Symmetry Groupoids and Signatures of Geometric Objects

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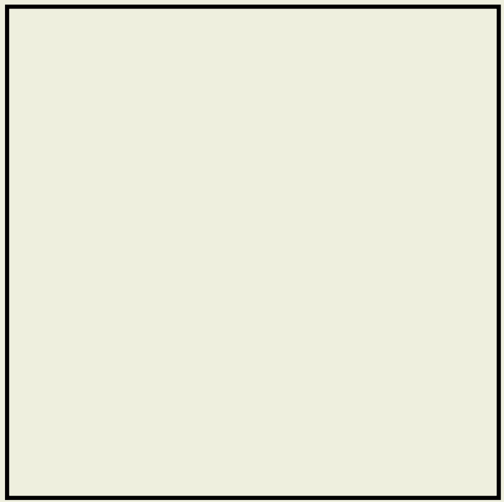
Symmetry

Definition. A symmetry of a set S is a transformation that preserves it:

$$g \cdot S = S$$

What is the Symmetry Group?



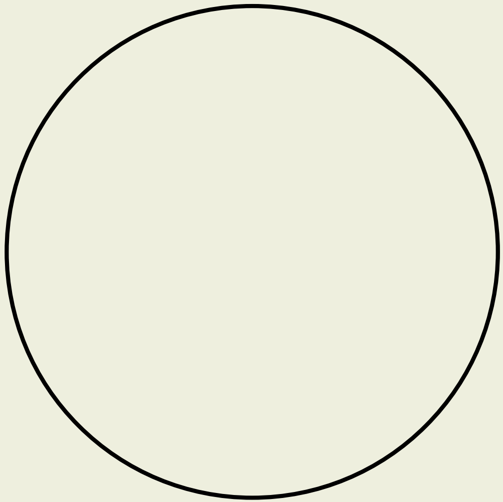
Rotations by 90°:

$$G_S = \mathbb{Z}_4$$

Rotations + reflections:

$$G_S = \mathbb{Z}_4 \ltimes \mathbb{Z}_4$$

What is the Symmetry Group?



Rotations:

$$G_S = \mathrm{SO}(2)$$

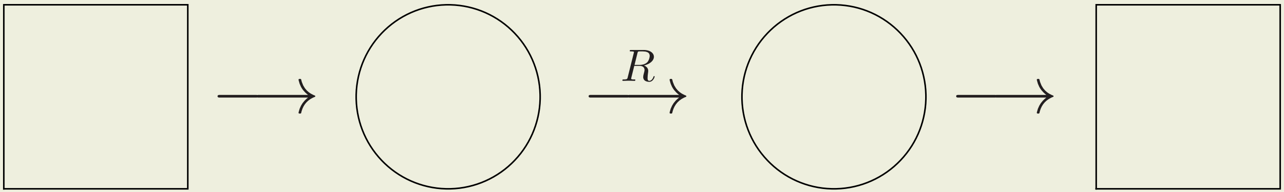
Rotations + reflections:

$$G_S = \mathrm{O}(2)$$

Conformal Inversions:

$$\overline{x} = \frac{x}{x^2 + y^2} \quad \overline{y} = \frac{y}{x^2 + y^2}$$

Continuous Symmetries of a Square



Symmetry

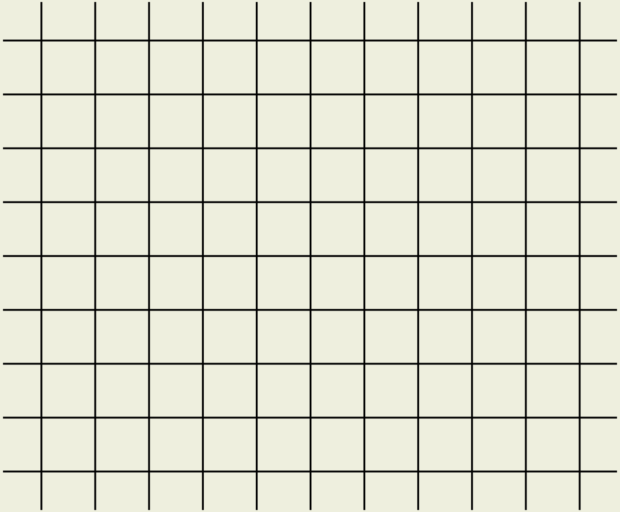
⋆ To define the set of symmetries requires a priori specification of the allowable transformations or, equivalently, the underlying geometry.

G — transformation group or pseudo-group of allowable transformations of the ambient space M

Definition. A symmetry of a subset $S \subset M$ is an allowable transformation $g \in G$ that preserves it:

$$g \cdot S = S$$

What is the Symmetry Group?



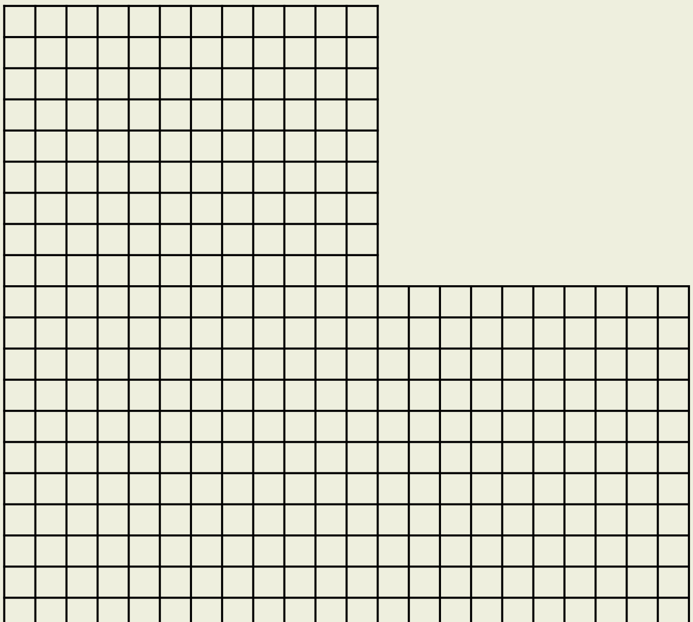
Allowable transformations:

Rigid motions

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$G_S = \mathbb{Z}_4 \ltimes \mathbb{Z}^2$$

What is the Symmetry Group?



Allowable transformations:

Rigid motions

$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

$G_S = \{e\}$

Local Symmetries

Definition. $g \in G$ is a local symmetry of $S \subset M$ based at a point $z \in S$ if there is an open neighborhood $z \in U \subset M$ such that

$$g \cdot (S \cap U) = S \cap (g \cdot U)$$

$G_z \subset G$ — the set of local symmetries based at z.

Global symmetries are local symmetries at all $z \in S$:

$$G_S \subset G_z \qquad G_S = \bigcap_{z \in S} G_z$$

$\star\star$ The set of all local symmetries forms a groupoid!

Groupoids

Definition. A groupoid is a small category such that every morphism has an inverse.

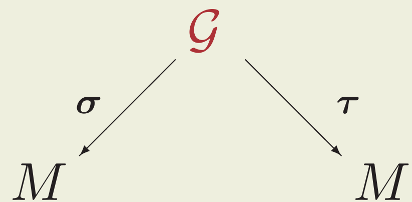
$\Longrightarrow$ Brandt (quadratic forms), Ehresmann (Lie pseudo-groups)

Mackenzie, R. Brown, A. Weinstein

Groupoids form the appropriate framework for studying objects with variable symmetry.

Groupoids

Double fibration:



$\boldsymbol{\sigma}$ — source map $\qquad$ $\boldsymbol{\tau}$ — target map

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$\star\star$ You are only allowed to multiply $\alpha \cdot \beta \in \mathcal{G}$ if

$$\boldsymbol{\sigma}(\alpha) = \boldsymbol{\tau}(\beta)$$

Groupoids

- *Source and target of products:*

$$\boldsymbol{\sigma}(\alpha \cdot \beta) = \boldsymbol{\sigma}(\beta) \qquad \boldsymbol{\tau}(\alpha \cdot \beta) = \boldsymbol{\tau}(\alpha) \qquad \text{when} \qquad \boldsymbol{\sigma}(\alpha) = \boldsymbol{\tau}(\beta)$$

- *Associativity:*

$$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma \qquad \text{when defined}$$

- *Identity section:* $e : M \to \mathcal{G} \qquad \boldsymbol{\sigma}(e(x)) = x = \boldsymbol{\tau}(e(x))$

$$\alpha \cdot e(\boldsymbol{\sigma}(\alpha)) = \alpha = e(\boldsymbol{\tau}(\alpha)) \cdot \alpha$$

- *Inverses:* $\boldsymbol{\sigma}(\alpha) = x = \boldsymbol{\tau}(\alpha^{-1}), \qquad \boldsymbol{\tau}(\alpha) = y = \boldsymbol{\sigma}(\alpha^{-1}),$

$$\alpha^{-1} \cdot \alpha = e(x), \qquad \alpha \cdot \alpha^{-1} = e(y)$$

Jet Groupoids

$\Longrightarrow$ Ehresmann

The set of Taylor polynomials of degree $\leq n$, or Taylor series ($n = \infty$) of local diffeomorphisms $\Psi : M \to M$ forms a groupoid.

