

Equilibria with a nontrivial nodal set and the dynamics of parabolic equations on symmetric domains

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Abstract

We consider the Dirichlet problem for a class of semilinear parabolic equations on a bounded domain which is reflectionally symmetric about a hyperplane H . The equations consist of a symmetric time-autonomous part and a nonsymmetric perturbation which decays to zero as time approaches infinity. In our first theorem, we prove the asymptotic symmetry of each bounded positive solution of this asymptotically symmetric problem. The novelty of this result is that the solutions considered are not assumed uniformly positive, which prevents one from applying common techniques based on the method of moving hyperplanes. In our second main theorem, we classify the positive entire solutions of the unperturbed time-autonomous problems. In particular, we characterize all entire solutions, which are not symmetrically decreasing in the direction orthogonal to H , as connecting orbits from an equilibrium with a nontrivial nodal set to another invariant set.

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1 Introduction

In this paper, we consider two classes of semilinear parabolic problems,

$$u_t = \Delta u + f(x, u, \nabla u), \quad (x, t) \in \Omega \times (0, \infty), \quad (1.1)$$

$$u = 0, \quad (x, t) \in \partial\Omega \times (0, \infty), \quad (1.2)$$

and

$$u_t = \Delta u + f(x, u, \nabla u) + h(x, t), \quad (x, t) \in \Omega \times (0, \infty), \quad (1.3)$$

$$u = 0, \quad (x, t) \in \partial\Omega \times (0, \infty). \quad (1.4)$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain and $f : (x, u, p) \mapsto f(x, u, p) : \bar{\Omega} \times [0, \infty) \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous function, which is Lipschitz in u and p . In (1.3), h is a bounded continuous function which decays to 0 as $t \rightarrow \infty$. Thus (1.3) is an asymptotically autonomous equation, (1.1) being its “limit” autonomous equation. We only consider nonnegative solutions: by not defining $f(x, u, p)$ for $u < 0$, we postulate that only nonnegative functions can be solutions of (1.1), (1.3).

We further assume that Ω is symmetric about the hyperplane

$$H_0 := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$$

and convex in the direction of the x_1 -axis. Also we make a symmetry assumption on f (see (F2) in the next section), which makes equation (1.1) equivariant under the reflections about the hyperplanes parallel to H_0 . We then examine the dynamics of global solutions of (1.1), (1.2), and (1.3), (1.4) from the symmetry point of view. One of our main objectives is to clarify whether all global bounded solutions of (1.3), (1.4) are asymptotically

symmetric. Toward that goal, we investigate the dynamics of the limit problem (1.1), (1.2); in particular, we want to understand how the dynamics is affected by the presence of equilibria with nontrivial nodal set.

To spell our goals out, let us briefly summarize earlier results pertinent to our study. It is well known that positive steady states of (1.1), (1.2) are symmetric about H_0 and strictly decreasing with increasing $|x_1|$, see [6, 11, 17, 26] (for related symmetry results see also [5, 10, 23, 29, 35] and references therein). The steady states which are nonnegative, but not strictly positive, obviously fail to have the strict monotonicity property. However, as shown in [32], they still enjoy the symmetry about H_0 and, in addition, they are reflectionally symmetric within their nodal domains (in particular, the nodal domains themselves are reflectionally symmetric). Examples of nonnegative steady states with a nontrivial nodal set can be found in [32, 34].

For time-dependent solutions of (1.1), (1.2), two kinds of symmetry results are available, one dealing with *entire solutions*, that is, solutions defined for all $t \in \mathbb{R}$, the other one with *global solutions*, that is, solutions defined for all $t > 0$.

For entire solutions, the spatial symmetry about H_0 and monotonicity in $x_1 > 0$ was established in [2, 4]. The hypotheses of this result include in particular a uniform positivity condition requiring the solutions in question to stay away from zero at each point $x \in \Omega$, uniformly in time.

In the case of global solutions, the symmetry at all times cannot be established unless the initial condition is symmetric. Rather, it has been proved that global positive solutions are asymptotically symmetric in the sense that all their limit profiles as $t \rightarrow \infty$, or elements of their ω -limit sets, are symmetric about H_0 and monotone nondecreasing in $x_1 > 0$ (see [2, 4, 20, 30], related results can be found in [12, 31, 36]). Similarly as for elliptic equations [6], the symmetry results for parabolic equations concerning the entire solutions [4] as well as the global solutions [15, 30], have been proved for fully nonlinear equations on nonsmooth domains. Moreover, in the results for the parabolic equations, the nonlinearities are allowed to depend on t , and some asymptotic symmetry results can also be established if the equation is asymptotically symmetric, not necessarily symmetric at all times [13, 15]. Some sort of uniform positivity condition is always used in the proof of these results and, alongside the asymptotic symmetry of the solutions, one also proves their asymptotic monotonicity with respect to x_1 (for $x_1 \geq 0$).

This brings us to the main topic of this paper. We want to address the question whether the asymptotic symmetry of a solution u can be proved

even if its ω -limit set may possibly contain functions which are not monotone in $x_1 > 0$. Consider for example the situation when problem (1.1), (1.2) possesses nonnegative equilibria with nontrivial nodal sets. If u is a positive global solution of (1.1), (1.2) or (1.3), (1.4), then, without any uniform positivity assumption on u , some of these equilibria can appear in the ω -limit set of u (an example where this happens can be found in [30, Example 2.3]). This prevents u from being asymptotically monotone in $x_1 > 0$, which makes it clear that the asymptotic symmetry of u cannot be established by a direct application of the moving plane method. In this regard, this symmetry problem is quite different from the ones discussed above. Also note that even for the time-autonomous equation (1.1), there is no obvious gradient structure and a given solution may not approach a set of equilibria. Therefore, it is not a priori clear whether the symmetry of equilibria, as proved in [32], has any significance for the asymptotic symmetry of general positive solutions.

Letting the asymptotic symmetry problem aside for a while, the question of how the equilibria with nontrivial nodal sets enter into the asymptotic behavior of positive solutions is quite interesting itself. Clearly, the presence of such an equilibrium φ in the ω -limit set of a solution u means that at some times $t_k \rightarrow \infty$, the function $u(\cdot, t_k)$ has a “near-nodal set” resembling the nodal set of φ . If two such equilibria are contained in $\omega(u)$, then the graph of $u(\cdot, t)$ forms two different near-nodal patterns and repeatedly transfers from one to the other, as $t \rightarrow \infty$. Surprisingly perhaps, we show that this does not happen. One of our main results, Theorem 2.2, addresses this issue as well as the asymptotic symmetry problem. It says that if u is a bounded global solution of (1.3), (1.4), then the following alternative concerning its ω -limit set holds. Either $\omega(u)$ consists of functions which are symmetric about H_0 and monotone nonincreasing in $x_1 > 0$, or it consists of a single nonnegative equilibrium with a nontrivial nodal set. Since the nonnegative equilibria are all symmetric about H_0 , this result implies the asymptotic symmetry of the solution u . Also it shows that the only way u may fail to be asymptotically monotone in $x_1 > 0$ is that it converges to an equilibrium with a nontrivial nodal set. In particular, if u has a nontrivial asymptotic nodal pattern, then the pattern is unique.

Our second main result concerns entire solutions of (1.1), (1.2). Such solutions play a distinguished role in the global dynamics of (1.1), (1.2). For example, if a global attractor of (1.1), (1.2) exists, then it is formed by entire solutions (see [18, 37]). It is also well-known that the ω -limit sets of solutions of (1.3), (1.4) consist of entire solutions of (1.1), (1.2). This is relevant for

the asymptotic symmetry result discussed above. Namely, the asymptotic symmetry of bounded positive solutions of (1.3), (1.4) would be proved if one could establish the symmetry of all nonnegative entire solutions of (1.1), (1.2). The problem whether all entire solutions of (1.1), (1.2), even those which are not monotone in $x_1 > 0$, are symmetric about H_0 at all times is open and we do not resolve it in this paper. It appears to be much harder than in the case of equilibria. In particular, the method of [32], which works well for nonnegative solutions of elliptic equations, does not extend to time dependent solutions of parabolic equations. What we can prove for a general nonnegative entire solution U is that one of the following possibilities occurs:

- (i) $U(\cdot, t)$ is symmetric about H_0 and monotone in $x_1 > 0$ for each $t \in \mathbb{R}$,
- (ii) U is a nonnegative equilibrium with a nontrivial nodal set,
- (iii) $\omega(U)$ consists of functions, which are symmetric about H_0 and monotone nonincreasing in $x_1 > 0$, and $\alpha(U)$ consists of a single nonnegative equilibrium with a nontrivial nodal set,
- (iv) U is a connecting orbit between two equilibria, each having a nontrivial nodal set.

See Theorem 2.3 and Remark 2.4(a) below for more precise statements. Here $\alpha(U)$ stands for the α -limit set of U , that is, the set of all limit profiles of $U(\cdot, t)$ as $t \rightarrow -\infty$.

The above result implies that either U is symmetric about H_0 (cases (i) and (ii)) or else $U(\cdot, t)$ converges to an equilibrium ψ as $t \rightarrow -\infty$ (cases (iii) and (iv)). Even with this additional information, it is not clear whether U is symmetric. While there are results on the symmetry of the unstable manifold of a positive equilibrium ψ (see [3, 19] or [31]), they depend on comparison arguments and the positivity of the function $-\partial_{x_1}\psi$ in $\{x \in \Omega : x_1 > 0\}$. No simple modification of such arguments applies if ψ has a nontrivial nodal set.

In any case, our theorem gives an interesting description of the behavior of a general nonnegative entire solution and, although it does not give the symmetry of all entire solutions, it is sufficient for the proof of the asymptotic symmetry result discussed above. Let us explain briefly how we use (i)-(iv) in the proof of the asymptotic symmetry. Inspired by [20], we consider a functional Λ arising in the process of moving hyperplanes. Roughly speaking, for a function z , $\Lambda(z)$ measures how far to the left one can move the hyperplane

$H_\lambda = \{x \in \mathbb{R}^N : x_1 = \lambda\}$, while preserving a relation between the graph of z and its reflection through the hyperplane H_λ . In [20], Λ was shown to be decreasing along positive solutions which are not symmetric from the start. Thus, Λ can be viewed as a strict Lyapunov functional and from this viewpoint the asymptotic symmetry result of [20] is a manifestation of the LaSalle invariance principle. In our more general setting, particularly due to the lack of any smoothness of Ω , we cannot prove that Λ is a strict Lyapunov functional, however, it serves us in a similar way. Specifically, we prove that if U is a connecting orbit as in (iv), then U connects an equilibrium with higher “energy” (the value of Λ) to an equilibrium with lower energy (see Remark 2.4(d) below). This conclusion, combined with a chain-recurrence property of $\omega(u)$, implies that if an equilibrium $\psi \in \omega(u)$ has a nontrivial nodal set, then necessarily $\omega(u) = \{\psi\}$. On the other hand, in view of the possibilities (i)-(iv), if $\omega(u)$ contains no such equilibrium, then it consists of entire solutions satisfying (i). This implies the conclusion of our asymptotic symmetry theorem.

We conclude the introduction with a few remarks concerning the structure of equations (1.1), (1.3). We work with autonomous or asymptotically autonomous equations mainly because our goal was to understand how the equilibria with a nontrivial nodal set fit into the dynamics of nonnegative solutions. We have chosen to consider semilinear equations, rather than fully nonlinear equations, in order not to obscure the key ideas by additional technicalities. Since the equations do not have a gradient structure, the main conceptual difficulties are already present in this study.

The remainder of the paper is organized as follows. In the next section we state our main results. Their proofs are given in Sections 4, 5. Section 3 contains preliminary material, including a discussion of linear equations arising in the process of moving hyperplanes and some symmetry results from earlier papers. Basic monotonicity properties of the functional Λ along solutions of nonautonomous symmetric equations are also derived in Section 3.

2 Main results

We start by precisely stating our hypotheses.

(D1) $\Omega \subset \mathbb{R}^N$ is a bounded domain, which is symmetric with respect to H_0 and convex in the direction of the x_1 -axis.

(D2) For each $\lambda > 0$, the set

$$\Omega_\lambda := \{x \in \Omega : x_1 > \lambda\}$$

has only finitely many connected components.

