i^\alpha \xi^i \right) + \sum_{i=1}^{p} u_{J,i}^\alpha \xi^i$$

$$\equiv \Phi_\alpha^J(x, u^{(n)}; \xi^{(n)}, \varphi^{(n)})$$

Infinitesimal invariance:

$$\mathbf{v}^{(n)}(\Delta_\nu) = 0 \qquad \text{whenever} \qquad \Delta = 0.$$

Infinitesimal determining equations:

$$\mathcal{L}(x, u; \xi^{(n)}, \varphi^{(n)}) = 0$$

$$\mathcal{L}(\ \ldots\ , x^i,\ \ldots\ , u^\alpha,\ \ldots\ , \xi_A^i,\ \ldots\ , \varphi_A^\alpha,\ \ldots\ ) = 0$$

$$\Longrightarrow \quad \text{involutive completion}$$

## The Korteweg–deVries equation

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Symmetry generator:

$$\mathbf{v} = \tau(t,x,u)\,\frac{\partial}{\partial t} + \xi(t,x,u)\,\frac{\partial}{\partial x} + \varphi(t,x,u)\,\frac{\partial}{\partial u}$$

Prolongation:

$$\mathbf{v}^{(3)} = \mathbf{v} + \varphi^t\,\frac{\partial}{\partial u_t} + \varphi^x\,\frac{\partial}{\partial u_x} + \ \cdots\ + \varphi^{xxx}\,\frac{\partial}{\partial u_{xxx}}$$

where

$$\varphi^t = \varphi_t + u_t\varphi_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x\xi_u$$

$$\varphi^x = \varphi_x + u_x\varphi_u - u_t\tau_x - u_t u_x\tau_u - u_x\xi_x - u_x^2\xi_u$$

$$\varphi^{xxx} = \varphi_{xxx} + 3\,u_x\varphi_u + \ \cdots$$

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Infinitesimal invariance:

$$\mathbf{v}^{(3)}(u_t + u_{xxx} + u u_x) = \varphi^t + \varphi^{xxx} + u\,\varphi^x + u_x\,\varphi = 0$$

on solutions

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Infinitesimal determining equations:

$$\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0$$

$$\varphi = \xi_t - \tfrac{2}{3}\, u \tau_t \qquad \varphi_u = -\tfrac{2}{3}\,\tau_t = -\,2\,\xi_x$$

$$\tau_{tt} = \tau_{tx} = \tau_{xx} = \ \cdots \ = \varphi_{uu} = 0$$

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General solution:

$$\tau = c_1 + 3\,c_4 t, \qquad \xi = c_2 + c_3 t + c_4 x, \qquad \varphi = c_3 - 2 c_4 u.$$

Basis for symmetry algebra:

$$\partial_t, \qquad \partial_x, \qquad t\,\partial_x + \partial_u, \qquad 3\,t\,\partial_t + x\,\partial_x - 2\,u\,\partial_u.$$

The symmetry group $\mathcal{G}_{KdV}$ is four-dimensional

$$(x, t, u) \ \longmapsto \ (\lambda^3 t + a, \lambda x + c\,t + b, \lambda^{-2} u + c)$$

# Differential Invariants

$\mathcal{G}$ — transformation group acting on $p$-dimensional submanifolds $N = \{ u = f(x) \} \subset M$

$\mathcal{G}^{(n)}$ — prolonged action on the submanifold jet space $\mathrm{J}^n = \mathrm{J}^n(M, p)$

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Differential invariant $\qquad I : \mathrm{J}^n \to \mathbb{R}$

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$\Longrightarrow$ curvature, torsion, ...

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

$\Longrightarrow$ arc length derivative

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$\star\star$ $\mathcal{I}(\mathcal{G})$ — the algebra of differential invariants $\star\star$

# The Basis Theorem

**Theorem.** Assume $\mathcal{G}^{(n)}$ acts locally freely on a dense open subset of $\mathrm{J}^n$ for all sufficiently large $n$. Then its differential invariant algebra $\mathcal{I}(\mathcal{G})$ is generated by a finite number of differential invariants $I_1, \ldots, I_\ell$, meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa.$$

$\Longrightarrow$ *Lie, Tresse, Ovsiannikov, Kumpera*

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$\diamondsuit\ \diamondsuit$ functional independence $\diamondsuit\ \diamondsuit$

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## $\star\star$ Constructive Version $\star\star$

$\Longrightarrow$ Computational algebra & Gröbner bases

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# Main Goals

Given a system of partial differential equations:

- Find the structure of its symmetry (pseudo-) group $\mathcal{G}$

  directly from the determining equations.

- Find and classify its differential invariants.

- Determine the structure of the

  differential invariant algebra $\mathcal{I}(\mathcal{G})$:

  ◇ Generating invariants: $I_1, \ldots, I_\ell$

  ◇ Invariant differential operators: $\mathcal{D}_1, \ldots, \mathcal{D}_p$

  ◇ Commutation relations $[\,\mathcal{D}_j, \mathcal{D}_k\,] = \sum K^i_{j,k}\,\mathcal{D}_i$

  ◇ Syzygies: $H(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) \equiv 0$

  $\Longrightarrow$ Gauss–Codazzi relations

# Basic Themes

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- The structure of a (connected) pseudo-group is fixed by its Lie algebra of *infinitesimal generators*: $\mathfrak{g}$
- The infinitesimal generators satisfy an overdetermined system of linear partial differential equations — the *determining equations*: $F = 0$
- The basic structure of an overdetermined system of PDEs is fixed by the algebraic structure of its *symbol module*: $\mathcal{I}$

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- The structure of the differential invariant algebra $\mathcal{I}(\mathcal{G})$ is fixed by the *prolonged infinitesimal generators*: $\mathfrak{g}^{(\infty)}$
- These satisfy an overdetermined system of partial differential equations — the *prolonged determining equations*: $H = 0$
- The basic structure of the prolonged determining equations is fixed by the algebraic structure of its *prolonged symbol module*: $\mathcal{J}$

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# Pseudo-groups

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**Definition.** A *pseudo-group* is a collection of local diffeomorphisms $\varphi \colon M \to M$ such that

- *Identity*: $\mathbf{1}_M \in \mathcal{G}$,
- *Inverses*: $\varphi^{-1} \in \mathcal{G}$,
- *Restriction*: $U \subset \operatorname{dom} \varphi \implies \varphi \mid U \in \mathcal{G}$,
- *Continuation*: $\operatorname{dom} \varphi = \bigcup U_\kappa$ and $\varphi \mid U_\kappa \in \mathcal{G} \implies \varphi \in \mathcal{G}$,
- *Composition*: $\operatorname{im} \varphi \subset \operatorname{dom} \psi \implies \psi \circ \varphi \in \mathcal{G}$.

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**Definition.** A *Lie pseudo-group* $\mathcal{G}$ is a pseudo-group whose transformations are the solutions to an involutive system of partial differential equations:

$$F(z, \varphi^{(n)}) = 0.$$

$\star$ Nonlinear determining equations
$\implies$ *analytic* (*Cartan–Kähler*)

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$\star\star$ Key complication: $\nexists$ Abstract object $\mathcal{G}$ ??? $\star\star$

## A Non-Lie Pseudo-group

Acting on $M = \mathbb{R}^2$:

$$\boxed{X = \varphi(x) \qquad Y = \varphi(y)}$$

where $\varphi \in \mathcal{D}(\mathbb{R})$.