$\diamondsuit$ Algebraic composition of Taylor polynomials/series is well-defined only when the source of the second matches the target of the first.

The Symmetry Groupoid

Definition. The symmetry groupoid of $S \subset M$ is

$$\mathcal{G}_S = \{\, (g, z) \mid z \in S,\; g \in G_z \,\} \subset G \times S$$

Source and target maps: $\boldsymbol{\sigma}(g, z) = z, \qquad \boldsymbol{\tau}(g, z) = g \cdot z.$

Groupoid multiplication and inversion:

$$(h, g \cdot z) \cdot (g, z) = (g \cdot h, z) \qquad (g, z)^{-1} = (g^{-1}, g \cdot z)$$

Identity map: $e(z) = (z, e) \in \mathcal{G}_S$

Local isotropy group of z:

$$G_z^* = \{\, g \in G_z \mid g \cdot z = z \,\}$$

$\Longrightarrow$ *vertex group*

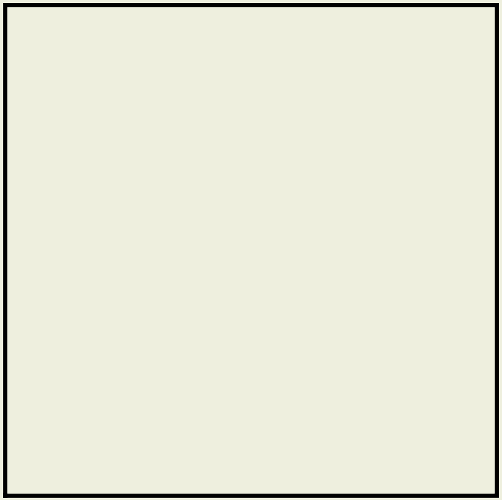
Lie Groupoids

$$\begin{array}{ccc} & \mathcal{G} & \\ {\scriptstyle\boldsymbol{\sigma}}\swarrow & & \searrow{\scriptstyle\boldsymbol{\tau}} \\ M & & M \end{array}$$

♡ A groupoid is a Lie groupoid if $\mathcal{G}$ and M are smooth manifolds, the source and target maps are smooth surjective submersions, and the identity and multiplication maps are smooth.

♠ Symmetry groupoids, even those of smooth submanifolds, are not necessarily Lie groupoids.

What is the Symmetry Groupoid?



$G = \mathrm{SE}(2)$

Corners:

$$G_z = G_S = \mathbb{Z}_4$$

Sides: G_z generated by

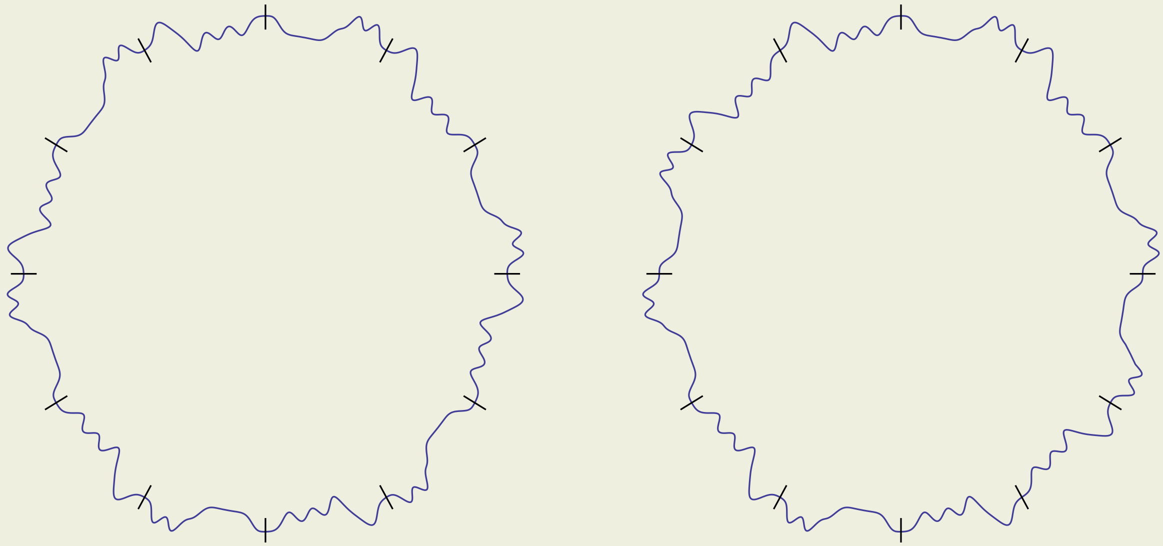
$G_S = \mathbb{Z}_4$

some translations

180° rotation around z

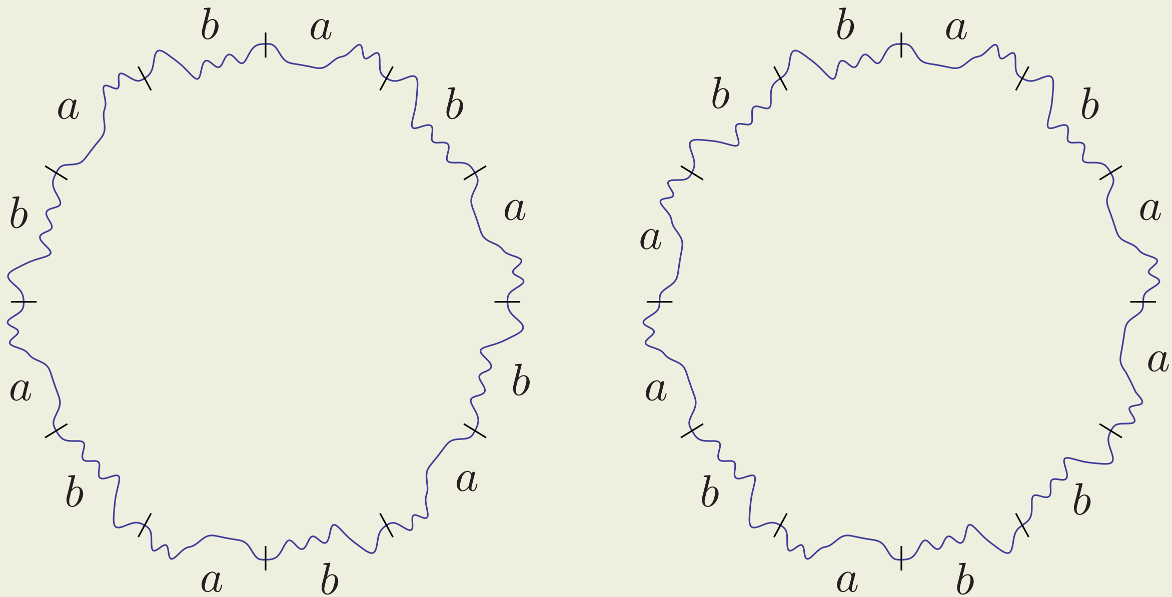
What is the Symmetry Groupoid?

Cogwheels $\Longrightarrow$ Musso–Nicoldi



What is the Symmetry Groupoid?

Cogwheels $\Longrightarrow$ Musso–Nicoldi



$G_S = \mathbb{Z}_6$ $\qquad$ $G_S = \mathbb{Z}_2$

Symmetry Orbits

$$\mathcal{O}_z = \boldsymbol{\tau}(\mathcal{G}_z) = \boldsymbol{\tau} \circ \boldsymbol{\sigma}^{-1}\{z\} = \{\, g \cdot z \mid g \in G_z \,\}.$$

$$\mathcal{O}_z \simeq G_z / G_z^*$$

Orbit equivalence:

$$z \sim \widehat{z} \text{ if and only } \widehat{z} = g \cdot z \text{ for some } g \in G_z$$

Symmetry moduli space: $S^{\mathcal{G}} = S/\sim$

The Equivalence Problem

$\Longrightarrow$ É Cartan

G — transformation group acting on M

Equivalence:

Determine when two subsets

$$S \quad \text{and} \quad \overline{S} \subset M$$

are congruent:

$$\overline{S} = g \cdot S \qquad \text{for} \qquad g \in G$$

Symmetry:

Find all symmetries or self-congruences:

$$S = g \cdot S$$

Tennis, Anyone?