(A) There exist numbers $\varsigma \in (0, 1)$ and $R > 0$ such that for each $x \in \partial\Omega$, $\rho \in (0, R)$, one has

$$|\Omega \cap B(x, \rho)| \leq \varsigma |B(x, \rho)|,$$

where $B(x, \rho)$ is the ball of radius ρ centered at x and $|\cdot|$ stands for the Lebesgue measure.

Condition (A) is a minor regularity requirement on Ω which allows us to use boundary Hölder estimates on the solutions of (1.3), (1.4). The technical condition (D2) is assumed in order to make the results of [32] applicable.

Concerning the functions $f : (x, u, p) \mapsto f(x, u, p) : \bar{\Omega} \times [0, \infty) \times \mathbb{R}^N \rightarrow \mathbb{R}$ and $h : \Omega \times (0, \infty) \rightarrow \mathbb{R}$ we assume the following.

(F1) (*Regularity*) f is continuous on $\bar{\Omega} \times [0, \infty) \times \mathbb{R}^N$,

$$f \in C_{\text{loc}}^{\alpha_0}(\Omega \times [0, \infty) \times \mathbb{R}^N) \quad (2.1)$$

for some $\alpha_0 \in (0, 1)$, and f is differentiable with bounded derivatives with respect to (u, p) . In particular, f is Lipschitz in (u, p) uniformly with respect to $x \in \bar{\Omega}$: there is $\beta_0 > 0$ such that

$$\sup_{x \in \bar{\Omega}} |f(x, u, p) - f(x, \tilde{u}, \tilde{p})| \leq \beta_0 |(u, p) - (\tilde{u}, \tilde{p})| \\ (x \in \bar{\Omega}, (u, p), (\tilde{u}, \tilde{p}) \in [0, \infty) \times \mathbb{R}^N).$$

(F2) (*Symmetry*) f is independent of x_1 and even in p_1 :

$$f(x, u, -p_1, p_2, \dots, p_N) = f(x, u, p_1, p_2, \dots, p_N) \\ ((x, u, p_1, p_2, \dots, p_N) \in \bar{\Omega} \times [0, \infty) \times \mathbb{R}^N).$$

(H) h is continuous and bounded, and

$$\lim_{t \rightarrow \infty} \|h(\cdot, t)\|_{L^\infty(\Omega)} = 0.$$

When dealing with nonlinear problems (1.1), (1.2) and (1.3), (1.4) (or problem (3.6) introduced below), we always consider classical solutions. In particular a global solution of (1.3), (1.4) is a function $u \in C_{\text{loc}}^{2,1}(\Omega \times (0, \infty)) \cap C(\bar{\Omega} \times [0, \infty))$ satisfying the equation and the boundary condition everywhere. We shall consider global solutions which are bounded (this simply means that u is a bounded function on $\Omega \times (0, \infty)$).

To formulate our main results, we need to introduce some notation. For $\lambda \in \mathbb{R}$, we set

$$\begin{aligned} H_\lambda &:= \{x \in \mathbb{R}^N : x_1 = \lambda\}, \\ \ell &:= \sup\{\lambda : \Omega \cap H_\lambda \neq \emptyset\}, \\ \Omega_\lambda &:= \{x \in \Omega : x_1 > \lambda\}, \end{aligned}$$

and let $P_\lambda : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be the reflection about H_λ , that is,

$$P_\lambda x := (2\lambda - x_1, x') \quad (x = (x_1, x') \in \mathbb{R}^N).$$

For any function $z \in C(\bar{\Omega})$, we define $V_\lambda z : \Omega_\lambda \rightarrow \mathbb{R}$ by

$$V_\lambda z(x) := z(P_\lambda x) - z(x).$$

Since Ω is convex in x_1 , $V_\lambda z$ is well defined for any $\lambda \geq 0$.

Let $C_0(\bar{\Omega})$ stand for the space of continuous functions on $\bar{\Omega}$ vanishing on $\partial\Omega$ equipped with the supremum norm. We define a functional $\Lambda : C_0(\bar{\Omega}) \rightarrow [0, \ell]$ by

$$\Lambda(z) := \inf\{\lambda \in (0, \ell] : V_\mu z(x) \geq 0 \quad (x \in \Omega_\mu, \mu > \lambda)\}.$$

Since $\Omega_\ell = \emptyset$, the set on the right hand side trivially contains $\lambda = \ell$, thus $\Lambda(z) \in [0, \ell]$ is well defined for each $z \in C_0(\bar{\Omega})$.

Remark 2.1. It is clear from the definition of $\Lambda(z)$ that z is nonincreasing in x_1 in $\Omega_{\Lambda(z)}$.

We denote by E the set of equilibria (time-independent solutions) of (1.1), (1.2). We allow ourselves a harmless ambiguity and view the equilibria as elements of $C_0(\bar{\Omega})$ or as functions on $\Omega \times \mathbb{R}$ constant in time, depending on the context. Set

$$\begin{aligned} E_0 &:= \{z \in E : \Lambda(z) = 0\}, \\ E_+ &:= E \setminus E_0 = \{z \in E : \Lambda(z) > 0\}. \end{aligned}$$

Recall that we only consider nonnegative solutions ($f(x, u, p)$ is not defined for $u < 0$). In particular, all equilibria of (1.1), (1.2) are nonnegative hence, by [32], they are all even in x_1 . By Remark 2.1, the set E_0 consists of the equilibria which are nonincreasing in $x_1 > 0$; in fact, each of them is either identically equal to zero or strictly positive and decreasing in $x_1 > 0$. It also follows from [32] that E_+ is the set of all equilibria whose nodal set in Ω is nontrivial (different from Ω and \emptyset).

As usual, we shall discuss the asymptotic behavior of a bounded solution u of (1.3), (1.4) in terms of its ω -limit set. For that we first note that the semi-orbit $\{u(\cdot, t) : t \in (1, \infty)\}$ is precompact in $C_0(\bar{\Omega})$. This is a consequence of Arzelà-Ascoli theorem and the following Hölder estimate

$$\sup_{\substack{x, \bar{x} \in \Omega, x \neq \bar{x} \\ t, \bar{t} \in [s, s+1], t \neq \bar{t} \\ s \in [1, \infty)}} \frac{|u(x, t) - u(\bar{x}, \bar{t})|}{|x - \bar{x}|^\alpha + |t - \bar{t}|^{\alpha/2}} < \infty. \quad (2.2)$$

The fact that any bounded solution satisfies (2.2) for some $\alpha \in (0, 1)$ is proved in [30, Proposition 2.7] for fully nonlinear equations, including (1.1) as a special case; the proof is really just a summary of well-known interior and boundary Hölder estimates (condition (A) is needed for the boundary estimates; the interior estimates hold irrespectively of condition (2.1) which is not assumed [30]). One just needs to verify an extra assumption in [30, Proposition 2.7], which in our case requires the boundedness of the function $(x, t) \mapsto f(x, 0, 0) + h(x, t)$ on $\Omega \times (0, \infty)$. This requirement is clearly met due to hypotheses (F1) and (H).

Once the precompactness of $\{u(\cdot, t) : t \in (1, \infty)\}$ has been established, it follows by standard results that the ω -limit set of u in $C_0(\bar{\Omega})$, that is, the set

$$\omega(u) := \bigcap_{t>0} \text{cl}_{C_0(\bar{\Omega})} \{u(\cdot, s) : s \geq t\},$$

is nonempty, compact, connected, and it attracts the semi-orbit of u :

$$\lim_{t \rightarrow \infty} \text{dist}_{C_0(\bar{\Omega})}(u(\cdot, t), \omega(u)) = 0. \quad (2.3)$$

Our main theorem concerning the bounded solutions of (1.3), (1.4) can now be stated as follows.

Theorem 2.2. *Assume (D1), (D2), (A), (F1), (F2), (H), and let u be a bounded global solution of (1.3), (1.4). Then each $z \in \omega(u)$ is even in x_1 :*

$$z(x_1, x') = z(-x_1, x') \quad ((x_1, x') \in \Omega).$$

Moreover, either $\omega(u) = \{z\}$ for some $z \in E_+$, or each $z \in \omega(u) \setminus \{0\}$ is (strictly) decreasing in x_1 on Ω_0 .

We next consider entire solutions of (1.1), (1.2). We usually use symbol U for an entire solution. Thus U satisfies (in the classical sense) the problem

$$\begin{aligned} U_t &= \Delta U + f(x, U, \nabla U), & (x, t) &\in \Omega \times \mathbb{R}, \\ U &= 0, & (x, t) &\in \partial\Omega \times \mathbb{R}. \end{aligned} \quad (2.4)$$

Denote

$$\mathcal{A} := \{U : U \text{ is a bounded (non-negative) entire solution of (2.4)}\}.$$

The equilibria of (2.4) are of course examples of entire solutions in \mathcal{A} , that is, $E \subset \mathcal{A}$.

As in the case of global solutions of (1.3), (1.4), Hölder estimates from [30, Proposition 2.7] give the following Hölder estimate for each bounded entire solution U of (2.4):

$$\sup_{\substack{x, \bar{x} \in \bar{\Omega}, x \neq \bar{x} \\ t, \bar{t} \in [s, s+1], t \neq \bar{t} \\ s \in \mathbb{R}}} \frac{|U(x, t) - U(\bar{x}, \bar{t})|}{|x - \bar{x}|^\alpha + |t - \bar{t}|^{\alpha/2}} < \infty. \quad (2.5)$$

Hence the orbit $\{U(\cdot, t) : t \in \mathbb{R}\}$ is precompact in $C_0(\bar{\Omega})$. Defining the α and ω -limit sets of U by

$$\alpha(U) := \bigcap_{t \leq 0} \text{cl}_{C_0(\bar{\Omega})} \{U(\cdot, s) : s \leq t\}, \quad (2.6)$$

$$\omega(U) := \bigcap_{t \geq 0} \text{cl}_{C_0(\bar{\Omega})} \{U(\cdot, s) : s \geq t\}, \quad (2.7)$$

we obtain by standard results that $\alpha(U)$ and $\omega(U)$ are nonempty, compact, connected sets in $C_0(\bar{\Omega})$, which attract the orbit of U in the following sense:

$$\lim_{t \rightarrow \infty} \text{dist}_{C_0(\bar{\Omega})}(U(\cdot, t), \omega(U)) = 0, \quad \lim_{t \rightarrow -\infty} \text{dist}_{C_0(\bar{\Omega})}(U(\cdot, t), \alpha(U)) = 0.$$

Here is our main result for the entire solutions of (2.4).

Theorem 2.3. *Assume (D1), (D2), (A), (F1), (F2), and let $U \in \mathcal{A}$. Then exactly one of the following possibilities occurs:*

- (i) $\Lambda(U(\cdot, t)) = 0$ for each $t \in \mathbb{R}$,
- (ii) $U \in E_+$,
- (iii) $\alpha(U) = \{\xi_*\}$ for some $\xi_* \in E_+$ and $\Lambda(z) = 0$ for each $z \in \omega(U)$,
- (iv) $\alpha(U) = \{\xi_*\}$ and $\omega(U) = \{\xi^*\}$ for some $\xi_*, \xi^* \in E_+$ with $\Lambda(\xi^*) < \Lambda(\xi_*)$.

If $f(\cdot, 0, 0) \geq 0$, then (i) is the case.

Remark 2.4. (a) Note that if (i) holds, then U is symmetric (even) in x_1 . To see this, first recall that (i) means that

$$U(P_0x, t) - U(x, t) \geq 0 \quad (x \in \Omega_0, t \in \mathbb{R}). \quad (2.8)$$

Consider now the function $\tilde{U} := U(P_0\cdot, \cdot)$. Clearly, $\tilde{U} \in \mathcal{A}$, hence one of the possibilities (i)-(iv) of Theorem 2.3 applies to \tilde{U} ; we claim that (i) does. Indeed, if not, then $\alpha(\tilde{U}) = \{z\}$ for some $z \in E_+$. But then also $\alpha(U) = \{z\}$, because $z \circ P_0 = z$ by [32]. However, this contradicts Theorem 2.3 (if (i) holds for U , then none of the other possibilities can occur). So (i) holds for both U and \tilde{U} , which implies that (2.8) holds together with the opposite inequality, and therefore U is even in x_1 .