♠ Cannot be characterized by a system of partial differential equations

$$\Delta(x, y, X^{(n)}, Y^{(n)}) = 0$$

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**Theorem.** (Johnson, Itskov) Any non-Lie pseudo-group can be completed to a Lie pseudo-group with the same differential invariants.

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Completion of previous example:

$$\boxed{X = \varphi(x), \qquad Y = \psi(y)}$$

where $\varphi, \psi \in \mathcal{D}(\mathbb{R})$.

# Infinitesimal Generators

$\mathfrak{g}$ — Lie algebra of infinitesimal generators of
the pseudo-group $\mathcal{G}$

$z = (x, u)$ — local coordinates on $M$

Vector field:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha \frac{\partial}{\partial u^\alpha}$$

Vector field jet:

$$\mathrm{j}_n \mathbf{v} \longmapsto \zeta^{(n)} = (\ \ldots\ \zeta^b_A\ \ldots\ )$$

$$\zeta^b_A = \frac{\partial^{\#A} \zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

*Infinitesimal* (*Linearized*) *Determining Equations*

$$\mathcal{L}(z, \zeta^{(n)}) = 0 \qquad (*)$$

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*Remark*: If $\mathcal{G}$ is the symmetry group of a system of differential equations $\Delta(x, u^{(n)}) = 0$, then $(*)$ is the (involutive completion of) the usual Lie determining equations for the symmetry group.

# The Diffeomorphism Pseudogroup

$M$ — smooth $m$-dimensional manifold

$\mathcal{D} = \mathcal{D}(M)$

— pseudo-group of all local diffeomorphisms

$$Z = \varphi(z) \qquad \begin{cases} z = (x,u) \text{ — source coordinates} \\ Z = (X,U) \text{ — target coordinates} \end{cases}$$

$\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset \mathrm{J}^n(M,M)$ — $n^{\text{th}}$ order jets

$\Longrightarrow$ groupoid

Local coordinates on $\mathcal{D}^{(n)}$:

$$g^{(n)} = (z, Z^{(n)}) = (\ \ldots\ z^a\ \ldots\ Z^b_A\ \ldots\ )$$
$$= (\ \ldots\ x^i\ \ldots\ u^\alpha\ \ldots\ X^i_A\ \ldots\ U^\alpha_A\ \ldots\ )$$

The multi-indices $A$ indicate partial derivatives with respect to $z = (x,u)$

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$\Longrightarrow$ *The Maurer–Cartan forms for the diffeomorphism pseudo-group are the* right-invariant *contact forms on* $\mathcal{D}^{(\infty)}$:

$$R_\psi(\varphi) = \varphi \circ \psi$$

## Diffeomorphism Jets and the Variational Bicomplex

$$\mathcal{D}^{(\infty)} \subset \mathrm{J}^\infty(M,M)$$

Local coordinates:

$$\underbrace{z^1,\ \ldots\ ,z^m,}_{\text{source}} \quad \underbrace{Z^1,\ \ldots\ ,Z^m,}_{\text{target}} \quad \underbrace{\ldots\ Z^b_A,\ \ldots}_{\text{jet}}$$

Horizontal forms:

$$dz^1,\ \ldots\ ,dz^m$$

Contact forms:

$$\Theta^b_A = d_G\, Z^b_A = dZ^b_A - \sum_{a=1}^{m} Z^a_{A,a}\, dz^a$$

Maurer–Cartan forms:

$$\mu^b_A = \mathbb{D}^A_Z \Theta^b = \mathbb{D}_{Z^{a_1}} \cdots \mathbb{D}_{Z^{a_n}} \Theta^b$$

$$b = 1, \ldots, m, \ \#A \geq 0$$

## Maurer–Cartan forms for $\mathcal{D}^{(\infty)}$

*Key observation*: The target coordinate functions $Z^a$ are right-invariant.

Decompose

$$dZ^a = \underset{\text{horizontal}}{\sigma^a} + \underset{\text{contact}}{\mu^a}$$

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Invariant horizontal forms:

$$\sigma^a = d_M Z^a = \sum_{b=1}^{m} Z^a_b \, dz^b$$

Invariant total differentiation (dual operators):

$$\mathbb{D}_{Z^a} = \sum_{b=1}^{m} (Z^a_b)^{-1} \, \mathbb{D}_{z^b}$$

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Invariant contact forms:

$$\mu^b = d_G Z^b = \Theta^b = dZ^b - \sum_{a=1}^{m} Z^b_a \, dz^a$$

$$\mu^b_A = \mathbb{D}^A_Z \Theta^b = \mathbb{D}_{Z^{a_1}} \cdots \mathbb{D}_{Z^{a_n}} \Theta^b$$

$$b = 1, \ldots, m, \ \#A \geq 0$$

$\Longrightarrow$ *Maurer–Cartan forms*

## One-dimensional case: $M = \mathbb{R}$

Contact forms:

$$\Theta = d_G X = dX - X_x \, dx$$

$$\Theta_x = \mathbb{D}_x \Theta = dX_x - X_{xx} \, dx$$

$$\Theta_{xx} = \mathbb{D}_x^2 \Theta = dX_{xx} - X_{xxx} \, dx$$

Right-invariant horizontal form:

$$\sigma = d_M X = X_x \, dx$$

Invariant differentiation:

$$\mathbb{D}_X = \frac{1}{X_x} \, \mathbb{D}_x$$

Maurer–Cartan forms:

$$\mu = \Theta$$

$$\mu_X = \mathbb{D}_X \mu = \frac{\Theta_x}{X_x}$$

$$\mu_{XX} = \mathbb{D}_X^2 \mu = \frac{X_x \, \Theta_{xx} - X_{xx} \, \Theta_x}{X_x^3}$$

# Two-dimensional case: $M = \mathbb{R}^2$

Coordinates on $\mathcal{D}^{(\infty)}(\mathbb{R}^2)$:

$$(x, u, X, U, X_x, X_u, U_x, U_u, X_{xx}, X_{xu}, \ \ldots \ )$$

Contact forms on $\mathcal{D}^{(2)}(\mathbb{R}^2)$:

$$\Upsilon = dX - X_x\,dx - X_u\,du \qquad \Phi = dU - U_x\,dx - U_u\,du$$

$$\Upsilon_x = dX_x - X_{xx}\,dx - X_{xu}\,du \qquad \Phi_x = dU_x - U_{xx}\,dx - U_{xu}\,du$$

$$\Upsilon_u = dX_u - X_{xu}\,dx - X_{uu}\,du \qquad \Phi_u = dU_u - U_{xu}\,dx - U_{uu}\,du$$

Maurer–Cartan forms:

$$\sigma = d_M\,X = X_x\,dx + X_u\,du, \qquad \tau = d_M\,U = U_x\,dx + U_u\,du,$$

$$\mu = \Upsilon = \ d_G\,X \qquad \nu = \Phi = \ d_G\,U$$

$$\mu_X = \mathbb{D}_X\mu = \frac{U_u\,\Upsilon_x - U_x\,\Upsilon_u}{X_xU_u - X_uU_x} \qquad \nu_X = \mathbb{D}_X\nu = \frac{U_u\,\Phi_x - U_x\,\Phi_u}{X_xU_u - X_uU_x}$$

$$\mu_U = \mathbb{D}_U\mu = \frac{X_x\,\Upsilon_u - X_u\,\Upsilon_x}{X_xU_u - X_uU_x} \qquad \nu_U = \mathbb{D}_U\nu = \frac{X_x\,\Phi_u - X_u\,\Phi_x}{X_xU_u - X_uU_x}$$