Invariants

The solution to an equivalence problem rests on understanding its invariants.

Definition. If G is a group acting on M, then an invariant is a real-valued function $I\colon M \to \mathbb{R}$ that does not change under the action of G:

$$I(g \cdot z) = I(z) \qquad \text{for all} \qquad g \in G, \quad z \in M$$

Differential Invariants

Given a submanifold (curve, surface, . . .)

$$S \subset M$$

a differential invariant is an invariant of the prolonged action of G on its Taylor coefficients (jets):

$$\boxed{I(g \cdot z^{(k)}) \;=\; I(z^{(k)})}$$

Euclidean Plane Curves

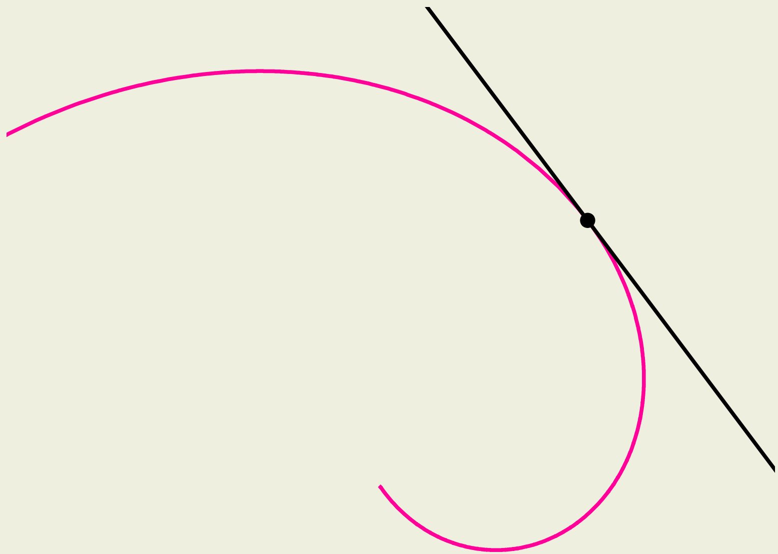
$G = \mathrm{SE}(2)$ acts on curves $C \subset M = \mathbb{R}^2$

The simplest differential invariant is the curvature, defined as the reciprocal of the radius of the osculating circle:

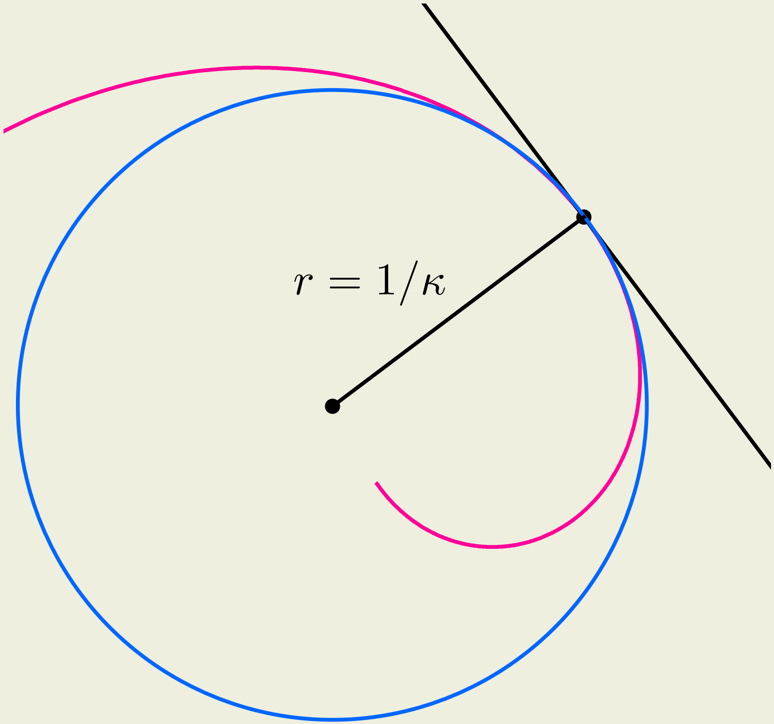
$$\kappa = \frac{1}{r}$$

Curvature

Curvature



Curvature



Euclidean Plane Curves: $G = \mathrm{SE}(2)$

Differentiation with respect to the Euclidean-invariant arc length element ds is an invariant differential operator, meaning that it maps differential invariants to differential invariants.

Thus, starting with curvature κ, we can generate an infinite collection of higher order Euclidean differential invariants:

$$\kappa, \quad \frac{d\kappa}{ds}, \quad \frac{d^2\kappa}{ds^2}, \quad \frac{d^3\kappa}{ds^3}, \quad \cdots$$

Theorem. All Euclidean differential invariants are functions of the derivatives of curvature with respect to arc length: $\kappa, \ \kappa_s, \ \kappa_{ss}, \ \cdots$

Euclidean Plane Curves: $G = \mathrm{SE}(2)$

Assume the curve $C \subset M$ is a graph: $\quad y = u(x)$

Differential invariants:

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \qquad \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}, \qquad \frac{d^2\kappa}{ds^2} = \cdots$$

Arc length (invariant one-form):

$$ds = \sqrt{1+u_x^2}\; dx, \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Equi-affine Plane Curves: $G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$

Equi-affine curvature:

$$\kappa = \frac{5\, u_{xx} u_{xxxx} - 3\, u_{xxx}^2}{9\, u_{xx}^{8/3}} \qquad \frac{d\kappa}{ds} = \cdots$$

Equi-affine arc length:

$$ds = \sqrt[3]{u_{xx}}\; dx \qquad \frac{d}{ds} = \frac{1}{\sqrt[3]{u_{xx}}}\, \frac{d}{dx}$$

Theorem. All equi-affine differential invariants are functions of the derivatives of equi-affine curvature with respect to equi-affine arc length: $\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots$

Plane Curves

Theorem. Let G be an ordinary* Lie group acting on $M = \mathbb{R}^2$. Then for curves $C \subset M$, there exists a unique (up to functions thereof) lowest order differential invariant κ and a unique (up to constant multiple) invariant differential form ds. Every other differential invariant can be written as a function of the "curvature" invariant and its derivatives with respect to "arc length": $\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots \quad .$

* ordinary = transitive + no pseudo-stabilization.

Moving Frames

The equivariant method of moving frames provides a systematic and algorithmic calculus for determining complete systems of differential invariants, invariant differential forms, invariant differential operators, etc., and the structure of the non-commutative differential algebra they generate.

Equivalence & Invariants

- Equivalent submanifolds $S \approx \overline{S}$
  must have the same invariants: $I = \overline{I}$.

Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

Syzygies

However, a functional dependency or syzygy among the invariants *is* intrinsic:

$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Theorem. (Cartan)

Two regular submanifolds are locally equivalent if and only if they have identical syzygies among *all* their differential invariants.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order differential invariants are always generated by invariant differentiation from a finite collection of basic differential invariants, and the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or ...) plane curves we need only know a single syzygy between κ and κ_s in order to establish equivalence!

Signature Curves

Definition. The signature curve $\Sigma \subset \mathbb{R}^2$ of a plane curve $C \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\chi \; : \; C \longrightarrow \Sigma = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$

$\Longrightarrow$ *Calabi, PJO, Shakiban, Tannenbaum, Haker*

Theorem. Two regular curves C and $\overline{C}$ are locally equivalent:

$$\overline{C} = g \cdot C$$

if and only if their signature curves are identical:

$$\overline{\Sigma} = \Sigma$$

$\Longrightarrow$ regular: $(\kappa_s, \kappa_{ss}) \neq 0$.