(b) If (iv) holds, then U is a positive heteroclinic solution between two equilibria in E_+ . We do not have an example of an equation where such a heteroclinic solution occurs. It cannot occur if, for example, Ω is convex in all variables, for in that case E_+ contains at most one element (see [33]). On the other hand, it is not difficult to find examples with a heteroclinic connection from an equilibrium in E_+ to an equilibrium in E_0 . We sketch an example in dimension $N = 1$. Take $\Omega = (-3\pi, 3\pi)$ and let $f(u) = u - 1$ for $u \leq 2$ and $f(u) < 0$ for $u \geq 3$. Then E_+ contains the equilibrium $\xi(x) = 1 + \cos x$. This equilibrium is unstable and its fast unstable manifold (an invariant manifold tangent at ξ to a positive function) contains an entire solution U monotonically increasing in time and such that $U(\cdot, t) \rightarrow \xi$ as $t \rightarrow -\infty$. This solution is positive and bounded (by the condition $f(u) < 0$ for $u \geq 3$) and its limit as $t \rightarrow \infty$ is a strictly positive equilibrium, hence an element of E_0 . This illustrates that the possibility (iii) can occur. Possibility (i) or (ii) occurs, for example, if U is an equilibrium in E_0 or E_+ , respectively.

(c) The fact that none of the conditions (ii)-(iv) can hold if $f(\cdot, 0, 0) \geq 0$ follows from the strong comparison principle: each equilibrium either vanishes identically or is strictly positive in Ω . In particular, $E_+ = \emptyset$.

(d) Theorem 2.3 shows that unless U is an equilibrium or is symmetric, the value of Λ on $\omega(U)$ is strictly smaller than its value on $\alpha(U)$. In this regard, Λ behaves as a strict Lyapunov functional.

3 Linear equations and moving hyperplanes

This section has four parts. In Subsections 3.1, 3.2 we recall some useful estimates for solutions of linear parabolic equations and show how linear equations arise in the process of moving hyperplanes. In Subsection 3.3 we use the estimates for linear equations to derive basic properties of the functional Λ . Finally, in Subsection 3.4, we recall two symmetry results concerning symmetric equations with time-dependent nonlinearities.

In this section, as in the whole paper, Ω is a fixed domain satisfying conditions (D1), (D2), and (A).

3.1 Linear equations

We use the following standard notation. For a bounded set G in \mathbb{R}^N or \mathbb{R}^{N+1} , $\text{diam } G$ denotes the diameter of G and $|G|$ for the Lebesgue measure of G (if it is measurable). By $B(x, r)$ we denote the open ball in \mathbb{R}^N centered at x with radius r and symbols f^+ and f^- stand for the positive and negative parts of a function f : $f^\pm := (|f| \pm f)/2 \geq 0$. For a domain $D \subset \mathbb{R}^N$, we define the inner radius of D by

$$\text{inrad}(D) := \sup\{\rho > 0 : B(x_0, \rho) \subset D \text{ for some } x_0 \in D\},$$

and if D is an open set, we let $\text{inrad}(D)$ stand for the infimum of inner radii of the connected components of D .

For any open bounded $S \subset \mathbb{R}^{n+1}$ and any bounded, continuous function $f : S \rightarrow \mathbb{R}$ define

$$[f]_{p,S} := \left(\frac{1}{|S|} \int_S |f|^p dx dt \right)^{\frac{1}{p}} \quad (p > 0).$$

Definition 3.1. Given an open set $Q \subset \mathbb{R}^{N+1}$, and $\beta_0 > 0$, we say that a differential operator L belongs to $\mathcal{E}(\beta_0, Q)$ if

$$L(x, t) = \Delta + \sum_{k=1}^N b_k(x, t) \frac{\partial}{\partial x_k} + c(x, t),$$

where b_k, c are measurable functions defined on Q such that

$$|b_k(x, t)|, |c(x, t)| \leq \beta_0 \quad ((x, t) \in Q, k = 1, \dots, N).$$

Given an open set $G \subset \Omega$ and $\tau < T$, consider the linear parabolic problem

$$v_t = L(x, t)v + h(x, t), \quad (x, t) \in G \times (\tau, T), \quad (3.1)$$

$$v = 0, \quad (x, t) \in \partial G \times (\tau, T), \quad (3.2)$$

where $L \in \mathcal{E}(\beta_0, G \times (\tau, T))$ for some $\beta_0 > 0$ and $h \in L^\infty(G \times (\tau, T))$. We say that v is a supersolution of (3.1), (3.2) if $v \in W_{N+1,loc}^{2,1}(G \times (\tau, T)) \cap C(\bar{G} \times [\tau, T])$, $v \geq 0$ on $\partial G \times [\tau, T]$, and (3.1) holds almost everywhere in $G \times (\tau, T)$ with ‘=’ replaced by ‘ \geq ’. We say v is a subsolution of (3.1), (3.2) if $-v$ is a supersolution and we say v is a solution of (3.1), (3.2) if it is both supersolution and subsolution.

In addition to the standard maximum principle, we shall also use the following estimate (see [7, 24, 27, 38]).

Theorem 3.2. *If v is a supersolution of (3.1), (3.2), then*

$$\|v^-(\cdot, t)\|_{L^\infty(G)} \leq C^*(\|v^-(\cdot, \tau)\|_{L^\infty(G)} + \|h^-\|_{L^\infty(G \times (\tau, t))}) \quad (t \in (\tau, T)),$$

where $C^* = C^*(N, \beta_0, T - \tau)$ is a positive constant.

The following result is the maximum principle on small domains. For the proof see [12, 30].

Lemma 3.3. *There exists $\delta = \delta(N, \beta_0)$ such that if $|G| < \delta$ and v is a super-solution of (3.1), (3.2) with $h \equiv 0$, then*

$$\|v^-(\cdot, t)\|_{L^\infty(G)} \leq 2e^{-(t-\tau)} \|v^-(\cdot, \tau)\|_{L^\infty(G)} \quad (t \in (\tau, T)).$$

For the proof of the next theorem see [30].

Theorem 3.4. *Given any $\rho > 0$, $d > 0$, and $\theta > 0$, there exist positive constants $\delta = \delta(N, \text{diam } \Omega, \beta_0, \rho)$, $p = p(N, \text{diam } \Omega, \beta_0, d, \theta, \rho)$, and $\tilde{\mu} = \tilde{\mu}(N, \text{diam } \Omega, \beta_0, d, \theta, \rho)$ with the following properties. If $D \subset G \subset \Omega$ are open sets satisfying*

$$\text{inrad}(D) > \rho, \quad |G \setminus \bar{D}| < \delta, \quad \text{dist}(\bar{D}, \partial G) \geq d,$$

if v is a super-solution of (3.1), (3.2) with $T = \infty$ and $f \equiv 0$, and if

$$\begin{aligned} v(x, t) &> 0 \quad ((x, t) \in \bar{D} \times (\tau, \tau + 8\theta)), \\ \|v^-(\cdot, \tau)\|_{L^\infty(G \setminus \bar{D})} &\leq \tilde{\mu}[v]_{p, D_0 \times (\tau + \theta, \tau + 2\theta)}, \end{aligned} \quad (3.3)$$

for each connected component D_0 of D , then the following statements hold true:

$$v(x, t) > 0 \quad ((x, t) \in \bar{D} \times [\tau, \infty)), \quad (3.4)$$

$$\|v^-(\cdot, t)\|_{L^\infty(G)} \leq 2e^{-(t-\tau)} \|v^-(\cdot, \tau)\|_{L^\infty(G)} \quad (t > \tau). \quad (3.5)$$

Notice that out of the quantities d , θ , and ρ , the constant δ depends only by ρ , whereas the constants p and $\tilde{\mu}$ also depend on d and θ .

3.2 From nonlinear to linear equations

Let us now recall how linear problems of the form (3.1), (3.2) arise in the process of moving hyperplanes. Below we need to apply this process to problem (2.4) as well as to a transformation of (2.4) which is no longer time-autonomous. Therefore, we introduce the following more general problem:

$$\begin{aligned} U_t &= \Delta U + g(t, x, U, \nabla U), & (x, t) &\in \Omega \times \mathbb{R}, \\ U &= 0, & (x, t) &\in \partial\Omega \times \mathbb{R}. \end{aligned} \quad (3.6)$$

We assume that the function $g : (t, x, u, p) \mapsto g(t, x, u, p) : \mathbb{R} \times \bar{\Omega} \times [0, \infty) \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies the following two conditions similar to (F1), (F2) ((G1) requires slightly less regularity than (F1)):

(G1) (*Regularity*) g is continuous in all variables and Lipschitz in (u, p) : there is $\beta_0 > 0$ such that

$$\begin{aligned} \sup_{x \in \bar{\Omega}} |g(t, x, u, p) - g(t, x, \tilde{u}, \tilde{p})| &\leq \beta_0 |(u, p) - (\tilde{u}, \tilde{p})| \\ (t \in \mathbb{R}, x \in \bar{\Omega}, (u, p), (\tilde{u}, \tilde{p})) &\in [0, \infty) \times \mathbb{R}^N. \end{aligned}$$

(G2) (*Symmetry*) g is independent of x_1 and even in p_1 .

Also we assume the following boundedness condition

(G3) The function $g(t, x, 0, 0)$ is bounded on $\mathbb{R} \times \Omega$.

Condition (G3) guarantees that the Hölder estimate (2.5) applies to any bounded entire solution U of (3.6) (see [30, Proposition 2.7]). Consequently, the trajectory of U and its α and ω -limit sets have the properties discussed in Section 2 (see the paragraph containing (2.6), (2.7)).

Denote

$$\mathcal{A}^* := \{U : U \text{ is a bounded nonnegative entire solution of (3.6)}\}.$$

Given $U \in \mathcal{A}^*$ and $\lambda \in [0, \ell)$, define $U^\lambda : \bar{\Omega}_\lambda \times \mathbb{R} \rightarrow \mathbb{R}$ by $U^\lambda(x, t) := U(P_\lambda x, t)$. By (G2)

$$\partial_t U^\lambda = \Delta U^\lambda + g(t, x, U^\lambda, \nabla U^\lambda), \quad (x, t) \in \Omega_\lambda \times \mathbb{R}.$$

Hence, the function $w^\lambda : \bar{\Omega}_\lambda \times \mathbb{R} \rightarrow \mathbb{R}$,

$$w^\lambda(x, t) := U^\lambda(x, t) - U(x, t) \tag{3.7}$$

satisfies

$$\partial_t w^\lambda = \Delta w^\lambda + g(t, x, U^\lambda, \nabla U^\lambda) - g(t, x, U, \nabla U), \quad (x, t) \in \Omega_\lambda \times \mathbb{R}. \tag{3.8}$$

Using the Hadamard formula, we can rewrite (3.8) as

$$\partial_t w^\lambda = L^\lambda(x, t)w^\lambda, \quad (x, t) \in \Omega_\lambda \times \mathbb{R}, \tag{3.9}$$

where $L^\lambda \in \mathcal{E}(\beta_0, \Omega_\lambda \times \mathbb{R})$, with β_0 as in (G1).

Also, since $U \geq 0$ in Ω , w^λ satisfies

$$w^\lambda(x, t) \geq 0 \quad ((x, t) \in \partial\Omega_\lambda \times \mathbb{R}). \tag{3.10}$$

Hence, w^λ is a supersolution of (3.1), (3.2), with $G = \Omega_\lambda$ and $L = L^\lambda$.

We shall also encounter different linear equations associated with (3.6). For example, if U, \tilde{U} are two solutions of (3.6), then $w = U - \tilde{U}$ is a solution of a linear problem (3.1), (3.2) on $\Omega \times \mathbb{R}$ with $L \in \mathcal{E}(\beta_0, \Omega \times \mathbb{R})$.

3.3 Basic properties of the functional Λ

We now use the estimates from Subsection 3.1 to examine the behavior of the functional Λ along entire solutions of the nonautonomous problem (3.6).

We assume that the nonlinearity g satisfies conditions (G1)-(G3). Of course, all the results proved here apply to the bounded entire solutions of the more specific problem (2.4).