Right-invariant differentiations:

$$\mathbb{D}_X = \frac{U_u\,\mathbb{D}_x - U_x\,\mathbb{D}_u}{X_xU_u - X_uU_x}\,, \qquad \mathbb{D}_U = \frac{-\,X_u\,\mathbb{D}_x + X_x\,\mathbb{D}_u}{X_xU_u - X_uU_x}\,.$$

## The Universal Diffeomorphism

## Structure Equations

Maurer–Cartan formal series:

$$\mu^b[\![\, H \,]\!] = \sum_A \frac{1}{A!}\, \mu^b_A \, H^A$$

$$\Longrightarrow \quad H = (H^1, \ldots, H^m) \;\text{ — parameters}$$

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*Universal Structure Equations*

$$d\mu[\![\, H \,]\!] = \nabla_H \mu[\![\, H \,]\!] \wedge (\, \mu[\![\, H \,]\!] - dZ \,)$$

$$d\sigma = -\, d\mu[\![\, 0 \,]\!] = \nabla_H \mu[\![\, 0 \,]\!] \wedge \sigma$$

$\Longrightarrow$ equate powers of $H$:

$$d\mu^b_A = \sum \, C^{b,D,F}_{A,c,e}\, \mu^c_D \wedge \mu^e_F$$

## One-dimensional case: $M = \mathbb{R}$

Structure equations:

$$d\sigma = \mu_X \wedge \sigma \qquad d\mu[\![ H ]\!] = \mu_H[\![ H ]\!] \wedge (\mu[\![ H ]\!] - dZ)$$

where

$$\sigma = X_x\, dx = dX - \mu$$

$$\mu[\![ H ]\!] = \mu + \mu_X\, H + \tfrac{1}{2}\, \mu_{XX}\, H^2 + \cdots$$

$$\mu[\![ H ]\!] - dZ = -\,\sigma + \mu_X\, H + \tfrac{1}{2}\, \mu_{XX}\, H^2 + \ \cdots$$

$$\mu_H[\![ H ]\!] = \mu_X + \mu_{XX}\, H + \tfrac{1}{2}\, \mu_{XXX}\, H^2 + \ \cdots$$

In components:

$$d\sigma = \mu_1 \wedge \sigma$$

$$d\mu_n = -\,\mu_{n+1} \wedge \sigma \ + \ \sum_{i=0}^{n-1} \binom{n}{i}\ \mu_{i+1} \wedge \mu_{n-i}$$

$$= \sigma \wedge \mu_{n+1} - \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \frac{n-2j+1}{n+1} \binom{n+1}{j}\ \mu_j \wedge \mu_{n+1-j}.$$

$\Longrightarrow$ Cartan

## **Two-dimensional case:** $M = \mathbb{R}^2$

Maurer–Cartan form series:

$$dX = \sigma + \mu \qquad\qquad dY = \tau + \nu$$

$$\mu[\![ H, K ]\!] = \mu + \mu_H H + \mu_K K + \tfrac{1}{2}\mu_{HH} H^2 + \mu_{HK} H K + \tfrac{1}{2}\mu_{KK} K^2 + \cdots ,$$

$$\nu[\![ H, K ]\!] = \nu + \nu_H H + \nu_K K + \tfrac{1}{2}\nu_{HH} H^2 + \nu_{HK} H K + \tfrac{1}{2}\nu_{KK} K^2 + \cdots ,$$

Structure equations:

$$\begin{pmatrix} d\mu[\![ H, K ]\!] \\ d\nu[\![ H, K ]\!] \end{pmatrix} = \begin{pmatrix} \mu_H[\![ H, K ]\!] & \mu_K[\![ H, K ]\!] \\ \nu_H[\![ H, K ]\!] & \nu_K[\![ H, K ]\!] \end{pmatrix} \wedge \begin{pmatrix} \mu[\![ H, K ]\!] - dX \\ \nu[\![ H, K ]\!] - dU \end{pmatrix}$$

First order structure equations:

$$d\mu = -\,d\sigma = -\,\mu_X \wedge \sigma - \mu_U \wedge \tau,$$

$$d\nu = -\,d\tau = -\,\nu_X \wedge \sigma - \nu_U \wedge \tau,$$

$$d\mu_X = -\,\mu_{XX} \wedge \sigma - \mu_{XU} \wedge \tau + \mu_U \wedge \nu_X,$$

$$d\nu_X = -\,\nu_{XX} \wedge \sigma - \nu_{XU} \wedge \tau + \nu_X \wedge (\mu_X - \nu_U),$$

$$d\mu_U = -\,\mu_{XU} \wedge \sigma - \mu_{UU} \wedge \tau + (\mu_X - \nu_U) \wedge \mu_U,$$

$$d\nu_U = -\,\nu_{XU} \wedge \sigma - \nu_{UU} \wedge \tau + \nu_X \wedge \mu_U$$

# The Structure Equations for a Lie Pseudo-group

Infinitesimal determining equations:

$$\mathcal{L}(\ \ldots\ z^a\ \ldots\ \zeta^b_A\ \ldots\ ) = 0 \qquad (\star)$$

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*The Maurer–Cartan forms for $\mathcal{G}$ are obtained by restricting the diffeomorphism Maurer–Cartan forms $\mu^b_A$ to $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.*

The resulting forms $\mu^b_A$ are no longer linearly independent:

**Theorem.** The Maurer–Cartan forms on $\mathcal{G}^{(\infty)}$ satisfy the invariant infinitesimal determining equations

$$\mathcal{L}(\ \ldots\ Z^a\ \ldots\ \mu^b_A\ \ldots\ ) = 0 \qquad (\star\star)$$

obtained from the infinitesimal determining equations $(\star)$ by replacing

- source variables $z^a$ by target variables $Z^a$
- derivatives of vector field coefficients $\zeta^b_A$ by right-invariant Maurer–Cartan forms $\mu^b_A$

# The Fundamental Structure Theorem

**Theorem.** The structure equations for the pseudo-group $\mathcal{G}$ are obtained by restricting the universal diffeomorphism structure equations

$$d\mu[\![\, H \,]\!] = \nabla_H \mu[\![\, H \,]\!] \wedge (\, \mu[\![\, H \,]\!] - dZ \,)$$

to the solution space of the linearized involutive system

$$\mathcal{L}(\ \ldots\ Z^a,\ \ldots\ \mu^b_A,\ \ldots\ ) = 0.$$

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♠ The structure equations are on the principal bundle $\mathcal{G}^{(\infty)}$; if $G$ is a finite-dimensional Lie group, then $\mathcal{G}^{(\infty)} \simeq M \times G$, and the usual Lie group structure equations are found by neestriction to the target fibers $\{\, Z = c \} \simeq G$.