Continuous Symmetries of Curves

Theorem. For a connected curve, the following are equivalent:

- All the differential invariants are constant on C:
$$\kappa = c, \qquad \kappa_s = 0, \qquad \ldots$$
- The signature Σ degenerates to a point: $\dim \Sigma = 0$
- C is a piece of an orbit of a 1-dimensional subgroup $H \subset G$
- The local symmetry sets $G_z \subset G$ of $z \in C$ are all one-dimensional, and in fact, contained in a common one-dimensional subgroup $G_z \subset H \subset G$

Discrete Symmetries of Curves

Definition. The index of a completely regular point $\zeta \in \Sigma$ equals the number of points in C which map to it:

$$i_\zeta = \#\, \chi^{-1}\{\zeta\}$$

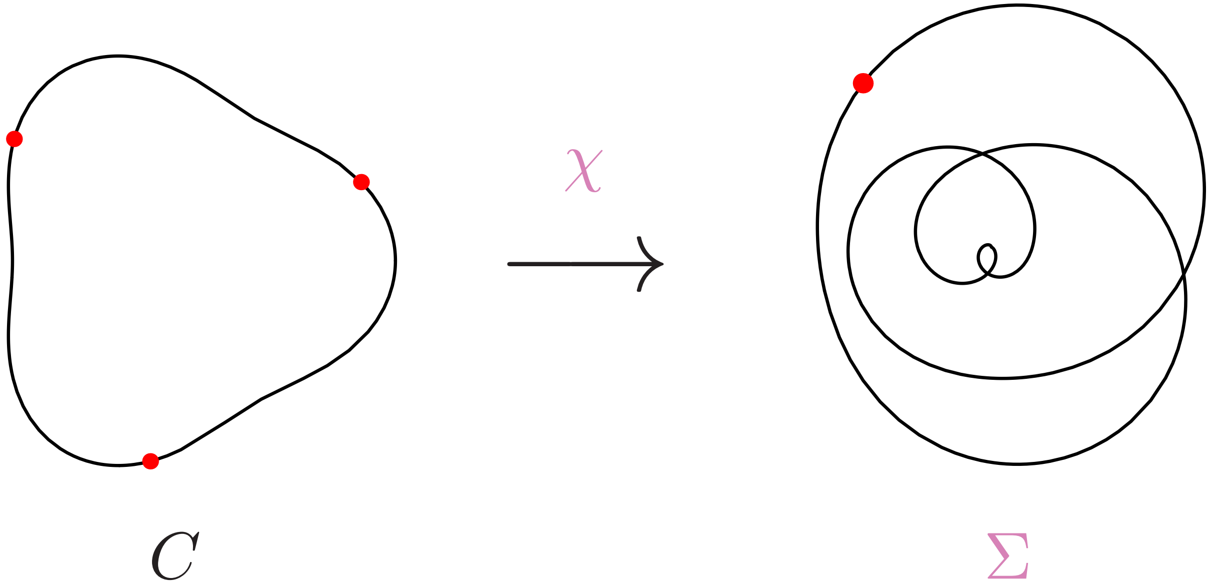
Regular means that, in a neighborhood of ζ, the signature is an embedded curve — no self-intersections.

Theorem. If $\chi(z) = \zeta$ is completely regular, then its index counts the number of discrete local symmetries of C that move z:

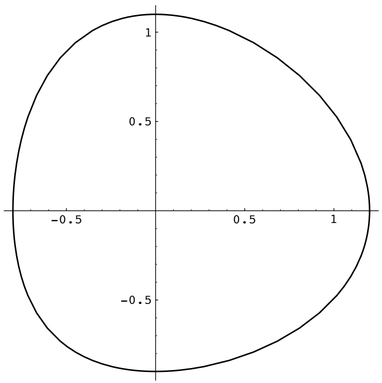
$$i_\zeta = \#\,(G_z / G_z^*)$$

G_z^* — isotropy group of z

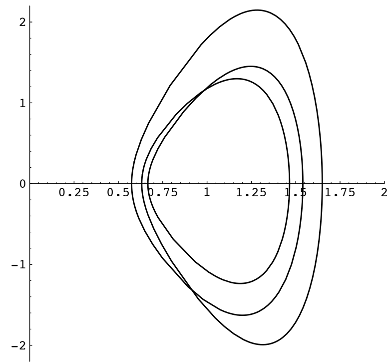
The Index



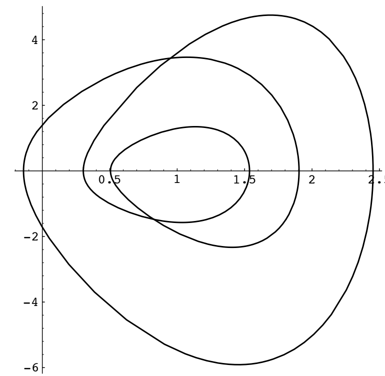
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

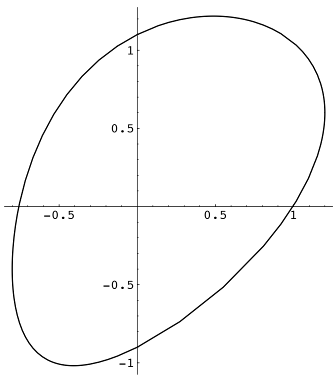


Euclidean Signature

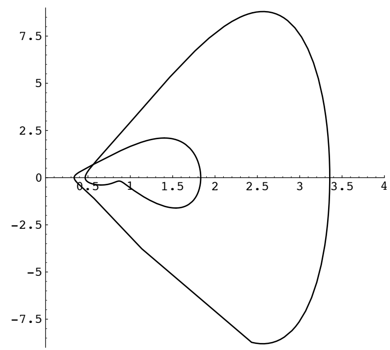


Equi-affine Signature

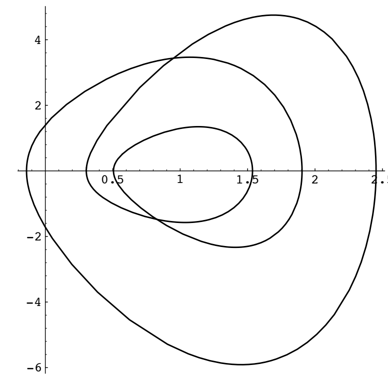
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \;\; y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