Lemma 3.5. *There is $\varepsilon_0 > 0$ such that for any $U \in \mathcal{A}^*$ one has*

$$\Lambda(U(\cdot, t)) < \ell - \varepsilon_0 \quad (t \in \mathbb{R}).$$

Proof. Take $\delta > 0$ as in Lemma 3.3 and fix $\varepsilon_0 > 0$ such that $|\Omega_\lambda| < \delta$ for any $\lambda \in (\ell - \varepsilon_0, \ell)$. Let w^λ be defined as in (3.7). Then by Lemma 3.3 one has

$$\|(w^\lambda)^-(\cdot, t)\|_{L^\infty(\Omega_\lambda)} \leq 2e^{-(t-\tau)} \|(w^\lambda)^-(\cdot, \tau)\|_{L^\infty(\Omega_\lambda)} \quad (\tau \leq t, \lambda \in (\ell - \varepsilon_0, \ell)).$$

Taking $\tau \rightarrow -\infty$ and using the boundedness of U , we obtain $w^\lambda(\cdot, t) \geq 0$ for any $\lambda \in (\ell - \varepsilon_0, \ell)$, which proves the lemma. \square

Lemma 3.6. *For any $U \in \mathcal{A}^*$, the function $t \mapsto \Lambda(U(\cdot, t))$ is nonincreasing and*

$$\Lambda(z) \leq \lim_{t \rightarrow \infty} \Lambda(U(\cdot, t)) \quad (z \in \omega(U)), \quad (3.11)$$

$$\Lambda(z) \leq \lim_{t \rightarrow -\infty} \Lambda(U(\cdot, t)) \quad (z \in \alpha(U)). \quad (3.12)$$

Proof. Given any $\tau \in \mathbb{R}$, denote $\lambda_0 := \Lambda(U(\cdot, \tau))$ and fix an arbitrary $\lambda \in (\lambda_0, \ell)$. If w^λ is as in (3.7), then w^λ satisfies (3.9), (3.10) and, by our choice of λ , it also satisfies

$$w^\lambda(x, \tau) \geq 0 \quad (x \in \Omega_\lambda). \quad (3.13)$$

By the maximum principle, $w^\lambda(\cdot, t) \geq 0$ for each $t \geq \tau$. Since $\lambda \in (\lambda_0, \ell)$ was arbitrary, we obtain that $\Lambda(U(\cdot, t)) \leq \lambda_0$ for any $t \geq \tau$ and the monotonicity property follows.

The monotonicity implies that relation (3.11) is equivalent to

$$\Lambda(z) \leq \Lambda(U(\cdot, t)) \quad (z \in \omega(U), t \in \mathbb{R}). \quad (3.14)$$

To prove (3.14), fix $t \in \mathbb{R}$, $z \in \omega(U)$, and $\lambda \geq \Lambda(U(\cdot, t))$. Then, for some sequence $(t_n)_{n \in \mathbb{N}}$ with $t_n > t$, $t_n \rightarrow \infty$, we have $U(\cdot, t_n) \rightarrow z$. Consequently,

$V_\lambda z = \lim_{n \rightarrow \infty} V_\lambda U(\cdot, t_n) \geq 0$ on Ω_λ , since $\Lambda(U(\cdot, t_n)) \leq \Lambda(U(\cdot, t)) \leq \lambda$. This proves (3.14).

Now fix an arbitrary $z \in \alpha(U)$. If $\Lambda(z) = 0$, then the relation in (3.12) holds trivially. Assume $\Lambda(z) > 0$. By the definition of Λ , arbitrarily close to $\Lambda(z)$ there is $\lambda < \Lambda(z)$ such that $V_\lambda z(x_\lambda) < 0$ for some $x_\lambda \in \Omega_\lambda$. Since $z \in \alpha(U)$, by a choice of a large negative τ we can make $U(\cdot, \tau)$ so close to z that $V_\lambda U(x_\lambda, \tau) < 0$. Consequently, $\Lambda(U(\cdot, \tau)) > \lambda$ and by the monotonicity $\Lambda(U(\cdot, t)) > \lambda$ for each $t \leq \tau$. This proves that $\Lambda(z) \leq \lim_{t \rightarrow -\infty} \Lambda(U(\cdot, t))$, as desired. \square

Lemma 3.7. *Let $U \in \mathcal{A}^*$ and $\lambda_0 \in (0, \ell)$. Then the following two statements are valid:*

(i) *If for some $\tau_0 \in \mathbb{R}$ one has $\Lambda(U(\cdot, \tau_0)) \leq \lambda_0$ and $V_{\lambda_0} U(\cdot, \tau_0) \not\equiv 0$ on each connected component of Ω_{λ_0} , then there exists $\varepsilon_0 > 0$ such that $\Lambda(z) \leq \lambda_0 - \varepsilon_0$ for each $z \in \omega(U)$.*

(ii) *If for some $z \in \alpha(U)$ one has $V_\lambda z > 0$ on Ω_λ for each $\lambda \in [\lambda_0, \ell)$, then there exists $\varepsilon_0 > 0$ such that $\Lambda(U(\cdot, t)) \leq \lambda_0 - \varepsilon_0$ for each $t \in \mathbb{R}$.*

Proof. In this proof we apply Theorem 3.4 in much the same way as in [30, Proof of Lemma 4.3].

The proofs of (i) and (ii) use similar arguments. The following is a common part to both proofs. By (D2), Ω_{λ_0} has finitely many connected components, and therefore $\rho := \text{inrad}(\Omega_{\lambda_0})/2 > 0$. To this ρ (and β_0 as in (G1)), there is $\delta > 0$ as in Theorem 3.4. Fix an open set D such that $\bar{D} \subset \Omega_{\lambda_0}$, $D \cap M$ is a domain for each connected component M of Ω_{λ_0} , $\text{inrad}(D) > \rho$, and $|\Omega_{\lambda_0} \setminus D| < \delta/2$. Then for sufficiently small $\varepsilon_0 > 0$ one has $|\Omega_\lambda \setminus D| < \delta$ for each $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$. Below we assume that $\varepsilon_0 > 0$ has this property, but we may need to make it even smaller. Denote $d := \text{dist}(D, \partial\Omega_{\lambda_0})$ and observe that $d \leq \text{dist}(D, \partial\Omega_\lambda)$ for each $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$.

For any $\lambda \geq 0$, let w^λ be as in (3.7). Recall that w^λ satisfies (3.9)–(3.10).

We now prove statement (i). Since $\lambda_0 \geq \Lambda(U(\cdot, \tau_0))$, we have

$$w^{\lambda_0}(x, \tau_0) \geq 0 \quad (x \in \Omega_{\lambda_0}). \quad (3.15)$$

The maximum principle and the assumption in (i) therefore imply that $w^{\lambda_0} > 0$ on $\Omega_{\lambda_0} \times (\tau_0, \infty)$. Fix any $\tau > \tau_0$ and denote

$$r_1 := \frac{1}{2} \inf_{\bar{D}} w^{\lambda_0}(\cdot, \tau) > 0.$$

Then, by continuity, choosing a sufficiently small $\theta > 0$ and making $\varepsilon_0 > 0$ smaller if necessary, we achieve that

$$\inf_{\bar{D}} w^\lambda(\cdot, t) > r_1 \quad (t \in [\tau, \tau + 8\theta], \lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]). \quad (3.16)$$

Having fixed d, ρ, θ , let p and $\tilde{\mu}$ be as in Theorem 3.4.

By (3.15) and the Hölder estimate (2.5), one has

$$\begin{aligned} w^\lambda(x, \tau) &= w^{\lambda_0}(x, \tau) + u(P_\lambda x, t) - u(P_{\lambda_0} x, t) \geq -C|P_\lambda x - P_{\lambda_0} x|^\alpha \\ &= -2C|\lambda_0 - \lambda|^\alpha \quad (x \in \Omega_{\lambda_0}) \end{aligned}$$

and

$$w^\lambda(x, \tau) \geq -C|P_\lambda x - x|^\alpha \geq -2C|\lambda_0 - \lambda|^\alpha \quad (x \in \Omega_\lambda \setminus \Omega_{\lambda_0}).$$

Thus decreasing $\varepsilon_0 > 0$ further if needed, one achieves

$$\|(w^\lambda)^-(\cdot, \tau)\|_{L^\infty(\Omega_\lambda)} < \tilde{\mu}r_1 \quad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]). \quad (3.17)$$

Relations (3.16) and (3.17) show that the second inequality in (3.3) is satisfied with $v = w^\lambda$ and $G = \Omega_\lambda$. Consequently, (3.4) and (3.5) hold true. By (3.5) we obtain

$$\begin{aligned} \|(w^\lambda)^-(\cdot, t)\|_{L^\infty(\Omega_\lambda)} &\leq 2e^{-(t-\tau)} \|(w^\lambda)^-(\cdot, \tau)\|_{L^\infty(G)} \\ &\quad (t > \tau, \lambda \in (\lambda_0 - \varepsilon_0, \lambda_0]), \end{aligned} \quad (3.18)$$

and therefore, by passing to the limit as $t \rightarrow \infty$, $V_\lambda z \geq 0$ in Ω_λ for each $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0]$ and each $z \in \omega(U)$. Combining this result with Lemma 3.6, we conclude that $\Lambda(z) \leq \lambda_0 - \varepsilon_0$ for each $z \in \omega(U)$. Statement (i) is proved.

Next we prove statement (ii). Choose a sequence $(t_n)_{n \in \mathbb{N}}$ such that $t_n \rightarrow \infty$ and $U(\cdot, -t_n) \rightarrow z$ as $n \rightarrow \infty$. Denote

$$r_1 := \frac{1}{2} \inf_{\bar{D}} V_{\lambda_0} z > 0.$$

Then, by (2.5), there are θ and n_0 such that, possibly after $\varepsilon_0 > 0$ is made smaller, one has

$$\inf_{\bar{D}} w^\lambda(\cdot, t) \geq r_1 \quad (t \in [-t_n, -t_n + 8\theta], \lambda \in (\lambda_0 - \varepsilon_0, \lambda_0], n \geq n_0).$$

Having fixed θ (and d, ρ), let p and $\tilde{\mu}$ be as in Theorem 3.4.

We can now argue as in the proof of statement (i), taking $\tau = -t_n$, with $n \geq n_0$, in the arguments following (3.16). Specifically, we first make $\varepsilon_0 > 0$ yet smaller (independently of n) to achieve that (3.17) holds. This implies that (3.18) holds with $\tau = -t_n$. Since w^λ is bounded, passing to the limit as $n \rightarrow \infty$, we obtain $w^\lambda(\cdot, t) \geq 0$ in Ω_λ for each $t \in \mathbb{R}$ and $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0]$. In particular, $w^{\lambda_0}(\cdot, t) \geq 0$ in Ω_{λ_0} . Clearly, in view of the assumption of (ii), we can replace λ_0 with any other value $\tilde{\lambda}_0 \in (\lambda_0, \ell)$, hence $w^\lambda(\cdot, t) \geq 0$ in Ω_λ for all $t \in \mathbb{R}$ and $\lambda \in (\lambda_0 - \varepsilon_0, \ell]$. This proves that $\Lambda(U(\cdot, t)) \leq \lambda_0 - \varepsilon_0$ for each $t \in \mathbb{R}$. \square

3.4 Two symmetry results for nonautonomous equations

In this subsection we state two symmetry results from [30, 31], as they apply to entire solutions of (3.6). The results in [30, 31] concern fully nonlinear equations of which (3.6) (hence also (2.4)) is a special case. We will use these results in the proof of Theorem 2.3.

We assume that g is a function satisfying conditions (G1)-(G3).

Theorem 3.8. *Let $U \in \mathcal{A}^*$ and*

$$\lambda^* = \sup\{\Lambda(z) : z \in \omega(U)\}.$$

Then $\lambda^ \in [0, \ell)$ and for each $z \in \omega(U)$ one has $V_{\lambda^*} z \equiv 0$ on some connected component of Ω_{λ^*} . If $\omega(u)$ contains a function z_0 such that $z_0 > 0$ in Ω , then $\lambda^* = 0$.*

The existence of $\lambda^* \in [0, \ell)$ satisfying the first conclusion is proved in [30, Theorem 2.4]. As shown there (see the beginning of the proof of Lemma 4.2 in [30]), λ^* is given by

$$\lambda^* = \inf\{\mu \geq 0 : V_\lambda(z) \geq 0 \text{ in } \Omega_\lambda \text{ for all } \lambda \in (\mu, \ell) \text{ and } z \in \omega(U)\}$$

which is the same as

$$\lambda^* = \inf\{\mu \geq 0 : \Lambda(z) \leq \mu \ (z \in \omega(U))\} = \sup\{\Lambda(z) : z \in \omega(U)\}.$$

The fact that $\lambda^* = 0$ if $\omega(u)$ contains a strictly positive function is proved in [30, Theorem 2.2]. In this case, each $z \in \omega(U)$ is even in x_1 and monotone nonincreasing in $x_1 > 0$.