# Korteweg–deVries Equation

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Diffeomorphism Maurer–Cartan forms:

$$\mu^t,\ \mu^x,\ \mu^u,\ \mu^t_T,\ \mu^t_X,\ \mu^t_U,\ \mu^x_T,\ \ldots,\ \mu^u_U,\ \mu^t_{TT},\ \mu^T_{TX},\ \ldots$$

Maurer–Cartan determining equations:

$$\mu^t_X = \mu^t_U = \mu^x_U = \mu^u_T = \mu^u_X = 0,$$

$$\mu^u = \mu^x_T - \tfrac{2}{3}\, U \mu^t_T, \qquad \mu^u_U = -\,\tfrac{2}{3}\,\mu^t_T = -\,2\,\mu^x_X,$$

$$\mu^t_{TT} = \mu^t_{TX} = \mu^t_{XX} = \ \cdots\ = \mu^u_{UU} = \ \ldots\ = 0.$$

Basis ($\dim \mathcal{G}_{KdV} = 4$):

$$\mu^1 = \mu^t, \qquad \mu^2 = \mu^x, \qquad \mu^3 = \mu^u, \qquad \mu^4 = \mu^t_T.$$

Structure equations:

$$d\mu^1 = -\,\mu^1 \wedge \mu^4,$$

$$d\mu^2 = -\mu^1 \wedge \mu^3 - \tfrac{2}{3}\, U\, \mu^1 \wedge \mu^4 - \tfrac{1}{3}\,\mu^2 \wedge \mu^4,$$

$$d\mu^3 = \tfrac{2}{3}\,\mu^3 \wedge \mu^4,$$

$$d\mu^4 = 0.$$

$$\boxed{d\mu^i = C^i_{jk}\, \mu^j \wedge \mu^k}$$

# Lie–Kumpera Example

$$\boxed{X = f(x) \qquad U = \frac{u}{f'(x)}}$$

Linearized determining system

$$\xi_x = -\frac{\varphi}{u} \qquad \xi_u = 0 \qquad \varphi_u = \frac{\varphi}{u}$$

Maurer–Cartan forms:

$$\sigma = \frac{u}{U}\,dx = f_x\,dx, \qquad \tau = U_x\,dx + \frac{U}{u}\,du = \frac{-\,u\,f_{xx}\,dx + f_x\,du}{f_x^2}$$

$$\mu = dX - \frac{U}{u}\,dx = df - f_x\,dx, \qquad \nu = dU - U_x\,dx - \frac{U}{u}\,du = -\frac{u}{f_x^2}\,(\,df_x - f_{xx}\,dx\,)$$

$$\mu_X = \frac{du}{u} - \frac{dU - U_x\,dx}{U} = \frac{df_x - f_{xx}\,dx}{f_x}\,, \qquad \mu_U = 0$$

$$\nu_X = \frac{U}{u}\,(dU_x - U_{xx}\,dx) - \frac{U_x}{u}\,(dU - U_x\,dx)$$

$$= -\frac{u}{f_x^3}(\,df_{xx} - f_{xxx}\,dx\,) + \frac{u\,f_{xx}}{f_x^4}(\,df_x - f_{xx}\,dx\,)$$

$$\nu_U = -\frac{du}{u} + \frac{dU - U_x\,dx}{U} = -\frac{df_x - f_{xx}\,dx}{f_x}$$

Right-invariant linearized system:

$$\mu_X = -\frac{\nu}{U} \qquad \mu_U = 0 \qquad \nu_U = \frac{\nu}{U}$$

First order structure equations:

$$d\mu = -\,d\sigma = \frac{\nu \wedge \sigma}{U}\,, \qquad d\nu = -\,\nu_X \wedge \sigma \; - \; \frac{\nu \wedge \tau}{U}$$

$$d\nu_X = -\,\nu_{XX} \wedge \sigma \; - \; \frac{\nu_X \wedge (\tau + 2\,\nu)}{U}$$

## Action of Pseudo-groups on Submanifolds

$\mathcal{G}$ — Lie pseudo-group acting on $p$-dimensional submanifolds — solutions to differential equations:

$$N = \{ u = f(x) \} \subset M$$

$\mathrm{J}^n = \mathrm{J}^n(M,p)$ — $n^{\text{th}}$ order jet bundle

Local coordinates

$$z^{(n)} = (x, u^{(n)}) = ( \ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$$

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## Prolongation

Prolonged action of $\mathcal{G}^{(n)}$ on submanifolds:

$$(x, u^{(n)}) \quad \longmapsto \quad (X, \widehat{U}^{(n)})$$

Coordinate formulae:

$$\widehat{U}^\alpha_J = F^\alpha_J(x, u^{(n)}, g^{(n)})$$

$\Longrightarrow$ Implicit differentiation.

# Moving Frames for Pseudo–Groups

In the finite-dimensional Lie group case, a moving frame is defined as an equivariant map

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow G$$

However, we no longer have an abstract object to represent our pseudo-group $\mathcal{G}$, and so the moving frame will be an equivariant section of the pulled-back pseudo-group principal jet bundle:

$$\begin{array}{ccc} \mathcal{G}^{(n)} & \longleftarrow & \mathcal{H}^{(n)} \\ \downarrow & & \downarrow \\ M & \longleftarrow & \mathrm{J}^n. \end{array}$$

**Definition.** A (right) *moving frame* of *order* $n$ is a right-equivariant section $\rho^{(n)} : V^n \to \mathcal{H}^{(n)}$ defined on an open subset $V^n \subset \mathrm{J}^n$.

# Freeness

**Proposition.** A moving frame of order $n$ exists if and only if $\mathcal{G}^{(n)}$ acts *freely* and regularly.

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♣ In finite-dimensions, freeness means no isotropy. For infinite-dimensional pseudo-groups, one must restrict to the transformation jets of order $n$.

---

Isotropy subgroup

$$\mathcal{G}^{(n)}_{z^{(n)}} = \left\{ g^{(n)} \in \mathcal{G}^{(n)}_z \;\middle|\; g^{(n)} \cdot z^{(n)} = z^{(n)} \right\}$$

**Definition.** The pseudo-group $\mathcal{G}$ acts

- *freely* at $z^{(n)} \in \mathrm{J}^n$ if $\mathcal{G}^{(n)}_{z^{(n)}} = \{ \mathbf{1}^{(n)}_z \}$
- *locally freely* if
  - $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}_z$
  - the orbits have $\dim = r_n = \dim \mathcal{G}^{(n)}_z$

# Freeness Theorem

**Theorem.** If $n \geq 1$ and $\mathcal{G}^{(n)}$ acts locally freely at $z^{(n)} \in \mathrm{J}^n$, then it acts locally freely at any $z^{(k)} \in \mathrm{J}^k$ with $\widetilde{\pi}^k_n(z^{(k)}) = z^{(n)}$ for all $k > n$.

# Normalization

♠ To construct a moving frame :

I. Choose a cross-section to the pseudo-group orbits:

$$u^{\alpha_\kappa}_{J_\kappa} = c_\kappa, \qquad \kappa = 1, \ldots, r_n = \text{fiber dim } \mathcal{G}^{(n)}$$

II. Solve the normalization equations

$$F^{\alpha_\kappa}_{J_\kappa}(x, u^{(n)}, g^{(n)}) = c_\kappa$$

for the pseudo-group parameters

$$g^{(n)} = \rho^{(n)}(x, u^{(n)})$$

III. Invariantization maps differential functions to differential invariants:

$$\iota : F(x, u^{(n)}) \longmapsto I(x, u^{(n)}) = F(\, \rho^{(n)}(x, u^{(n)}) \cdot (x, u^{(n)})\,)$$

$\Longrightarrow$ an algebra morphism and a projection:

$$\iota \circ \iota = \iota$$

$$I(x, u^{(n)}) = \iota(\, I(x, u^{(n)})\,)$$

# Invariantization

A moving frame induces an invariantization process, denoted $\iota$, that projects functions to invariants, differential operators to invariant differential operators; differential forms to invariant differential forms, etc.

Geometrically, the invariantization of an object is the unique invariant version that has the same cross-section values.

Algebraically, invariantization amounts to replacing the group parameters in the transformed object by their moving frame formulas.