The Original Curve



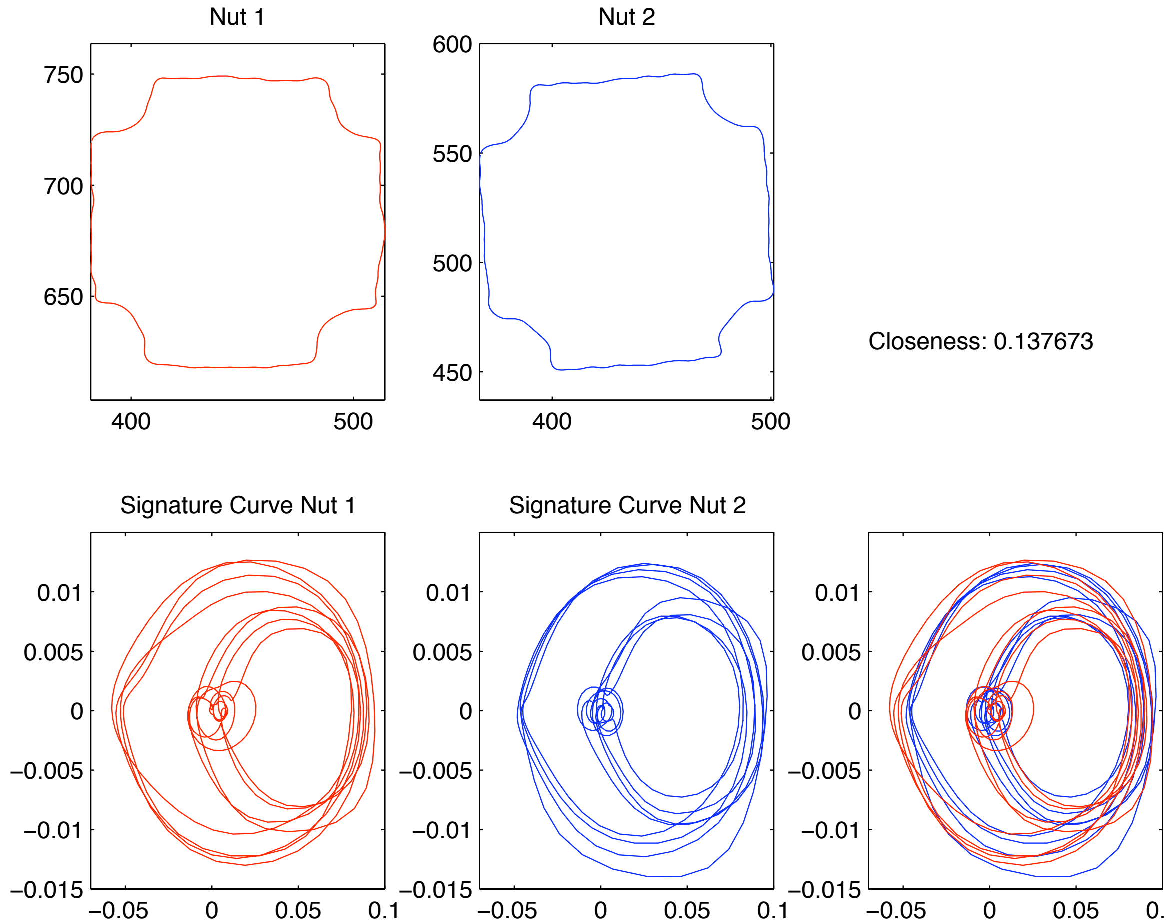
Euclidean Signature

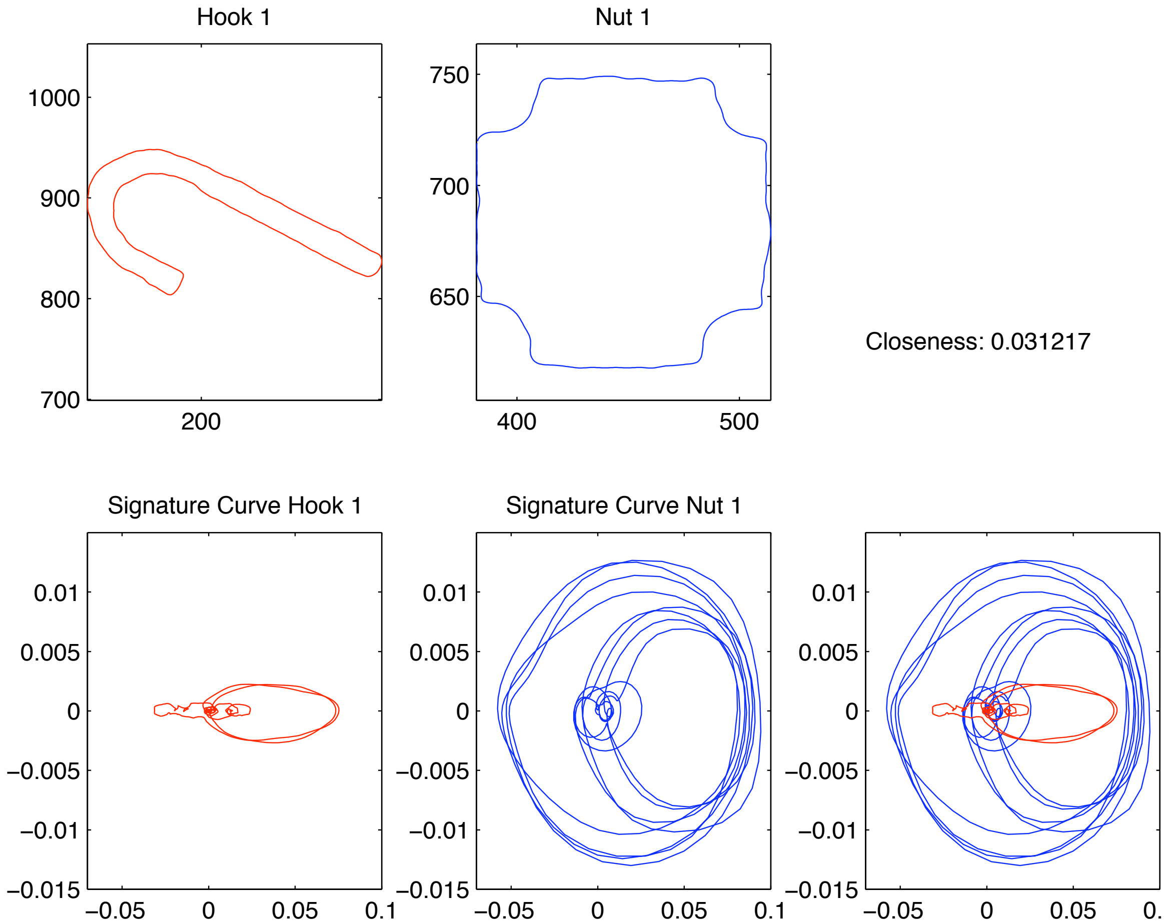


Equi-affine Signature

Object Recognition

$\Longrightarrow$ Steve Haker


Nut 1
750
700
650
400
500
Nut 2
600
550
500
450
400
500
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1


Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
1000
900
800
700
200
750
700
650
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

3D Signatures

Euclidean space curves: $C \subset \mathbb{R}^3$

$$\Sigma = \{ \, (\, \kappa \, , \, \kappa_s \, , \, \tau \,) \, \} \; \subset \; \mathbb{R}^3$$

- κ — curvature, τ — torsion

Euclidean surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \left(H \, , \, K \, , \, H_{,1} \, , \, H_{,2} \, , \, K_{,1} \, , \, K_{,2} \right) \right\} \; \subset \; \mathbb{R}^6$$

or $\widehat{\Sigma} = \left\{ \left(H \, , \, H_{,1} \, , \, H_{,2} \, , \, H_{,11} \right) \right\} \; \subset \; \mathbb{R}^4$

- H — mean curvature, K — Gauss curvature

Equi–affine surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \left(P \, , \, P_{,1} \, , \, P_{,2} , \, P_{,11} \right) \right\} \; \subset \; \mathbb{R}^4$$

- P — Pick invariant

Advantages of the Signature Curve

- Purely local — no ambiguities
- Local symmetries and approximate symmetries
- Extends to surfaces and higher dimensional submanifolds
- Occlusions and reconstruction
- Partial matching and puzzles

Main disadvantage: Noise sensitivity due to dependence on high order derivatives.

Vertices of Euclidean Curves

Ordinary vertex: local extremum of curvature

Generalized vertex: $\kappa_s \equiv 0$

- critical point
- circular arc
- straight line segment

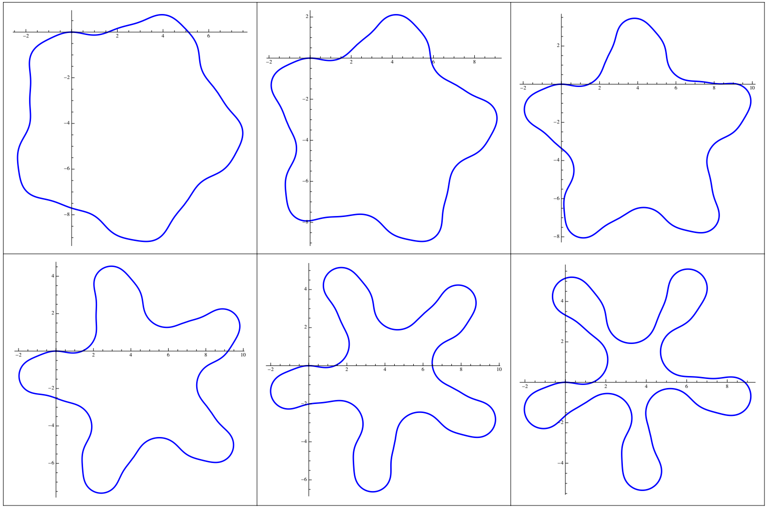
Mukhopadhya's Four Vertex Theorem:

A simple closed, non-circular plane curve has $n \geq 4$ generalized vertices.

"Counterexamples"

⋆ Generalized vertices map to a single point of the signature.

Hence, the (degenerate) curves obtained by replace ordinary vertices with circular arcs of the same radius all have *identical* signature:

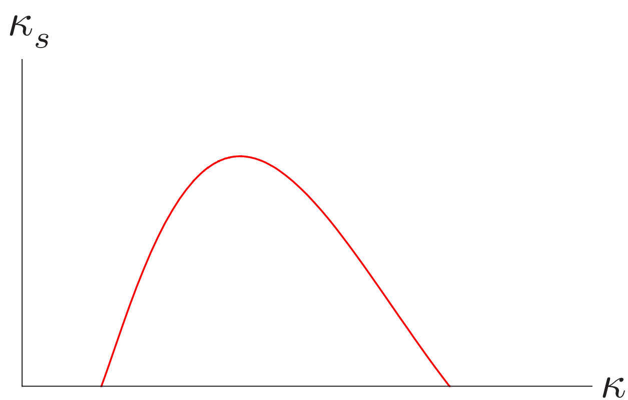


$\Longrightarrow$ Musso–Nicoldi

Bivertex Arcs

Bivertex arc: $\kappa_s \neq 0$ everywhere
except $\kappa_s = 0$ at the two endpoints

The signature Σ of a bivertex arc is a single arc that starts and ends on the κ–axis.