Theorem 3.9. *Let $U \in \mathcal{A}^*$. If $\alpha(U)$ contains a function z_0 such that $z_0 > 0$ in Ω , then for each $t \in \mathbb{R}$ one has $\Lambda(U(\cdot, t)) = 0$ and the function $U(\cdot, t)$ is even in x_1 and decreasing in x_1 .*

This is stated in [31, Theorem 3.4]. As indicated in [31, Sections 4.2, 6], the theorem can be proved by similar techniques as Theorem 3.8 if the general scheme of the proof is suitably adjusted to deal with the symmetry at all times rather than with the asymptotic symmetry as $t \rightarrow \infty$. For the reader's convenience, in Appendix A we give an alternative proof based on the properties of the functional Λ established in the previous subsection and a generalized Harnack inequality from [30, Theorem 2.2]. Under stronger hypotheses requiring in particular the strict positivity of all elements of $\alpha(u)$ and a sign condition on the nonlinearity, the symmetry result is proved in [2, 4].

4 Proof of Theorem 2.3

Throughout the section we assume the hypotheses (D1), (D2), (A), (F1), and (F2) to be satisfied.

We start with two results concerning equilibria. The proof of the following lemma can be found in [32].

Lemma 4.1. *Let $z \in E$. Then the following statements are valid.*

- (i) *$z \in E_+$ if and only if $z \not\equiv 0$ and z vanishes somewhere in Ω .*
- (ii) *If $z \not\equiv 0$, then z does not vanish on any nonempty open subset of Ω and $\lambda = \Lambda(z)$ is the maximal number in $(0, \ell)$ with the property that the function $V_\lambda z$ vanishes identically in a connected component of Ω_λ .*

Lemma 4.2. *The set E_+ has only finitely many elements. Moreover, for each $z \in E_+$ one has $\text{dist}_{C_0(\bar{\Omega})}(z, E_0) > 0$.*

Proof. The second statement follows from the fact that each element of E_0 is monotone in x_1 in the set Ω_0 , whereas $z \in E_+$ is obviously not monotone.

The first statement is a result of [33, Theorem 2.1]; however, a remark on the applicability of [33] is necessary. In [33] fully nonlinear equations were considered. To guarantee that a linearization of the equation has Lipschitz continuous coefficients in the principal part, a slightly higher regularity of U

had to be required. In the present case, the leading part of the equation is always given by the Laplacian, hence the extra regularity assumption is not needed. \square

We next recall an invariance property of the ω and α -limit sets.

Lemma 4.3. *If $U \in \mathcal{A}$ and $z \in \omega(U)$ (or $z \in \alpha(U)$), then there is $Z \in \mathcal{A}$ with $z = Z(\cdot, 0)$ and $Z(\cdot, t) \in \omega(U)$ (or, respectively, $Z(\cdot, t) \in \alpha(U)$) for any $t \in \mathbb{R}$.*

Proof. This is quite a standard result, however, since we do not assume any smoothness of $\partial\Omega$, we need to deal with some regularity issues. We carry out the details for $\omega(U)$, $\alpha(U)$ can be dealt with similarly.

We will employ the global Hölder estimate (2.5) as well as the following interior L^p -estimate. For any $p \in (1, \infty)$, $T \in (0, \infty)$, and any domain Ω whose closure is contained in Ω one has

$$\sup_{s \in \mathbb{R}} \|U\|_{W_p^{2,1}(\tilde{\Omega} \times (s-T, s+T))} < \infty. \quad (4.1)$$

To prove (4.1), rewrite the equation for U as follows:

$$\begin{aligned} U_t &= \Delta U + f(x, U, \nabla U) - f(x, 0, 0) + f(x, 0, 0) \\ &= L(x, t)U + f(x, 0, 0), \end{aligned} \quad (4.2)$$

where $L \in \mathcal{E}(\beta_0, \Omega \times \mathbb{R})$ and β_0 is as in (F1). The function $f(x, 0, 0)$ is bounded by (F1). Applying to this linear nonhomogeneous equation the interior L^p -estimates (see [25, 27]), one obtains (4.1).

Let now $z \in \omega(U)$. There is a sequence $s_m \rightarrow \infty$ such that $U(\cdot, s_m) \rightarrow z$ in $C_0(\tilde{\Omega})$. Using (2.5) and a diagonalization procedure, replacing s_m by a subsequence if necessary, one shows that the limit

$$Z := \lim_{m \rightarrow \infty} U(\cdot, \cdot + s_m) \quad (4.3)$$

exists, pointwise and uniformly on the compact subsets of $\tilde{\Omega} \times \mathbb{R}$. Clearly, $Z(\cdot, 0) = z$ and $Z(\cdot, t) \in \omega(U)$ for each t . Using the reflexivity of the Sobolev spaces $W_p^{2,1}(\tilde{\Omega} \times (s-T, s+T))$, $p \in (1, \infty)$, and their continuous imbedding in Hölder spaces [25], one further obtains, replacing s_m by a subsequence again, that the limit in (4.3) takes place in $C_{\text{loc}}^{1+\beta, \beta/2}(\Omega \times \mathbb{R})$ for each $\beta \in (0, 1)$, as

well as weakly in $W_p^{2,1}(\tilde{\Omega} \times (s-T, s+T))$ for each $p \in (1, \infty)$, $T > 0$, and $\tilde{\Omega}$ as above. In particular, $Z \in C_{\text{loc}}^{1+\beta, \beta/2}(\Omega \times \mathbb{R})$ and

$$f(\cdot, U(\cdot, \cdot + s_m), \nabla U(\cdot, \cdot + s_m)) \rightarrow f(\cdot, Z, \nabla Z)$$

locally uniformly in $\Omega \times \mathbb{R}$. Using test functions and passing to the limit in the equation for U , we obtain that Z satisfies in the generalized sense the equation

$$Z_t = \Delta Z + \Phi(x, t), \quad x \in \Omega, t \in \mathbb{R}, \quad (4.4)$$

where $\Phi(x, t) = f(x, Z(x, t), \nabla Z(x, t))$. By (2.1), this function is Hölder continuous, hence by Schauder theory Z is a classical solution of (4.4). \square

Note that $0 \in \mathcal{A}$ if and only if 0 is an equilibrium of (2.4), that is, if and only if $f(\cdot, 0, 0) \equiv 0$. Below, when writing $U \in \mathcal{A} \setminus \{0\}$ we mean that $U \in \mathcal{A}$ and $U \not\equiv 0$ (in case $0 \in E$). The next lemma shows that this is the same as saying that $U \in \mathcal{A}$ and $U(\cdot, t) \not\equiv 0$ for any $t \in \mathbb{R}$ on any open subset of Ω .

Lemma 4.4. *Let $U \in \mathcal{A}$. If $U(\cdot, \tau) \equiv 0$ on an open nonempty subset G of Ω for some $\tau \in \mathbb{R}$, then $U \equiv 0$.*

Proof. Since $U \geq 0$, 0 is a local minimum of U . Thus, $U_t(\cdot, \tau) \equiv 0$ on G . Of course, one also has $|\nabla U(\cdot, \tau)| \equiv \Delta U(\cdot, \tau) \equiv 0$ in G . From (2.4), we obtain $f(x, 0, 0) = 0$ for each $x \in G$, and therefore U solves

$$U_t = \Delta U + f(x, U, \nabla U) - f(x, 0, 0) = L(x, t)U, \quad (x, t) \in G \times \mathbb{R},$$

where $L \in \mathcal{E}(\beta_0, G \times \mathbb{R})$. Since $U \geq 0$ and $U(\cdot, \tau) \equiv 0$ on G , the strong maximum principle yields $U \equiv 0$ on $G \times (-\infty, \tau)$. For any $\delta \in (0, 1)$ define $w(x, t) := U(x, t + \delta) - U(x, t)$. Then w solves

$$\begin{aligned} w_t &= L(x, t)w, & (x, t) &\in \Omega \times \mathbb{R}, \\ w &= 0, & (x, t) &\in \partial\Omega \times \mathbb{R}, \end{aligned}$$

and $w \equiv 0$ on $G \times (-\infty, \tau - 1)$. Therefore, by the weak unique continuation theorem (see e.g. [1]) $w \equiv 0$ on $\Omega \times (-\infty, \tau - 1)$. Consequently, by the uniqueness for the Dirichlet initial-boundary value problem (which follows from the maximum principle), $w \equiv 0$ on $\Omega \times (-\infty, \infty)$. Hence $U(\cdot, \cdot) \equiv U(\cdot, \cdot + \delta)$ for each $\delta \in (0, 1)$, and therefore U is an equilibrium. Since $U \equiv 0$ on G , Lemma 4.1 implies that $U \equiv 0$ in Ω . \square

The following lemma is crucial for our further arguments. It shows that entire solutions with certain additional properties have to be equilibria.

Lemma 4.5. *Let $U \in \mathcal{A} \setminus \{0\}$, $\lambda_0 \in (0, \ell)$, and let I be an open interval. Assume that for each $t \in I$, one has $V_{\lambda_0}(U(\cdot, t)) \equiv 0$ on some connected component D^t of Ω_{λ_0} . Then $U \in E_+$.*

Proof. Recall that by hypothesis (D2), Ω_{λ_0} has only finitely many connected components. Since the function $V_{\lambda_0}U$ is continuous, shrinking the interval I if necessary, we may assume that the connected component $D = D^t$ is independent of t . Since $\lambda_0 > 0$, $\mathcal{M} \neq \emptyset$, where

$$\mathcal{M} := P_{\lambda_0}(\partial D \cap \partial \Omega) \cap \Omega.$$

Now by standard interior Schauder estimates, $U \in C_{\text{loc}}^{2+\alpha_0, 1+\alpha_0/2}(\Omega \times I)$, where $\alpha_0 > 0$ is as in (F1). Next, we see that

$$U(x, t) = U_t(x, t) = |\nabla U(x, t)| = 0 \quad ((x, t) \in \mathcal{M} \times I). \quad (4.5)$$

Indeed, we have $U = 0$ on $\mathcal{M} \times I$, since $V_{\lambda_0}U \equiv 0$ on D and U satisfies the Dirichlet boundary condition. Since $U \geq 0$ in $\Omega \times I$, any point in $\mathcal{M} \times I$ is a local minimizer of U . Thus $U_t \equiv |\nabla U| \equiv 0$ on $\mathcal{M} \times I$.

We next prove the following claim.

Claim. \mathcal{M} contains an $(N - 1)$ -dimensional $C^{1+\alpha_0}$ manifold Υ .

Let $\mathcal{P} : \mathbb{R}^N \mapsto H_0$, be the orthogonal projection to H_0 . Clearly, $\mathcal{P}(D) = \mathcal{P}(\mathcal{M})$ and $\mathcal{P}(D)$ has nonempty relative interior in H_0 .

We show that there is $(x_0, t_0) \in \mathcal{M} \times I$ such that $D^2U(x_0, t_0) \neq 0$. Otherwise, $\Delta U = 0$ on $\mathcal{M} \times I$, and combined with (4.5) this gives $f(x, 0, 0) = 0$ for each $x \in \mathcal{M}$. Since f is independent of x_1 , one has $f(\cdot, 0, 0) \equiv 0$ on the open cylinder $\mathbb{R} \times \mathcal{P}(D) \subset \mathbb{R}^N$. Fix $x_0 \in \mathcal{M}$ and $R > 0$ such that $B(x_0, R) \subset (\mathbb{R} \times \mathcal{P}(D)) \cap \Omega$. Then U satisfies

$$U_t = \Delta U + f(x, U, \nabla U) - f(x, 0, 0) = L(x, t)U, \quad (x, t) \in B(x_0, R) \times \mathbb{R},$$

where $L \in \mathcal{E}(\beta_0, B(x_0, R) \times \mathbb{R})$, and $U \geq 0$, $U(x_0, t) = 0$ for $t \in I$. Thus by the strong maximum principle $U \equiv 0$ on $B(x_0, R) \times I$, and consequently, by Lemma 4.4, $U \equiv 0$ everywhere, a contradiction.