# Invariantization

In particular, invariantization of the jet coordinates leads to a complete system of functionally independent differential invariants:

$$\iota(x^i) = H^i \qquad \iota(u_J^\alpha) = I_J^\alpha$$

- Phantom differential invariants: $I_{J_\kappa}^{\alpha_\kappa} = c_\kappa$
- The non-constant invariants form a functionally independent generating set for the differential invariant algebra $\mathcal{I}(\mathcal{G})$
- Replacement Theorem

$$I(\ \ldots\ x^i\ \ldots\ u_J^\alpha\ \ldots\ ) = \iota(\,I(\ \ldots\ x^i\ \ldots\ u_J^\alpha\ \ldots\ )\,)$$
$$= I(\ \ldots\ H^i\ \ldots\ I_J^\alpha\ \ldots\ )$$

---

$\diamondsuit$ Differential forms $\Longrightarrow$ invariant differential forms

$$\iota(\,dx^i) = \omega^i \qquad\qquad i = 1, \ldots, p$$

$\diamondsuit$ Differential operators $\Longrightarrow$

invariant differential operators

$$\iota\,(\,\mathrm{D}_{x^i}\,) = \mathcal{D}_i \qquad\qquad i = 1, \ldots, p$$

# Recurrence Formulae

⋆⋆ Invariantization and differentiation do not commute ⋆⋆

---

The *recurrence formulae* connect the differentiated invariants with their invariantized counterparts:

$$\boxed{\boxed{\mathcal{D}_i I^\alpha_J = I^\alpha_{J,i} + M^\alpha_{J,i}}}$$

$\Longrightarrow$ $M^\alpha_{J,i}$ — correction terms

♡ Once established, they completely prescribe the structure of the differential invariant algebra $\mathcal{I}(\mathcal{G})$ — thanks to the functional independence of the non-phantom normalized differential invariants.

⋆⋆ The recurrence formulae can be explicitly determined using only the infinitesimal generators and linear differential algebra!

## The Key Formula

$$d_H I^\alpha_J \;=\; \sum_{i=1}^{p} (\mathcal{D}_i I^\alpha_J)\,\omega^i \;=\; \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}^{\,\alpha}_J$$

where

$$\widehat{\psi}^{\,\alpha}_J = \iota(\,\widehat{\varphi}^{\,\alpha}_J\,) = \Phi^\alpha_J(\;\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\;;\;\ldots\;\gamma^b_A\;\ldots\;)$$

are the invariantized prolonged vector field coefficients, which are particular linear combinations of

$\gamma^b_A \;=\; \iota(\zeta^b_A)$ — invariantized Maurer–Cartan forms prescribed by the invariantized prolongation map.

**Proposition.**

The invariantized Maurer–Cartan forms are subject to the *invariantized determining equations*:

$$\mathcal{L}(H^1,\ldots,H^p,I^1,\ldots,I^q,\;\ldots\;,\gamma^b_A,\;\ldots\;) = 0$$

$$d_H I^\alpha_J \;=\; \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}\,^\alpha_J(\,\ldots\;\gamma^b_A\;\ldots\,)$$

**Step 1:** Solve the phantom recurrence formulas

$$0 = d_H I^\alpha_J \;=\; \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}\,^\alpha_J(\,\ldots\;\gamma^b_A\;\ldots\,)$$

for the invariantized Maurer–Cartan forms:

$$\gamma^b_A = \sum_{i=1}^{p} J^b_{A,i}\,\omega^i \qquad (*)$$

**Step 2:** Substitute $(*)$ into the non-phantom recurrence formulae to obtain the explicit correction terms.

♢ Only uses linear differential algebra based on the specification of cross-section.

♡ Does not require explicit formulas for the moving frame, the differential invariants, the invariant differential operators, or even the Maurer–Cartan forms!

# Korteweg–deVries equation

Symmetry Group Action:

$$T = e^{3\lambda_4}(t + \lambda_1) = 0$$

$$X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2) = 0$$

$$U = e^{-2\lambda_4}(u + \lambda_3) = 0$$

Prolonged Action:

$$\begin{aligned}
U_T &= e^{-5\lambda_4}(u_t - \lambda_3 u_x), \\
U_X &= e^{-3\lambda_4}u_x, \\
U_{TT} &= e^{-8\lambda_4}(u_{tt} - 2\lambda_3 u_{tx} + \lambda_3^2 u_{xx}), \\
U_{TX} &= D_X D_T U = e^{-6\lambda_4}(u_{tx} - \lambda_3 u_{xx}), \\
U_{XX} &= e^{-4\lambda_4}u_{xx}, \\
&\ \ \vdots
\end{aligned}$$

Cross Section:

$$T = e^{3\lambda_4}(t + \lambda_1) = 0$$

$$X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2) = 0$$

$$U = e^{-2\lambda_4}(u + \lambda_3) = 0$$

$$U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x) = 1$$

Moving Frame:

$$\lambda_1 = -t, \qquad \lambda_2 = -x, \qquad \lambda_3 = -u, \qquad \lambda_4 = \tfrac{1}{5}\log(u_t + u u_x)$$

Phantom Invariants:

$$H^1 = \iota(t) = 0, \qquad I_{00} = \iota(u) = 0,$$
$$H^2 = \iota(x) = 0, \qquad I_{10} = \iota(u_t) = 1.$$

Normalized differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + uu_x)^{3/5}}$$

$$I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2 u_{xx}}{(u_t + uu_x)^{8/5}}$$

$$I_{11} = \iota(u_{tx}) = \frac{u_{tx} + uu_{xx}}{(u_t + uu_x)^{6/5}}$$

$$I_{02} = \iota(u_{xx}) = \frac{u_{xx}}{(u_t + uu_x)^{4/5}}$$

$$I_{03} = \iota(u_{xxx}) = \frac{u_{xxx}}{u_t + uu_x}$$
$$\vdots$$

Replacement Theorem:

$$0 = \iota(u_t + uu_x + u_{xxx}) = 1 + I_{03} = \frac{u_t + uu_x + u_{xxx}}{u_t + uu_x}.$$

Invariant horizontal one-forms:

$$\omega^1 = \iota(dt) = (u_t + uu_x)^{3/5}\, dt,$$

$$\omega^2 = \iota(dx) = -u(u_t + uu_x)^{1/5}\, dt + (u_t + uu_x)^{1/5}\, dx.$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_t) = (u_t + uu_x)^{-3/5} D_t + u(u_t + uu_x)^{-3/5} D_x,$$

$$\mathcal{D}_2 = \iota(D_x) = (u_t + uu_x)^{-1/5} D_x.$$

Recurrence formula:

$$dI_{jk} = I_{j+1,k}\,\omega^1 + I_{j,k+1}\,\omega^2 + \iota(\varphi^{jk})$$

Invariantized Maurer–Cartan forms:

$$\iota(\tau) = \lambda, \qquad \iota(\xi) = \mu, \qquad \iota(\varphi) = \psi = \nu, \qquad \iota(\tau_t) = \psi^t = \lambda_t, \qquad \ldots$$

Invariantized determining equations:

$$\lambda_x = \lambda_u = \mu_u = \nu_t = \nu_x = 0$$

$$\nu = \mu_t \qquad \nu_u = -\,2\,\mu_x = -\,\tfrac{2}{3}\,\lambda_t$$

$$\lambda_{tt} = \lambda_{tx} = \lambda_{xx} = \;\cdots\; = \nu_{uu} = \;\cdots\; = 0$$

Invariantizations of prolonged vector field coefficients:

$$\iota(\tau) = \lambda, \qquad \iota(\xi) = \mu, \qquad \iota(\varphi) = \nu, \qquad \iota(\varphi^t) = -I_{01}\nu - \tfrac{5}{3}\,\lambda_t,$$

$$\iota(\varphi^x) = -I_{01}\lambda_t, \qquad \iota(\varphi^{tt}) = -2I_{11}\nu - \tfrac{8}{3}I_{20}\lambda_t, \qquad \ldots$$