Bivertex Decomposition

v-regular curve — finitely many generalized vertices

$$C = \bigcup_{j=1}^{m} B_j \cup \bigcup_{k=1}^{n} V_k$$

$B_1, \ldots, B_m$ — bivertex arcs

$V_1, \ldots, V_n$ — generalized vertices: $n \geq 4$

Main Idea: Compare individual bivertex arcs, and then decide whether the rigid equivalences are (approximately) the same.

D. Hoff & PJO, Extensions of invariant signatures for object recognition, *J. Math. Imaging Vision* **45** (2013), 176–185.

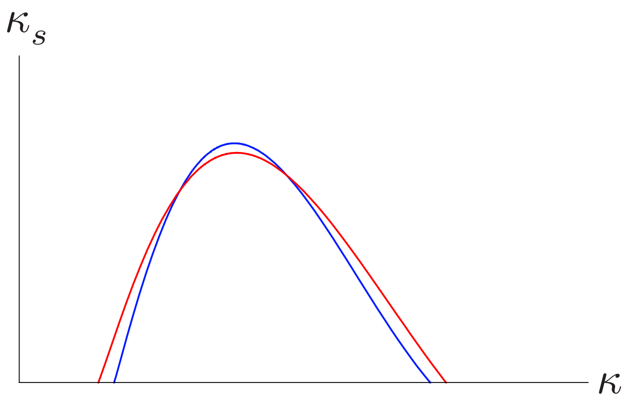
Signature Metrics

Used to compare signatures:

- Hausdorff
- Monge–Kantorovich transport
- Electrostatic/gravitational attraction
- Latent semantic analysis
- Histograms
- Gromov–Hausdorff & Gromov–Wasserstein

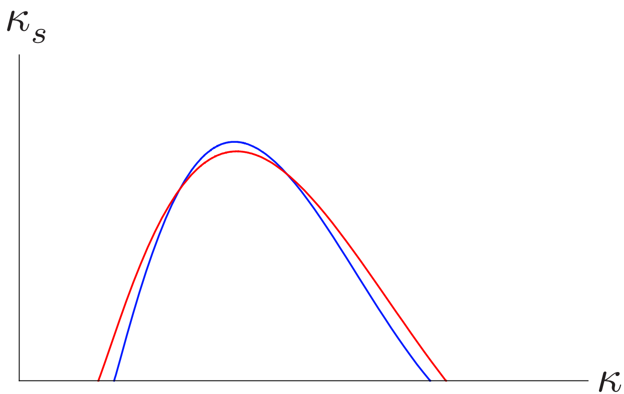
Gravitational/Electrostatic Attraction

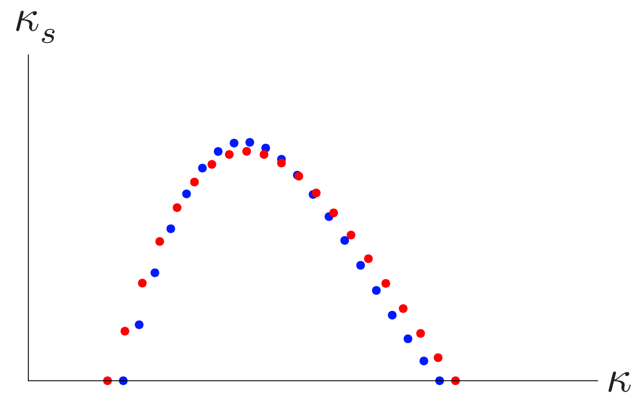
★ Treat the two signature curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.



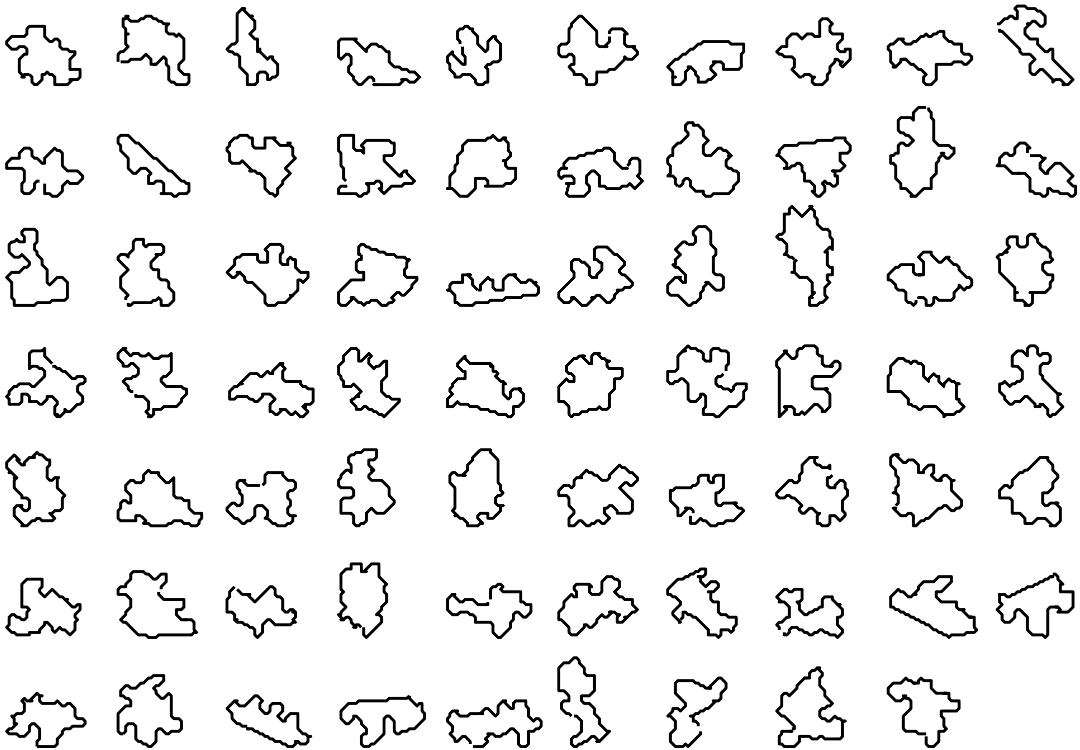
Gravitational/Electrostatic Attraction

- ★ Treat the two signature curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.
- ★ In practice, we are dealing with discrete data (pixels) and so treat the curves and signatures as point masses/charges.