Hence, we have showed that there are $(x_0, t_0) \in \mathcal{M} \times I$ and $k, l \in \{1, \dots, N\}$ such that $\partial_{x_k x_l} U(x_0, t_0) \neq 0$. Set $v := (U)_{x_k}$. Then one has

$v \in C_{\text{loc}}^{1+\alpha_0, \alpha_0/2}(\Omega \times I)$, $|\nabla v(x_0, t_0)| \neq 0$, and, by (4.5), $v \equiv 0$ on $\mathcal{M} \times I$. Denote $\mathcal{Z} := \{x \in \Omega : v(x, t_0) = 0\}$. By the implicit function theorem, there is $r > 0$ and an $(N-1)$ -dimensional $C^{1+\alpha_0}$ manifold Υ such that $\mathcal{Z} \cap B(x_0, r) = \Upsilon \cap B(x_0, r)$. We can also assume that Υ divides $B(x_0, r)$ into exactly 2 connected components. By (4.5), $\mathcal{M} \cap B(x_0, r) \subset \Upsilon \cap B(x_0, r)$. We finish the proof of the Claim by showing that $\mathcal{M} \cap B(x_0, r) = \Upsilon \cap B(x_0, r)$. Indeed, if it is not true, then the set $B(x_0, r) \setminus \mathcal{M}$ is connected. Consequently, $P_{\lambda_0}(B(x_0, r) \setminus \mathcal{M})$ is connected as well. Clearly this open connected set contains points of Ω , and therefore it cannot contain any points of $\mathbb{R}^N \setminus \bar{\Omega}$. Thus $P_{\lambda_0}(B(x_0, r) \setminus \mathcal{M}) \subset \Omega$. Now, near $P_{\lambda_0}(x_0)$, $\partial\Omega \subset P_{\lambda_0}(B(x_0, r) \cap \mathcal{M})$ is a subset of the $(N-1)$ -dimensional manifold $P_{\lambda_0}(\Upsilon)$, we obtain a contradiction to condition (A). The Claim is proved.

To continue, we fix a ball $G \subset \Omega$ such that Υ divides G into two connected components G_+ and G_- . Denote by τ, T the boundary points of the interval $I: I = (\tau, T)$, and set $I_0 := (\tau, (T + \tau)/2)$. Fix any $\delta \in (0, (T - \tau)/2)$ and denote $w(x, t) := U(x, t + \delta) - U(x, t)$ for $(x, t) \in \Omega \times \mathbb{R}$. Notice that $t + \delta \in I$ whenever $t \in I_0$. Then $w \in C^{2+\alpha_0, 1+\alpha_0/2}(\Omega \times \mathbb{R})$ satisfies

$$w_t = L(x, t)w, \quad (x, t) \in \Omega \times \mathbb{R},$$

where $L \in \mathcal{E}(\beta_0, \Omega \times \mathbb{R})$. We next consider a new operator $L^* \in \mathcal{E}(\beta_0, \Omega \times \mathbb{R})$ defined by $L^* = L$ on $G_+ \times I_0$ and $L^* = \Delta$ on $G_- \times I_0$ (the lower order coefficients are equal to 0). Also define w^* such that $w^* = w$ on $G_+ \times I_0$ and $w^* \equiv 0$ on $G_- \times I_0$. Since w is a solution on G and $w = |\nabla w| = 0$ on Υ , an integration by parts shows that w^* is a weak solution of

$$w_t^* = L^*(x, t)w^*, \quad (x, t) \in G \times I_0.$$

Since $w^* \equiv 0$ on the open set $G_- \times I_0$, the unique continuation principle [1] yields $w^* \equiv 0$ on $G \times I_0$. Thus $w = w^* = 0$ on $G_+ \times I_0$ and the unique continuation for w yields $w \equiv 0$ on $\Omega \times I_0$. Consequently, $U \equiv U(\cdot, \cdot + \delta)$ on $\Omega \times I_0$ for any sufficiently small δ . The uniqueness for the initial value problem implies that $U \equiv U(\cdot, \cdot + \delta)$ on $\Omega \times (\tau, \infty)$ and the backward uniqueness for parabolic equations (see Remark 4.6 below) then gives $U \equiv U(\cdot, \cdot + \delta)$ on $\Omega \times \mathbb{R}$, hence $U \in E$. Since $U \not\equiv 0$, the assumption that $V_{\lambda_0}U \equiv 0$ on D implies that U has nontrivial nodal set, thus $U \in E_+$ (see Lemma 4.1). \square

Remark 4.6. This remark is to justify the use of backward uniqueness in the previous proof and in an argument given in the next section. The backward

uniqueness theorems from [16, 37], for example, apply to the difference of any two solutions of (2.4) provided the following statement holds. Given any solution U of (2.4), the function $\tilde{U} : t \mapsto U(\cdot, t)$ satisfies

$$\tilde{U} \in C(\mathbb{R}, H_0^1(\Omega)) \cap L_{\text{loc}}^2(\mathbb{R}, D(\Delta)). \quad (4.6)$$

Here $D(\Delta)$ is the domain of the $L^2(\Omega)$ -realization of the Laplace operator with Dirichlet boundary conditions:

$$D(\Delta) = \{\varphi \in H_0^1(\Omega) : \Delta\varphi \in L^2(\Omega)\},$$

where Δ is viewed as an isomorphism of $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ (one has $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ if Ω is smooth). The space $D(\Delta)$ is equipped with the usual graph norm. For smooth domains Ω , (4.6) is well known. In the general case, (4.6) can be established by a rather standard approximation procedure. For the reader's convenience, we give the details in Appendix B.

Lemma 4.7. *Let $U \in \mathcal{A}$ and $\lambda_0 := \Lambda(U(\cdot, \tau)) > 0$ for some $\tau \in \mathbb{R}$. Then either $U \in E_+$ or there exists $\varepsilon > 0$ such that*

$$\Lambda(z) \leq \lambda_0 - \varepsilon \quad (z \in \omega(U)). \quad (4.7)$$

Proof. By Lemma 3.6, $\lambda_0 = \Lambda(U(\cdot, \tau)) \geq \Lambda(U(\cdot, \tau_0))$ for each $\tau_0 \geq \tau$. By Lemma 3.7, relation (4.7) holds for some $\varepsilon > 0$, provided there is $\tau_0 \geq \tau$ such that $V_{\lambda_0}(U(\cdot, \tau_0)) \not\equiv 0$ on any connected component of Ω_{λ_0} . On the other hand, if there is no such $\tau_0 \geq \tau$, then Lemma 4.5 applies and we conclude that $U \in E_+$. \square

Lemma 4.8. *If $U \in \mathcal{A} \setminus E_+$, $\tau \in \mathbb{R}$, and $\lambda_0 := \Lambda(U(\cdot, \tau))$, then either $\omega(U) = \{z_0\}$ for some $z_0 \in E_+$ with $\Lambda(z_0) < \lambda_0$, or*

$$\Lambda(z) = 0 \text{ for each } z \in \omega(U). \quad (4.8)$$

Proof. By Lemma 3.6, (4.8) holds if $\lambda_0 = 0$.

Assume that $\lambda_0 > 0$ and (4.8) does not hold, that is, there is $z_0 \in \omega(U)$ with $\Lambda(z_0) > 0$. Let $\lambda^* \geq 0$ be as in Theorem 3.8. Then, by Lemma 4.7, $\lambda^* < \lambda_0$ and, obviously, $\lambda^* \geq \Lambda(z_0) > 0$. Once we know that $\lambda^* > 0$, we can apply Lemma 4.5 to each nonzero entire solution in $\omega(U)$ (cp. Lemma 4.3). Indeed, any such entire solution Z satisfies $Z(\cdot, t) \in \omega(U)$, and therefore for each $t \in \mathbb{R}$ one has $V_{\lambda^*}Z(\cdot, t) \equiv 0$ on a connected component of Ω_{λ^*} , as stated in Theorem 3.8. By Lemma 4.5, $\omega(U) \setminus \{0\} \subset E_+$. Hence, by Lemma 4.2, $\omega(U)$ is a finite set and, as it is connected, $\omega(U) = \{z_0\}$. Since $\Lambda(z_0) > 0$, we have $z_0 \in E_+$ and the relations $\lambda_0 > \lambda^* \geq \Lambda(z_0)$ complete the proof. \square

The above results are mainly concerned with $\omega(U)$. Next we intend to consider $\alpha(U)$. For that the following lemma will be useful.

Lemma 4.9. *Let $Z \in \mathcal{A} \setminus \{0\}$ and $\tau \in \mathbb{R}$. Set $\lambda_0 := \Lambda(Z(\cdot, \tau))$. Then for each $\lambda \in (\lambda_0, \ell)$ one has $V_\lambda Z > 0$ in $\Omega_\lambda \times (\tau, \infty)$. Moreover if $\lambda_0 > 0$, then either $V_{\lambda_0} Z > 0$ in $\Omega_{\lambda_0} \times (\tau, \infty)$ or $Z \in E_+$.*

Proof. For $\lambda \geq \lambda_0$ let w^λ be as in (3.7); it satisfies (3.9), (3.10), and (3.13). By the maximum principle, either

$$w^\lambda > 0 \text{ in } \Omega_\lambda \times (\tau, \infty) \quad (4.9)$$

or there are $\epsilon > 0$ and a connected component D_λ of Ω_λ such that $w^\lambda \equiv 0$ in $D_\lambda \times (\tau, \tau + \epsilon)$. The second possibility cannot hold if $\lambda > \lambda_0$. For if it did, then, by Lemma 4.5, $Z \in E_+$, and we would have a contradiction to Lemma 4.1(ii). Hence, (4.9) holds for $\lambda > \lambda_0$. If $\lambda = \lambda_0$ and $\lambda_0 > 0$, then either (4.9) holds or Lemma 4.5 implies $Z \in E_+$. The lemma is proved. \square

Lemma 4.10. *Let $U \in \mathcal{A}$ and assume that $\mu_0 := \Lambda(U(\cdot, \tau)) > 0$ for some $\tau \in \mathbb{R}$. Then $\alpha(U) = \{z\}$ for some $z \in E$, and either $z \equiv 0$, or $z \in E_+$ and $\Lambda(z) \geq \mu_0$.*

Proof. By Lemma 3.6, $\sigma := \lim_{\tau \rightarrow -\infty} \Lambda(U(\cdot, \tau)) \geq \mu_0 > 0$ and

$$\sigma \geq \Lambda(z) \quad (z \in \alpha(U)). \quad (4.10)$$

We claim that

$$\Lambda(z) = \sigma \quad (z \in \alpha(U) \setminus \{0\}). \quad (4.11)$$

To prove this, take any $z_0 \in \alpha(U) \setminus \{0\}$. There is $Z \in \mathcal{A}$ with $Z(\cdot, 0) = z_0$ and $Z(\cdot, t) \in \alpha(U)$ for all $t \in \mathbb{R}$ (cp. Lemma 4.3). Then $\Lambda(Z(\cdot, 1)) \leq \Lambda(z_0)$, by Lemma 3.6. By Lemma 4.9, we have $V_\lambda Z(\cdot, 1) > 0$ in Ω_λ for any $\lambda \in (\Lambda(z_0), \ell)$. Applying Lemma 3.7(ii) with $z = Z(\cdot, 1)$, we obtain $\Lambda(U(\cdot, t)) < \lambda$ for each $t \in \mathbb{R}$, hence $\sigma \leq \lambda$. Since $\lambda \in (\Lambda(z_0), \ell)$ was arbitrary, it follows that $\sigma \leq \Lambda(z_0)$. Combined with (4.10), this gives $\Lambda(z_0) = \sigma$. Hence (4.11) is proved.

We next prove that $\alpha(U) \setminus \{0\} \subset E_+$. Take again any $z_0 \in \alpha(U) \setminus \{0\}$ and let Z have the same meaning as in the previous paragraph. Observe that $Z(\cdot, 1) \not\equiv 0$ (otherwise, $0 \equiv Z(\cdot, 0) \equiv z_0$ by Lemma 4.4). Thus, (4.11) gives $\Lambda(Z(\cdot, 1)) = \sigma > 0$. Assume $Z(\cdot, 1) \notin E_+$ and set $\lambda_0 := \Lambda(z_0)$. By Lemma 4.9, for each $\lambda \in [\lambda_0, \ell)$ one has $V_\lambda Z(\cdot, 1) > 0$ in Ω_λ , and consequently

we can apply Lemma 3.7(ii) with $z = Z(\cdot, 1)$. This yields $\varepsilon_0 > 0$ such that $\Lambda(U(\cdot, t)) \leq \lambda_0 - \varepsilon_0$ for each $t \in \mathbb{R}$ and therefore $\sigma \leq \Lambda(z_0) - \varepsilon_0$, in contradiction to (4.11). This contradiction shows that $Z(\cdot, 1) \in E_+$. Thus $z_0 = Z(\cdot, 0) = Z(\cdot, 1) \in E_+$, as desired.