Phantom recurrence formulae:

$$0 = d_H H^1 = \omega^1 + \lambda,$$

$$0 = d_H H^2 = \omega^2 + \mu,$$

$$0 = d_H I_{00} = I_{10}\omega^1 + I_{01}\omega^2 + \psi = \omega^1 + I_{01}\omega^2 + \nu,$$

$$0 = d_H I_{10} = I_{20}\omega^1 + I_{11}\omega^2 + \psi^t = I_{20}\omega^1 + I_{11}\omega^2 - I_{01}\nu - \tfrac{5}{3}\lambda_t,$$

$\Longrightarrow$ Solve for $\lambda = -\omega^1, \quad \mu = -\omega^2, \quad \nu = -\omega^1 - I_{01}\omega^2,$

$$\lambda_t = \tfrac{3}{5}(I_{20} + I_{01})\omega^1 + \tfrac{3}{5}(I_{11} + I_{01}^2)\omega^2.$$

Non-phantom recurrence formulae:

$$d_H I_{01} = I_{11}\omega^1 + I_{02}\omega^2 - I_{01}\lambda_t,$$

$$d_H I_{20} = I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\nu - \tfrac{8}{3}I_{20}\lambda_t,$$

$$d_H I_{11} = I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\nu - 2I_{11}\lambda_t,$$

$$d_H I_{02} = I_{12}\omega^1 + I_{03}\omega^2 - \tfrac{4}{3}I_{02}\lambda_t,$$

$$\vdots$$

$$\mathcal{D}_1 I_{01} = I_{11} - \tfrac{3}{5}I_{01}^2 - \tfrac{3}{5}I_{01}I_{20}, \qquad \mathcal{D}_2 I_{01} = I_{02} - \tfrac{3}{5}I_{01}^3 - \tfrac{3}{5}I_{01}I_{11},$$

$$\mathcal{D}_1 I_{20} = I_{30} + 2I_{11} - \tfrac{8}{5}I_{01}I_{20} - \tfrac{8}{5}I_{20}^2, \qquad \mathcal{D}_2 I_{20} = I_{21} + 2I_{01}I_{11} - \tfrac{8}{5}I_{01}^2I_{20} - \tfrac{8}{5}I_{11}I_{20},$$

$$\mathcal{D}_1 I_{11} = I_{21} + I_{02} - \tfrac{6}{5}I_{01}I_{11} - \tfrac{6}{5}I_{11}I_{20}, \qquad \mathcal{D}_2 I_{11} = I_{12} + I_{01}I_{02} - \tfrac{6}{5}I_{01}^2I_{11} - \tfrac{6}{5}I_{11}^2,$$

$$\mathcal{D}_1 I_{02} = I_{12} - \tfrac{4}{5}I_{01}I_{02} - \tfrac{4}{5}I_{02}I_{20}, \qquad \mathcal{D}_2 I_{02} = I_{03} - \tfrac{4}{5}I_{01}^2I_{02} - \tfrac{4}{5}I_{02}I_{11},$$

$$\vdots \qquad\qquad \vdots$$

Generating differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + u u_x)^{3/5}}, \qquad I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2u u_{tx} + u^2 u_{xx}}{(u_t + u u_x)^{8/5}}.$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_t) = (u_t + u u_x)^{-3/5} D_t + u(u_t + u u_x)^{-3/5} D_x,$$

$$\mathcal{D}_2 = \iota(D_x) = (u_t + u u_x)^{-1/5} D_x.$$

Commutation formula:

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = I_{01}\,\mathcal{D}_1$$

Fundamental syzygy:

$$\mathcal{D}_1^2 I_{01} + \tfrac{3}{5}\, I_{01} \mathcal{D}_1 I_{20} - \mathcal{D}_2 I_{20} + \left(\tfrac{1}{5}\, I_{20} + \tfrac{19}{5}\, I_{01}\right) \mathcal{D}_1 I_{01}$$
$$- \mathcal{D}_2 I_{01} - \tfrac{6}{25}\, I_{01} I_{20}^2 - \tfrac{7}{25}\, I_{01}^2 I_{20} + \tfrac{24}{25}\, I_{01}^3 = 0.$$

# Lie–Tresse–Kumpera Example

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

Horizontal coframe

$$d_H X = f_x \, dx, \qquad d_H Y = dy,$$

Implicit differentiations

$$\mathrm{D}_X = \frac{1}{f_x} \mathrm{D}_x, \qquad \mathrm{D}_Y = \mathrm{D}_y.$$

Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$

$$X = f \qquad Y = y \qquad U = \frac{u}{f_x}$$

$$U_X = \frac{u_x}{f_x^2} - \frac{u\, f_{xx}}{f_x^3} \qquad U_Y = \frac{u_y}{f_x}$$

$$U_{XX} = \frac{u_{xx}}{f_x^3} - \frac{3u_x\, f_{xx}}{f_x^4} - \frac{u\, f_{xxx}}{f_x^4} + \frac{3u\, f_{xx}^2}{f_x^5}$$

$$U_{XY} = \frac{u_{xy}}{f_x^2} - \frac{u_y\, f_{xx}}{f_x^3} \qquad U_{YY} = \frac{u_{yy}}{f_x}$$

$\Longrightarrow$ action is free at every order.

Coordinate cross-section

$$X = 0, \qquad U = 1, \qquad U_X = 0, \qquad U_{XX} = 0.$$

Moving frame

$$f = 0, \qquad f_x = u, \qquad f_{xx} = u_x, \qquad f_{xxx} = u_{xx}.$$

Differential invariants

$$U_Y \longmapsto J = \frac{u_y}{u}$$

$$U_{XY} \longmapsto J_1 = \frac{u u_{xy} - u_x u_y}{u^3} \qquad U_{YY} \longmapsto J_2 = \frac{u_{yy}}{u}$$

Horizontal invariant coframe

$$d_H X \longmapsto u\,dx, \qquad d_H Y \longmapsto dy,$$

Invariant differentiations

$$\mathcal{D}_1 = \frac{1}{u}\,\mathrm{D}_x \qquad \mathcal{D}_2 = \mathrm{D}_y$$

Higher order differential invariants: $\mathcal{D}_1^m\,\mathcal{D}_2^n\,J$

$$J_{,1} = \mathcal{D}_1 J = \frac{u u_{xy} - u_x u_y}{u^3} = J_1,$$

$$J_{,2} = \mathcal{D}_2 J = \frac{u u_{yy} - u_y^2}{u^2} = J_2 - J^2.$$

$\Longrightarrow$ All higher order differential invariants are obtained from $J$ by invariant differentiation

$$\mathbf{v}_\psi^{(\infty)} = \gamma\,\partial_x - u\,\gamma_1\,\partial_u - (\,u\,\gamma_2 + 2\,u_x\,\gamma_1\,)\,\partial_{u_x} - u_y\,\gamma_1\,\partial_{u_y} - \ \cdots$$

Phantom invariants:

$$0 = dH = \varpi^1 + \gamma, \qquad\qquad 0 = dI_{10} = J_1\,\varpi^2 + \vartheta_1 - \gamma_2,$$

$$0 = dI_{00} = J\,\varpi^2 + \vartheta - \gamma_1, \qquad\qquad 0 = dI_{20} = J_3\,\varpi^2 + \vartheta_3 - \gamma_3,$$