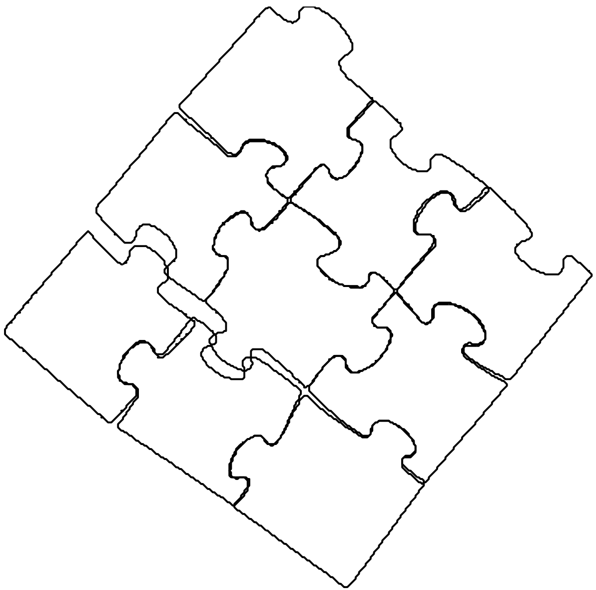


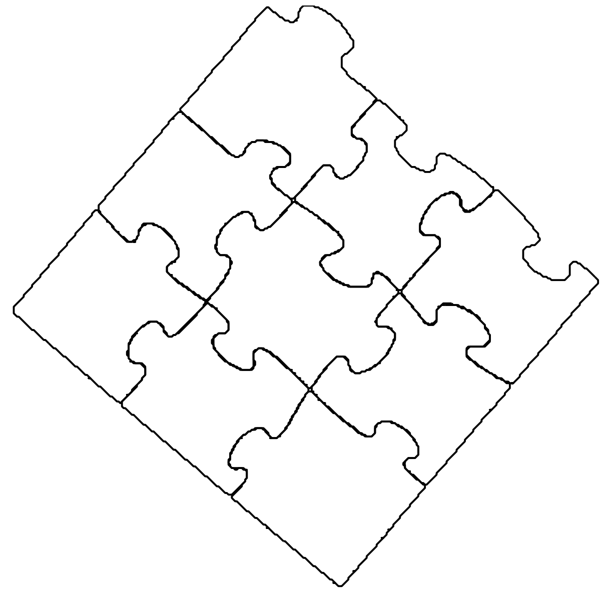


The Baffler Jigsaw Puzzle



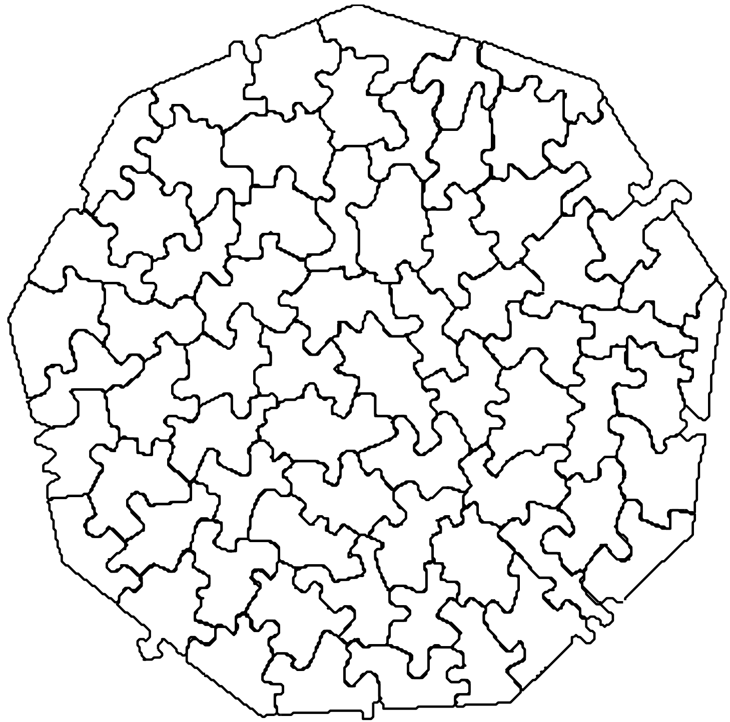
Piece Locking



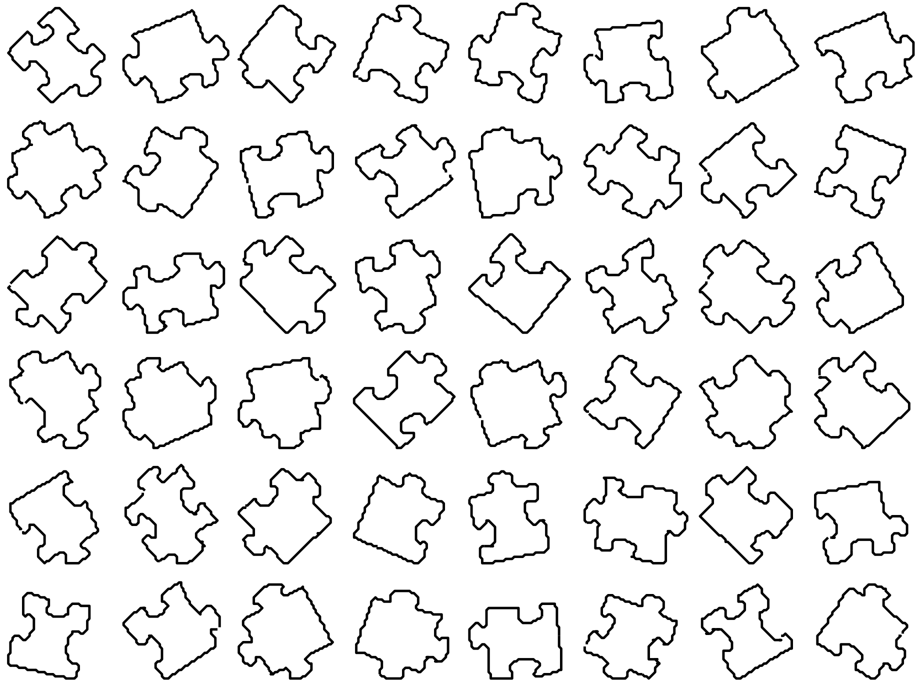


⋆⋆ Minimize force and torque based on gravitational attraction of the two matching edges.

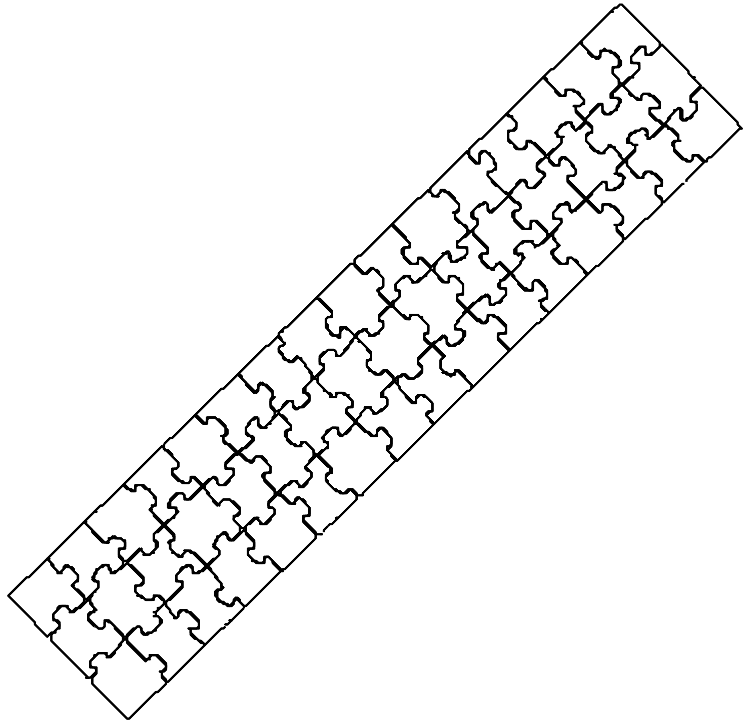
The Baffler Solved



The Rain Forest Giant Floor Puzzle

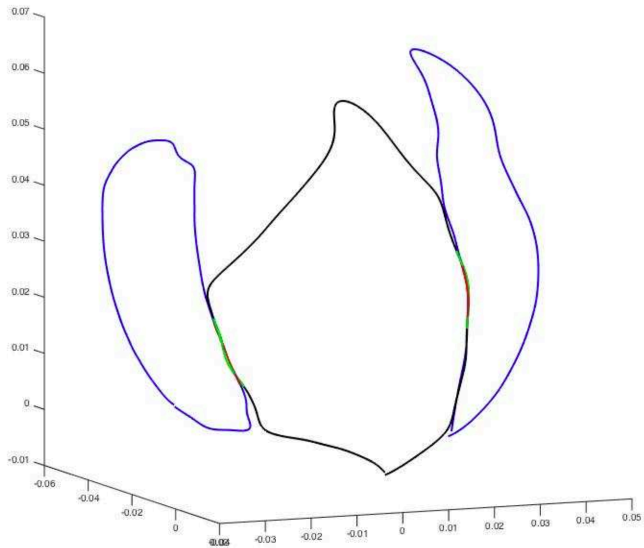


The Rain Forest Puzzle Solved



$\Longrightarrow$ D. Hoff & PJO, Automatic solution of jigsaw puzzles,
J. Math. Imaging Vision **49** (2014) 234–250.

Reassembling Humpty Dumpty



$\Longrightarrow$ Anna Grim, Ryan Schlecta

Broken ostrich egg shell — Marshall Bern

Signature and the Symmetry Groupoid

$\mathcal{G}_S$ — symmetry groupoid

Signature map: $\chi : S \to \Sigma$

If $g \in G_z$ is a local symmetry based at $z \in S$, then

$$\chi(g \cdot z) = \chi(z), \qquad \text{whenever} \qquad \alpha = (g, z) \in \mathcal{G}_S.$$

Thus, the signature map is constant on the symmetry groupoid orbits, and hence factors through the symmetry moduli space.