Once we know that $\alpha(U) \setminus \{0\} \subset E_+$, Lemma 4.2 implies that $\alpha(U)$ is a finite set. As it is connected, $\alpha(U)$ consists of a single equilibrium z and either $z \in E_+$ or $z \equiv 0$. \square

The next lemma treats the case $\alpha(U) = \{0\}$.

Lemma 4.11. *If $U \in \mathcal{A}$ and $\alpha(U) = \{0\}$, then $\Lambda(U(\cdot, t)) = 0$ for each $t \in \mathbb{R}$.*

We postpone the proof of this lemma until the next subsection.

We are now ready to complete the proof Theorem 2.3.

Proof of Theorem 2.3. It is obvious that at most one of the statements (i)-(iv) of Theorem 2.3 can hold.

If $\alpha(U) = \{0\}$, then Lemma 4.11 says that statement (i) holds.

Assume now that $\alpha(U) \neq \{0\}$ and none of the statements (i), (ii) holds, that is, $U \notin E_+$ and there is $\tau \in \mathbb{R}$ such that $\mu_0 = \Lambda(U(\cdot, \tau)) > 0$. Then, Lemmas 4.8, 4.10 imply that one of the statements (iii), (iv) holds.

The conclusion concerning the case $f(\cdot, 0, 0) \geq 0$ follows from the comparison principle, as already explained in Remark 2.4(c). \square

4.1 Proof of Lemma 4.11

Throughout this subsection we assume that $U \in \mathcal{A}$ and $\alpha(U) = \{0\}$. Lemmas 4.3 and 4.4, imply that $0 \in E$ and $f(\cdot, 0, 0) \equiv 0$. The conclusion of Lemma 4.11 trivially holds true if $U \equiv 0$, thus in the following we assume $U \not\equiv 0$. By Lemma 4.4, $U(\cdot, t) \not\equiv 0$ for any $t \in \mathbb{R}$, hence, by the strong comparison principle, $U(\cdot, t) > 0$ in Ω for each $t \in \mathbb{R}$.

Now using $f(\cdot, 0, 0) \equiv 0$, we have $U_t = \Delta U + f(x, U, \nabla U) - f(x, 0, 0)$. Therefore, by the Hadamard formula, U is a bounded positive entire solution of a linear problem

$$\begin{aligned} v_t &= \Delta v + L(x, t)v, & (x, t) &\in \Omega \times \mathbb{R}, \\ v &= 0, & (x, t) &\in \partial\Omega \times \mathbb{R}, \end{aligned} \tag{4.12}$$

where $L \in \mathcal{E}(\beta_0, \Omega \times \mathbb{R})$ and β_0 is as in (F1). We use this observation below to control the decay of $U(\cdot, t)$, as $t \rightarrow \infty$. Then we find a suitable transformation

of U which is uniformly positive. This will allow us to apply Theorem 3.9 to conclude that $\Lambda(U(\cdot, t)) = 0$ for all t .

Below, C^*, C_1, C_2, \dots denote positive constants independent of t and x .

Lemma 4.12. *One has*

$$0 < C_1 \leq \frac{\|U(\cdot, t + \tau)\|_{L^\infty(\Omega)}}{\|U(\cdot, t)\|_{L^\infty(\Omega)}} \leq C_2 < \infty \quad (t \in \mathbb{R}, \tau \in [0, 1]) \quad (4.13)$$

and

$$\inf_{t \in \mathbb{R}} \frac{U(x, t)}{\|U(\cdot, t)\|_{L^\infty(\Omega)}} > 0 \quad (x \in \Omega). \quad (4.14)$$

Proof. As remarked above, U is a positive bounded solution of a linear problem (4.12) with $L \in \mathcal{E}(\beta_0, \Omega \times \mathbb{R})$. By [21, Theorem 5.5], there exists a positive solution ϕ of (4.12) satisfying the following two conditions

$$0 < C_1 \leq \frac{\|\phi(\cdot, t + \tau)\|_{L^\infty(\Omega)}}{\|\phi(\cdot, t)\|_{L^\infty(\Omega)}} \leq C_2 < \infty \quad (t \in \mathbb{R}, \tau \in [0, 1]), \quad (4.15)$$

$$\inf_{t \in \mathbb{R}} \frac{\phi(x, t)}{\|\phi(\cdot, t)\|_{L^\infty(\Omega)}} > 0 \quad (x \in \Omega).$$

Note, in particular that (4.15) implies

$$\|\phi(\cdot, t)\|_{L^\infty(\Omega)} \leq C_3 e^{\gamma|t|} \quad (t \in \mathbb{R}) \quad (4.16)$$

for some $C_3, \gamma > 0$. By [21, Proposition 2.5], the positive solution satisfying (4.16) is unique up to scalar multiples. Since U is bounded and positive, $U = c\phi$ for some $c > 0$. This implies (4.13), (4.14). \square

Lemma 4.13. *There exists a smooth function $\gamma : \mathbb{R} \rightarrow (0, \infty)$ such that*

$$\frac{|\gamma'(t)|}{\gamma(t)} \leq C^* < \infty \quad (t \in \mathbb{R}), \quad (4.17)$$

$$0 < C_4 \leq \frac{\|U(\cdot, t)\|_{L^\infty(\Omega)}}{\gamma(t)} \leq C_5 < \infty \quad (t \in \mathbb{R}). \quad (4.18)$$

Proof. We follow [22, Proof of Lemma 6.3]. By (4.13),

$$|\log \|U(\cdot, k + 1)\|_{L^\infty(\Omega)} - \log \|U(\cdot, k)\|_{L^\infty(\Omega)}| \leq \frac{C_6}{2} \quad (k \in \mathbb{Z}).$$

It is therefore easy to find a smooth function $\eta : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\eta(k) := \log \|U(\cdot, k)\|_{L^\infty(\Omega)} \quad (k \in \mathbb{Z}), \quad |\eta'(t)| \leq C_6 \quad (t \in \mathbb{R}).$$

Set $\gamma(t) := e^{\eta(t)}$. Then

$$|\gamma'(t)| = \gamma(t)|\eta'(t)| \leq C_6\gamma(t) \quad (t \in \mathbb{R}).$$

Since $|\eta(t + \tau) - \eta(t)| \leq C_6$ for each $\tau \in [0, 1]$, we have

$$e^{-C_6} \leq e^{\eta(t) - \eta(t+\tau)} = \frac{\gamma(t)}{\gamma(t+\tau)} \leq e^{C_6} \quad (t \in \mathbb{R}, \tau \in [0, 1]). \quad (4.19)$$

From (4.13) and (4.19) we next obtain

$$C_1 e^{-C_6} \leq \frac{\gamma(k)}{\gamma(t)} \frac{\|U(\cdot, t)\|_{L^\infty(\Omega)}}{\|U(\cdot, k)\|_{L^\infty(\Omega)}} \leq C_2 e^{C_6} \quad (t \in [k, k+1], k \in \mathbb{Z}).$$

Since $\gamma(k) = \|U(\cdot, k)\|_{L^\infty(\Omega)}$ for each $k \in \mathbb{Z}$, (4.18) follows. \square

With γ as in Lemma 4.13, set $Z(x, t) := \frac{U(x, t)}{\gamma(t)}$. Then

$$Z_t = \frac{U_t}{\gamma(t)} - \frac{\gamma'(t)U}{\gamma^2(t)} = \Delta Z + \frac{1}{\gamma(t)} f(x, \gamma(t)Z, \gamma(t)\nabla Z) - \frac{\gamma'(t)}{\gamma(t)} Z.$$

Hence, Z is a solution of the problem

$$Z_t = \Delta Z + g(t, x, Z, \nabla Z), \quad (x, t) \in \Omega \times \mathbb{R}, \quad (4.20)$$

$$U = 0, \quad (x, t) \in \partial\Omega \times (0, \infty), \quad (4.21)$$

where

$$g(t, x, u, p) := \frac{1}{\gamma(t)} f(x, \gamma(t)u, \gamma(t)p) - \frac{\gamma'(t)}{\gamma(t)} u \\ (x \in \Omega, t \in \mathbb{R}, u \in [0, \infty), p \in \mathbb{R}^N).$$

We verify that g satisfies conditions (G1)-(G3) of Section 3.2. Since f is independent of x_1 and even in p_1 , so is g and (G2) is satisfied. Clearly, g is continuous. From (F1) and (4.17) we have

$$\sup_{x \in \bar{\Omega}, t \in \mathbb{R}} |g(x, t, u, p) - g(x, t, u', p')| \\ \leq \sup_{x \in \bar{\Omega}, t \in \mathbb{R}} \frac{1}{\gamma(t)} |f(x, \gamma(t)u, \gamma(t)p) - f(x, \gamma(t)u', \gamma(t)p')| + \left| \frac{\gamma'(t)}{\gamma(t)} \right| |u - u'| \\ \leq \tilde{\beta}_0 (|u - u'| + |p - p'|) \quad (u, u' \in \mathbb{R}_+, p, p' \in \mathbb{R}^N),$$

for some $\tilde{\beta}_0 > 0$. Thus g satisfies (G1). Also, since $f(\cdot, 0, 0) \equiv 0$,

$$g(t, x, 0, 0) = 0 \quad ((x, t) \in \Omega \times \mathbb{R}),$$

hence (G3) is satisfied as well.

Now, $\|Z\|_{L^\infty(\Omega \times \mathbb{R})} \leq C_5$, by (4.18). Hence Z is a positive bounded entire solution of (4.20), (4.21), or, in the notation of Section 3.2, $Z \in \mathcal{A}^*$. Moreover, for each $x \in \Omega$ relations (4.14) and (4.18) give

$$\inf_{t \in \mathbb{R}} Z(x, t) = \inf_{t \in \mathbb{R}} \frac{U(x, t)}{\|U(\cdot, t)\|_{L^\infty(\Omega)}} \frac{\|U(\cdot, t)\|_{L^\infty(\Omega)}}{\gamma(t)} > 0.$$

This implies that all functions in $\alpha(Z)$ are (strictly) positive on Ω . Applying Theorem 3.9 to Z , we obtain $\Lambda(Z(\cdot, t)) = 0$ for each $t \in \mathbb{R}$. Since, obviously, $\Lambda(U(\cdot, t)) = \Lambda(Z(\cdot, t))$, Lemma 4.11 is proved.

5 Proof of Theorem 2.2

Assume the hypotheses of Theorem 2.2 to be satisfied. Recall that $\omega(u)$ is a compact subset of $C_0(\Omega)$; we view it as compact metric space with the induced norm (the supremum norm). Our first concern is to show that there is a flow on $\omega(u)$ defined by elements of \mathcal{A} , that is, bounded entire solutions of problem (2.4). This is not completely obvious, for under our assumptions one cannot in general expect the initial-value problem for (1.1), (1.2) to be well-posed in $C_0(\Omega)$.

Fix any $z_0 \in \omega(u)$. Then there is $V \in \mathcal{A}$ such that $V(\cdot, 0) = z_0$ and $V(\cdot, t) \in \omega(u)$ for any $t \in \mathbb{R}$. This can be proved by a straightforward modification of the arguments given in the proof of Lemma 4.3: in addition to taking time intervals in $(0, \infty)$, rather than in $(-\infty, \infty)$, one needs to include the function h in the linear nonhomogeneous equation (4.2). Since h is bounded, the regularity estimates and the rest of the arguments go through (note in particular that, thanks to hypothesis (H), one obtains the same limit autonomous equation as in (4.4)).