Solve for pulled-back Maurer–Cartan forms:

$$\gamma = -\,\varpi^1, \qquad\qquad \gamma_2 = J_1\,\varpi^2 + \vartheta_1,$$

$$\gamma_1 = J\,\varpi^2 + \vartheta, \qquad\qquad \gamma_3 = J_3\,\varpi^2 + \vartheta_3,$$

Recurrence formulae: $\qquad dy = \varpi^2$

$$dJ = J_1\,\varpi^1 + (J_2 - J^2)\,\varpi^2 + \vartheta_2 - J\,\vartheta,$$

$$dJ_1 = J_3\,\varpi^1 + (J_4 - 3\,J\,J_1)\,\varpi^2 + \vartheta_4 - J\,\vartheta_1 - J_1\,\vartheta,$$

$$dJ_2 = J_4\,\varpi^1 + (J_5 - J\,J_2)\,\varpi^2 + \vartheta_5 - J_2\,\vartheta,$$

$$\mathcal{D}_1 J = J_1, \qquad \mathcal{D}_2 J = J_2 - J^2, \qquad d_{\mathcal{V}}\,J = \vartheta_2 - J\,\vartheta,$$

$$\mathcal{D}_1 J_1 = J_3, \qquad \mathcal{D}_2 J_1 = J_4 - 3\,J\,J_1, \qquad d_{\mathcal{V}}\,J_1 = \vartheta_4 - J\,\vartheta_{10} - J_1\,\vartheta,$$

$$\mathcal{D}_1 J_2 = J_4, \qquad \mathcal{D}_2 J_2 = J_5 - J\,J_2, \qquad d_{\mathcal{V}}\,J_2 = \vartheta_5 - J_2\,\vartheta,$$

$\Longrightarrow$ All higher order differential invariants are obtained from $J$ by invariant differentiation

Invariant horizontal forms

$$d\varpi^1 = -\,J\,\varpi^1 \wedge \varpi^2 + \vartheta \wedge \varpi^1, \qquad\qquad d\varpi^2 = 0.$$

Commmutation formula

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = J\,\mathcal{D}_1.$$

## Gröbner Basis Approach

Identify the cross-section variables with the complementary monomials to a certain algebraic module $\mathcal{J}$, which is the pull-back of the symbol module of the pseudo-group under a certain explicit linear map.

$\Longrightarrow$ Compatible term ordering.

$\Longrightarrow$ Algebraic specification of compatible moving frames of all orders $n > n^\star$.

---

**Theorem.** Suppose $\mathcal{G}$ acts freely at order $n^\star$. Then a system of generating differential invariants is contained in the non-phantom normalized differential invariants of order $n^\star$ and those differential invariants corresponding to a Gröbner basis for the module $\mathcal{J}^{>n^\star}$.

# The Symbol Module

Linearized determining equations

$$\mathcal{L}(z, \zeta^{(n)}) = 0$$

$$t = (t_1, \ldots, t_m), \qquad T = (T_1, \ldots, T_m)$$

$$\mathcal{T} = \left\{ P(t,T) = \sum_{a=1}^{m} P_a(t)\, T_a \right\} \simeq \mathbb{R}[t] \otimes \mathbb{R}^m \subset \mathbb{R}[t,T]$$

$$\mathcal{I} \subset \mathcal{T} \qquad \text{— symbol module}$$

$$s = (s_1, \ldots, s_p), \qquad S = (S_1, \ldots, S_q),$$

$$\widehat{\mathcal{S}} = \left\{ T(s,S) = \sum_{\alpha=1}^{q} T_\alpha(s)\, S^\alpha \right\} \simeq \mathbb{R}[s] \otimes \mathbb{R}^q \subset \mathbb{R}[s,S]$$

Define the linear map

$$s_i = \beta_i(t) = t_i + \sum_{\alpha=1}^{q} u_i^\alpha\, t_{p+\alpha}, \qquad i = 1, \ldots, p,$$

$$S^\alpha = B_\alpha(T) = T_{p+\alpha} - \sum_{i=1}^{p} u_i^\alpha\, T_i, \qquad \alpha = 1, \ldots, q.$$

Prolonged symbol module:

$$\boxed{\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I})}$$

$\mathcal{N}$ — leading monomials $s_J S^\alpha$

$\Longrightarrow$ normalized differential invariants $I_J^\alpha$

$\mathcal{K}$ — complementary monomials $s_K S^\beta$

$\Longrightarrow$ phantom differential invariants $I_K^\beta$

## The Symbol Module

Vector field:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^b(z) \frac{\partial}{\partial z^b}$$

Vector field jet:

$$\mathrm{j}_\infty \mathbf{v} \quad \Longleftrightarrow \quad \zeta^{(\infty)} = (\ \ldots\ \zeta^b_A\ \ldots\ )$$

$$\zeta^b_A = \frac{\partial^{\# A} \zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

*Determining Equations* for $\mathbf{v} \in \mathfrak{g}$

$$\mathcal{L}(z;\ \ldots\ \zeta^b_A\ \ldots\ ) = 0 \qquad (*)$$

# Duality

$$t = (t_1, \ldots, t_m) \qquad T = (T_1, \ldots, T_m)$$

Polynomial module:

$$\mathcal{T} = \left\{ P(t, T) = \sum_{a=1}^{m} P_a(t)\, T_a \right\} \simeq \mathbb{R}[t] \otimes \mathbb{R}^m \subset \mathbb{R}[t, T]$$

$$\mathcal{T} \;\simeq\; (\mathrm{J}^\infty TM|_z)^*$$

Dual pairing:

$$\left\langle\; \mathrm{j}_\infty \mathbf{v}\,;\, t_A T^b \;\right\rangle = \zeta^b_A.$$

Each polynomial

$$\eta(z; t, T) = \sum_{b=1}^{m} \sum_{\#A \leq n} h^A_b(z)\, t_A T^b \;\in \mathcal{T}$$

induces a linear partial differential equation

$$L(z, \zeta^{(n)}) = \left\langle\; \mathrm{j}_\infty \mathbf{v}\,;\, \eta(z; t, T) \;\right\rangle$$

$$= \sum_{b=1}^{m} \sum_{\#A \leq n} h^A_b(z)\, \zeta^b_A = 0$$

# The Linear Determining Equations

**Annihilator**:

$$\mathcal{L} = (\mathrm{J}^\infty \mathfrak{g})^\perp$$

**Determining Equations**

$$\left\langle\, \mathrm{j}_\infty \mathbf{v}\,;\, \eta \,\right\rangle = 0 \quad \text{for all} \quad \eta \in \mathcal{L} \quad \Longleftrightarrow \quad \mathbf{v} \in \mathfrak{g}$$

**Symbol** = highest degree terms:

$$\mathbf{\Sigma}[\, L(z, \zeta^{(n)}) \,] = \mathbf{H}[\, \eta(z; t, T) \,] = \sum_{b=1}^{m} \sum_{\#A=n} h_b^A(z)\, t_A T^b.$$

**Symbol submodule**:

$$\mathcal{I} = \mathbf{H}(\mathcal{L})$$

$\Longrightarrow$ Formal integrability (involutivity)

# Prolonged Duality

Prolonged vector field:

$$\mathbf{v}^{(\infty)} = \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha, J} \widehat{\varphi}\,{}^{\alpha}_{J}(x,u^{(k)})\,\frac{\partial}{\partial u^{\alpha}_{J}}$$

---

$$\widetilde{s} = (\widetilde{s}_1, \ldots, \widetilde{s}_p), \quad s = (s_1, \ldots, s_p), \quad S = (S_1, \ldots, S_q)$$