Signature Rank

Definition. The signature rank of a point $z \in S$ is the rank of the signature map at z:

$$r_z = \operatorname{rank} d\,\chi\,|_z.$$

A point $z \in S$ is called regular if the signature rank is constant in a neighborhood of z.

Proposition. If $z \in S$ is regular of rank k, then, near z, the signature Σ is a k-dimensional submanifold.

Cartan's Equivalence Theorem

Theorem. If $S, \widetilde{S} \subset M$ are regular, then locally there exists an equivalence map $g \in G$ with

$$\widetilde{S} \cap \widetilde{U} = g \cdot (S \cap U) \qquad g \in G$$

if and only if $S, \widetilde{S}$ have locally identical signatures:

$$\widetilde{\Sigma} = \widetilde{\chi}(\widetilde{S} \cap \widetilde{U}) = \chi(S \cap U) = \Sigma$$

Corollary. If $z \in S$ is regular, then $\widehat{z} = g \cdot z \in \mathcal{O}_z$ for $g \in G_z$ if and only if

$$\chi(S \cap U) = \chi(S \cap \widehat{U})$$

Pieces

Definition. A *piece* of the submanifold S is a connected subset $\widehat{S} \subset S$ whose interior is a non-empty submanifold of the same dimension $p = \dim \widehat{S} = \dim S$ and whose boundary $\partial\widehat{S}$ is a piecewise smooth submanifold of dimension $p - 1$.

Symmetry and Signature

$$\dim S = p$$

Assume $S \subset M$ is regular, connected, and of constant rank.

$$\begin{aligned} \operatorname{rank} S = k &= \dim \Sigma \\ &= \# \text{ functionally independent differential invariants} \end{aligned}$$

Then its local symmetry set at each $z \in S$ has

$$\dim G_z = p - k = \dim S - \dim \Sigma$$

Completion of Symmetry Groupoids

$$\dim S = p \qquad \dim \Sigma = k \qquad \dim G_z = p - k$$

★ If $k = p$ then G_z is discrete.

Theorem. If $k < p$, then G_z is a $(p-k)$-dimensional local Lie subgroup $G_z^* \subset G$ whose connected component containing the identity completion is a piece of a *common* $(p-k)$-dimensional Lie subgroup $G_z^* \subset G^* \subset G$, independent of $z \in S$.

Moreover, S is a union of a k parameter family of pieces of non-singular orbits of G^*:

$$S \subset G^* \cdot N \qquad \text{where} \qquad \dim N = k, \qquad \text{transverse to orbits}$$

Euclidean Surfaces

$G = \mathrm{SE}(3)$ acting on $M = \mathbb{R}^3$

$S \subset M$ — non-umbilic surface

Rank 0 Euclidean Surfaces

$\dim \Sigma = 0$

$G^* \simeq \mathrm{SO}(2) \ltimes \mathbb{R}$

$S \subset Z$ — piece of cylinder $Z = G^* \cdot z_0$ of radius $R > 0$

$H = 1/(2R), \quad K = 0 \quad \Longrightarrow \quad \Sigma = \{\zeta_0\}$

Rank 1 Euclidean Surfaces

$\dim \Sigma = 1$ $\qquad G^* \simeq \mathbb{R}$ or $\mathrm{SO}(2)$ or $\mathrm{SO}(2) + \mathbb{R}$

translations; rotations; screw motions

orbits:

- parallel straight lines;
- "concentric" circles with a common center axis
- "concentric" helices with a common axis

$S \subset Z$ is a piece of $Z = G^* \cdot C$ where C is a transversal curve:

- a surface of translation (traveling wave)
- a surface of revolution
- a helicoidal surface

Index

Definition. The index of a regular point $z \in S_{\text{reg}}$ is defined as the maximal number of connected components of $\chi^{-1}[\chi(S \cap U)]$ where $z \in U \subset M$ is a sufficiently small open neighborhood such that $S \cap U$ is connected.

Theorem. If $z \in S_{\text{reg}}$, its index $\operatorname{ind} z$ is equal to the number of connected components of the quotient G_z/G_z^*.

Weighted Signature

Basic idea: in numerical computations, one "uniformly" discretizes (samples) the original submanifold S. The signature invariants are then numerically approximated, perhaps using invariant numerical algorithms.

Ignoring numerical error, the result is a non-uniform sampling of the signature, and so we consider the images $\zeta_i = \chi(z_i) \in \Sigma$.

In the limit as the number of sample points $\longrightarrow \infty$ the original sample points z_i converge to the uniform G-invariant measure on S while the signature sample points ζ_i converge to the push forward of the uniform measure under the signature map:

$$\nu(\Gamma) = \mu(\chi^{-1}(\Gamma)) = \int_{\chi^{-1}(\Gamma)} |\,\Omega\,| \qquad \text{for} \qquad \Gamma \subset \Sigma.$$

Weighted Signatures of Plane Curves

$$\chi : C \subset \mathbb{R}^2 \to \Sigma \subset \mathbb{R}^2 \qquad \chi(z) = (\kappa, \kappa_s) = \zeta$$

If S has rank 1, then its signature Σ is locally a graph parametrized by κ, say. The weighted measure on Σ is given by

$$\boxed{d\nu = \chi^{\#}(ds) = \operatorname{ind}(\zeta)\,\frac{d\kappa}{|\,\kappa_s\,|}}$$

where $\operatorname{ind}(\zeta)$ denotes the index of the signature point ζ.

If S (connected) has rank 0, then it is a piece of an orbit of a one-parameter subgroup, and $\Sigma = \{\zeta_0\}$ is a single point. The weighted measure is atomic (delta measure) concentrated at ζ_0 with weight equal to the total length of S.

Weighted Signatures of Plane Curves

In general, when S has variable rank,

$$\boxed{\nu(\Gamma) = \int_\Gamma \operatorname{ind}(\zeta)\,\frac{d\kappa}{|\,\kappa_s\,|} + \sum_{\zeta\in\Gamma\,\cap\,\{\kappa_s=0\}} L(\chi^{-1}\{\zeta\})}$$

for $\Gamma \subset \Sigma$.

♠ The weighted signature does *not*, in general, uniquely determine the original curve, since the weight at any point $\zeta_0 = (\kappa_0, 0)$ only measures the total length of all the pieces haivng constant curvature κ_0 and not the number thereof nor how their individual lengths are apportioned.

Rank 2 Euclidean Surfaces

$\dim \Sigma = \dim S = 2$

$\exists$ 2 functionally independent differential invariants

$\Longrightarrow$ assume $dH \wedge dK \neq 0$

Weighted measure on Σ, parametrized by H, K:

$$d\nu = (\operatorname{ind} \zeta) \left| \frac{dH \wedge dK}{\mathcal{D}_1 H \, \mathcal{D}_2 K - \mathcal{D}_2 H \, \mathcal{D}_1 K} \right|$$

$\operatorname{ind} \zeta = \# \, G_z$

— number of discrete local symmetries at $z \in \chi^{-1}\{\zeta\}$.

Rank 0 Euclidean Surfaces

$S \subset Z$ — piece of a cylinder

$H = 1/(2\,R)$, $K = 0$ — $\Sigma = \{\,\zeta_0\,\}$

The weight of ζ_0 equals the area $A(S) = \iint_S \, dS$.

$$\nu = A(S)\,\delta_{\zeta_0}.$$

♠ The weighted signature only determines the area and radius of the cylindrical piece $S \subset Z$, and not its overall shape.

Euclidean Coarea Formula

Theorem. Let $S \subset G^* \cdot C_0$ be a surface of rank 1, such that $C_0 \subset S$ is a normal cross-section to the orbits $\mathcal{O}_z$ of the one-parameter subgroup $G^* \subset \mathrm{SE}(3)$:

$$TC_0|_z \cap T\mathcal{O}_z = \{0\}$$

Let

$$\ell(z) = L(\mathcal{O}_z \cap S) = \int_{\mathcal{O}_z \cap S} ds$$

denote the length of the piece of the orbit $\mathcal{O}_z$ through $z \in C_0$ (line segment, circular arc, or helical arc) that is contained in S. Then

$$A(S) = \int_{C_0} \ell(z(s))\, ds.$$

Corollary. The weighted signature of a surface of rank 1 is given by the push-forward via $\chi : C_0 \to \Sigma$ to its signature curve of the weighted arc length measure

$$\ell(z(s))\, ds$$

on the normal curve $C_0 \subset S$ multiplied by the index $\operatorname{ind} \zeta$.