"Prolonged" polynomial module:

$$\widehat{\mathcal{S}} = \left\{ \sigma(s, S, \widetilde{s}) = \sum_{i=1}^{p} c_i \widetilde{s}_i + \sum_{\alpha=1}^{q} \widehat{\sigma}_{\alpha}(s)\, S^{\alpha} \right\} \simeq \mathbb{R}^p \oplus (\mathbb{R}[s] \otimes \mathbb{R}^q)$$

$$\widehat{\mathcal{S}} \;\simeq\; T^* \mathrm{J}^{\infty}\big|_{z^{(\infty)}}$$

Dual pairing:

$$\left\langle \; \mathbf{v}^{(\infty)}\, ;\; \widetilde{s}_i \; \right\rangle = \xi^i$$

$$\left\langle \; \mathbf{v}^{(\infty)}\, ;\; S^{\alpha} \; \right\rangle = Q^{\alpha} = \varphi^{\alpha} - \sum_{i=1}^{p} u^{\alpha}_i\, \xi^i$$

$$\left\langle \; \mathbf{v}^{(\infty)}\, ;\; s_J S^{\alpha} \; \right\rangle = \widehat{\varphi}\,{}^{\alpha}_{J} = \Phi^{\alpha}_{J}(u^{(n)}; \zeta^{(n)})$$

# Algebraic Prolongation

Prolongation of vector fields:

$$\begin{aligned} \mathbf{p}:\ \mathrm{J}^{\infty}\mathfrak{g} &\longmapsto \mathfrak{g}^{(\infty)} \\ \mathrm{j}_{\infty}\mathbf{v} &\longmapsto \mathbf{v}^{(\infty)} \end{aligned}$$

Dual prolongation map:

$$\mathbf{p}^{*}:\ \mathcal{S} \longrightarrow \mathcal{T}$$

$$\left\langle\ \mathrm{j}_{\infty}\mathbf{v}\,;\,\mathbf{p}^{*}(\sigma)\ \right\rangle = \left\langle\ \mathbf{p}(\mathrm{j}_{\infty}\mathbf{v})\,;\,\sigma\ \right\rangle = \left\langle\ \mathbf{v}^{(\infty)}\,;\,\sigma\ \right\rangle$$

$\star\star$ On the symbol level, $\mathbf{p}^{*}$ is algebraic $\star\star$

# Prolongation Symbols

Define the linear map $\boldsymbol{\beta} : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^m$

$$s_i = \beta_i(t) = t_i + \sum_{\alpha=1}^{q} u_i^\alpha \, t_{p+\alpha}, \qquad i = 1, \ldots, p,$$

$$S^\alpha = B_\alpha(T) = T_{p+\alpha} - \sum_{i=1}^{p} u_i^\alpha \, T_i, \qquad \alpha = 1, \ldots, q.$$

Pull-back map

$$\begin{aligned} &\boldsymbol{\beta}^*[\, \sigma(s_1, \ldots, s_p, S_1, \ldots, S_q) \,] \\ &\qquad = \sigma(\, \beta_1(t), \ldots, \beta_p(t), B_1(T) \ldots, B_q(T) \,) \end{aligned}$$

**Lemma.** The symbols of the prolonged vector field coefficients are

$$\boldsymbol{\Sigma}(\xi^i) = T^i \qquad \boldsymbol{\Sigma}(\widehat{\varphi}^{\,\alpha}) = T^{\alpha+p}$$

$$\boldsymbol{\Sigma}(Q^\alpha) = \boldsymbol{\beta}^*(S^\alpha) = B_\alpha(T)$$

$$\begin{aligned} \boldsymbol{\Sigma}(\widehat{\varphi}^{\,\alpha}_J) = \boldsymbol{\beta}^*(s_J S^\alpha) &= \boldsymbol{\beta}^*(s_{j_1} \cdots s_{j_n} S^\alpha) \\ &= \beta_{j_1}(t) \, \cdots \, \beta_{j_n}(t) \, B_\alpha(T) \end{aligned}$$

**Prolonged annihilator:**

$$\mathcal{Z} = (\mathbf{p}^*)^{-1}\mathcal{L} = (\mathfrak{g}^{(\infty)})^{\perp}$$

$$\langle\, \mathbf{v}^{(\infty)}\,; \sigma \,\rangle = 0 \qquad \text{for all} \quad \mathbf{v} \in \mathfrak{g} \quad \Longleftrightarrow \quad \sigma \in \mathcal{Z}$$

**Prolonged symbol subbundle:**

$$\mathcal{U} \;=\; \mathbf{H}(\mathcal{Z}) \;\subset\; \mathrm{J}^{\infty}(M,p) \times \mathcal{S}$$

**Prolonged symbol module:**

$$\boxed{\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I})}$$

*Warning*:

$$\mathcal{U} \;\subseteq\; \mathcal{J}$$

But

$$\mathcal{U}^n \;=\; \mathcal{J}^n \qquad \text{when} \qquad n > n^\star$$

$n^\star$ — order of freeness.

# Algebraic Recurrence

Polynomial:

$$\widetilde{\sigma}(\mathbf{I}^{(k)}; s, S) = \sum_{\alpha, J} h_a^J(\mathbf{I}^{(k)})\ s_J S^\alpha \in \widehat{\mathcal{S}}$$

Differential invariant:

$$I_{\tilde{\sigma}} = \sum_{\alpha, J} h_a^J(\mathbf{I}^{(k)})\, I_J^\alpha$$

Recurrence:

$$\boxed{\boxed{\mathcal{D}_i\, I_{\tilde{\sigma}} = I_{\mathcal{D}_i \tilde{\sigma}} \equiv I_{s_i \tilde{\sigma}} + R_{i, \tilde{\sigma}}}}$$

$$\tilde{\sigma} \in \widetilde{\mathcal{J}}^n,\ n > n^\star \quad \Longrightarrow \quad \begin{aligned} \text{order}\, I_{\tilde{\sigma}} &= n \\ \text{order}\, I_{\mathcal{D}_i \tilde{\sigma}} &= n + 1 \\ \text{order}\, R_{i, \tilde{\sigma}} &\leq n \end{aligned}$$

## Algebra $\Longrightarrow$ Invariants

$\mathcal{I}$ — symbol module

- determining equations for $\mathfrak{g}$

$\mathcal{M} \simeq \mathcal{T} \,/\, \mathcal{I}$ — complementary monomials $t_A T^b$

- pseudo-group parameters
- Maurer–Cartan forms

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$\mathcal{N}$ — leading monomials $s_J S^\alpha$

- normalized differential invariants $I^\alpha_J$

$\mathcal{K} = \mathcal{S} \,/\, \mathcal{N}$ — complementary monomials $s_K S^\beta$

- cross-section coordinates $u^\beta_K = c^\beta_K$
- phantom differential invariants $I^\beta_K$

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$$\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I})$$

**Freeness:** $\boldsymbol{\beta}^* : \ \mathcal{K} \stackrel{\sim}{\longrightarrow} \mathcal{M}$

# Generating Differential Invariants

**Theorem.** The differential invariant algebra is generated by differential invariants that are in one-to-one correspondence with the Gröbner basis elements of the prolonged symbol module plus, possibly, a finite number of differential invariants of order $\leq n^\star$.

# Syzygies

**Theorem.** Every differential syzygy among the generating differential invariants is either a syzygy among those of order $\leq n^\star$, or arises from an algebraic syzygy among the Gröbner basis polynomials in $\widetilde{\mathcal{J}}$.