Next we show that V is uniquely defined. Indeed, if $V, \tilde{V} \in \mathcal{A}$ satisfy $V(\cdot, 0) = z_0 = \tilde{V}(\cdot, 0)$, then $w = V - \tilde{V}$ is a solution of a linear problem (3.1), (3.2) with $L \in \mathcal{E}(\beta_0, \Omega \times \mathbb{R})$ and $h \equiv 0$. Also $w(\cdot, 0) \equiv 0$. The maximum principle implies the uniqueness for the initial-boundary value problem: $w \equiv 0$ on $\Omega \times [0, \infty)$. To prove that $w \equiv 0$ on $\Omega \times (-\infty, 0]$ one uses backward uniqueness for parabolic equations (see Remark 4.6).

In view of the uniqueness of V , setting $S_t z_0 := V(\cdot, t)$ for $t \in \mathbb{R}$, we have defined a family S of maps on $\omega(u)$. We show that S is a flow, that is,

- (i) S_0 is the identity on $\omega(u)$,
- (ii) $S_{t+s} = S_t S_s \quad (s, t \in \mathbb{R})$,
- (iii) for each $t_0 \in \mathbb{R}$, the map S_{t_0} is continuous.

The fact that $S_0 = I$ is obvious. The group property (ii) follows from the uniqueness of V and the time-translation invariance of (2.4). To prove (iii), take first $t_0 > 0$. For any $z_1, z_2 \in \omega(u)$, the function $w(\cdot, t) = S_t z_1 - S_t z_2$ is a solution of a linear problem (3.1), (3.2) on $\Omega \times \mathbb{R}$ with $L \in \mathcal{E}(\beta_0, \Omega \times \mathbb{R})$ and $h \equiv 0$. By Theorem 3.2,

$$\|S_{t_0} z_1 - S_{t_0} z_2\|_{L^\infty(\Omega)} \leq C(t_0) \|z_1 - z_2\|_{L^\infty(\Omega)}$$

and the continuity of S_{t_0} follows. Now let $t_0 < 0$. Properties (i) and (ii) imply that S_{t_0} is the inverse to the continuous map S_{-t_0} . Since $\omega(u)$ is compact, the inverse is continuous.

In the next lemma, we show that $\omega(u)$ is *chain transitive* under the flow S . This means that for any $\phi, \psi \in \omega(u)$ and any $\varepsilon > 0$, $T > 0$ there exist an integer $k \geq 1$, real numbers $t_1, \dots, t_k \geq T$, and points $\phi_0, \phi_1, \dots, \phi_k \in \omega(u)$ with $\phi_0 = \phi$, $\phi_k = \psi$, such that

$$\|S_{t_{i+1}} \phi_i - \phi_{i+1}\|_{L^\infty(\Omega)} < \varepsilon \quad (0 \leq i < k). \quad (5.1)$$

This in particular means that $\omega(u)$ is *chain recurrent*, that is, the above condition is satisfied with $\psi = \phi$, for any $\phi \in \omega(u)$.

The following lemma is very similar to [8, Lemma 7.5], [14, Lemma 4.5] (see also [28]); however, we cannot directly apply those results here since the flow S is not defined outside $\omega(u)$.

Lemma 5.1. *The set $\omega(u)$ is chain transitive under the flow S .*

Proof. Fix any $\varepsilon, T > 0$ and $\phi, \psi \in \omega(u)$. Denote $I = [T, 2T]$ and let $C_0 = C^*(N, \beta_0, T)$, where C^* is as in Theorem 3.2. By (2.3) and (H) we can fix T_1 with

$$\text{dist}_{C_0(\bar{\Omega})}(u(\cdot, t), \omega(u)) < \frac{\varepsilon}{3C_0} \quad (t \geq T_1), \quad (5.2)$$

$$\|h(\cdot, t)\|_{L^\infty(\Omega)} < \frac{\varepsilon}{3C_0} \quad (t \geq T_1). \quad (5.3)$$

Since $\phi, \psi \in \omega(u)$, there are $s'_2 > s'_1 \geq T_1$ with $s'_2 - s'_1 > T$, such that $\|u(\cdot, s'_1) - \phi\|_{L^\infty(\Omega)} < \frac{\varepsilon}{3}$ and $\|u(\cdot, s'_2) - \psi\|_{L^\infty(\Omega)} < \frac{\varepsilon}{3}$. Clearly, there exist $k \in \mathbb{N}$ and an increasing finite sequence $(s_i)_{i=0}^k$ with $s_0 = s'_1$, $s_k = s'_2$, and $2T \geq s_{i+1} - s_i \geq T$. As $s_i \geq s'_1 \geq T_1$, (5.2) implies the existence of points $\phi_i \in \omega(u)$, $i \in \{0, \dots, k\}$, with $\phi_0 = \phi$, $\phi_k = \psi$, and $\|\phi_i - u(\cdot, s_i)\|_{L^\infty(\Omega)} \leq \frac{\varepsilon}{3C_0}$. We show that these points satisfy (5.1) with $t_i := s_i - s_{i-1} \in [T, 2T]$. Indeed,

$$\|S_{t_{i+1}}\phi_i - \phi_{i+1}\|_{L^\infty(\Omega)} \leq \|S_{t_{i+1}}\phi_i - u(\cdot, s_{i+1})\|_{L^\infty(\Omega)} + \|u(\cdot, s_{i+1}) - \phi_{i+1}\|_{L^\infty(\Omega)}.$$

Now, the function $w_i(x, t) := S_t\phi_i(x) - u(x, s_i + t)$ satisfies

$$\begin{aligned} (w_i)_t &= L_i(x, t)w_i + h(x, s_i + t), & (x, t) &\in \Omega \times (0, \infty), \\ w_i &= 0, & (x, t) &\in \partial\Omega \times (0, \infty), \\ w_i(\cdot, 0) &= u(\cdot, s_i) - \phi_i, & x &\in \Omega, \end{aligned}$$

where $L_i \in \mathcal{E}(\beta_0, \Omega \times (0, \infty))$. By Theorem 3.2, (5.2), and (5.3),

$$\begin{aligned} \|w_i(\cdot, t_{i+1})\|_{L^\infty(\Omega)} &\leq C_0(\|u(\cdot, s_i) - \phi_i\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega \times (s_i, s_i + 2T))}) \\ &< C_0 \left(\frac{\varepsilon}{3C_0} + \frac{\varepsilon}{3C_0} \right) = \frac{2\varepsilon}{3}. \end{aligned}$$

By the definition of ϕ_i we obtain

$$\|S_{t_{i+1}}\phi_i - \phi_{i+1}\|_{L^\infty(\Omega)} \leq \|w_i(\cdot, t_{i+1})\|_{L^\infty(\Omega)} + \|u(\cdot, s_{i+1}) - \phi_{i+1}\|_{L^\infty(\Omega)} < \varepsilon.$$

□

We are now ready to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. By Lemma 4.2, the set E_+ is finite. Let k be the number of elements of $E_+ \cap \omega(u)$. We write these elements in the order of decreasing values of Λ :

$$E_+ \cap \omega(u) = \{z_1, \dots, z_k\}, \quad \Lambda(z_1) \geq \dots \geq \Lambda(z_k)$$

(we choose an arbitrary order among the elements with the same value of Λ).

Now consider the following system of $k + 1$ subsets of $\omega(u)$:

$$M_1 := \{z_1\}, \dots, M_k := \{z_k\}, \quad M_{k+1} := \{z \in \omega(U) : \Lambda(z) = 0\}. \quad (5.4)$$

By Theorem 2.3, for each $U \in \omega(u) \setminus \bigcup_k M_k \subset \mathcal{A}$ one has $\alpha(U) \subset M_i$, $\omega(U) \subset M_j$, for some $i > j$. This means, in the terminology of [9], that the flow S admits a Morse decomposition with the Morse sets (5.4). By [9, Theorem II.7.A], every chain recurrent set of S is a subset of the union the Morse sets. Hence, by Lemma 5.1,

$$\omega(u) \subset \bigcup_{j=1, \dots, k+1} M_j.$$

Since $\omega(u)$ is connected, it must be equal to one of the sets M_j , which gives the conclusion of Theorem 2.2. \square

6 Appendix A: Proof of Theorem 3.9

Assume that g is a function satisfying conditions (G1)-(G3) and U is bounded (nonnegative) entire solution of (3.6). Recall from Section 3.2 that the Hölder estimate (2.5) holds and the trajectory $\{U(\cdot, t) : t \in \mathbb{R}\}$ is relatively compact in $C_0(\Omega)$.

Assume also that there is $z_0 \in \alpha(U)$ such that $z_0 > 0$ in Ω . To prove Theorem 3.9, we need to show that $\Lambda(U(\cdot, t)) = 0$ for each $t \in \mathbb{R}$.

Let $\lambda_0 := \Lambda(z_0)$.

We use the following lemma.

Lemma 6.1. *There is $z_1 \in \alpha(U)$ such that $\Lambda(z_1) \leq \lambda_0$ and $V_\lambda z_1 > 0$ in Ω_λ for each $\lambda \in [\lambda_0, \ell) \setminus \{0\}$.*

Suppose for a while that the statement in Lemma 6.1 is true. Let us show how it implies the desired conclusion.

By Lemma 3.6, $\sigma := \lim_{\tau \rightarrow -\infty} \Lambda(U(\cdot, \tau))$ satisfies $\Lambda(U(\cdot, t)) \leq \sigma$ for each t and

$$\sigma \geq \Lambda(z) \quad (z \in \alpha(U)). \quad (6.1)$$

If $\lambda_0 > 0$, then Lemma 6.1 in conjunction with Lemma 3.7(ii) implies that $\sigma < \lambda_0$ in contradiction to (6.1). Thus $\lambda_0 = 0$. Now for each $\lambda > 0$, Lemmas 6.1 and 3.7(ii) imply that $\sigma < \lambda$. Hence $\sigma = 0$, and consequently $\Lambda(U(\cdot, t)) = 0$ for each $t \in \mathbb{R}$.

It remains to prove the lemma.

Proof of Lemma 6.1. There is a sequence $t_n \rightarrow \infty$ such that $U(\cdot, -t_n) \rightarrow z_0$. Passing to a subsequence, we may also assume that $U(\cdot, -t_n + 1)$ converges to some z_1 in $C_0(\Omega)$ (this follows by the compactness of the trajectory of U). Of course, $z_1 \in \alpha(U)$. We show that z_1 has the properties stated in the lemma.

Pick any $\lambda \in [\lambda_0, \ell) \setminus \{0\}$. Since $z_0 > 0$ in Ω , we have $V_\lambda z_0 > 0$ on $\partial\Omega_\lambda \cap \partial\Omega$. Since Ω is convex in x_1 (hypothesis (D1)), this clearly implies that $V_\lambda z_0 \not\equiv 0$ on any connected component of Ω_λ . Also, $V_\lambda z_0 \geq 0$ on Ω_λ as $\lambda \geq \lambda_0 = \Lambda(z_0)$.

Let now G be any connected component of Ω_λ . The previous remarks imply that there exist a ball $B_0 \subset G$ and $r_0 > 0$ such that $V_\lambda z_0 > 3r_0$ on B_0 . Then $V_\lambda U(\cdot, -t_n) \geq 2r_0$ on B_0 for each sufficiently large n , and, by (2.5), there exists $\vartheta \in (0, 1/4)$ independent of n such that

$$V_\lambda U(\cdot, t) \geq r_0 \quad ((x, t) \in B_0 \times [-t_n, -t_n + 4\vartheta]). \quad (6.2)$$

We now show that $V_\lambda z_1 > 0$ in G . It is sufficient to prove that $V_\lambda z_1 > 0$ in D for any any subdomain of $D \subset G$ such that $\bar{D} \subset G$ and $B_0 \subset D$. Fix any such D . We use the following Harnack-type estimate on the function $V_\lambda U$ (recall that $w^\lambda = V_\lambda U$ is a solution of the linear problem (3.9), (3.10)):

$$\begin{aligned} V_\lambda U(x, -t_n + 1) \geq & \sup_{D \times (-t_n + \vartheta, -t_n + 2\vartheta)} \kappa_1 (V_\lambda U)^+ \\ & - \sup_{\partial_P(G \times (-t_n, -t_n + 1))} \kappa_2 (V_\lambda U)^- \quad (x \in D). \end{aligned} \quad (6.3)$$

Here κ_1, κ_2 are positive constants independent of n and ∂_P stands for the parabolic boundary:

$$\partial_P(G \times (-t_n, -t_n + 1)) := (\bar{G} \times \{-t_n\}) \cup (\partial G \times (-t_n, -t_n + 1)). \quad (6.4)$$

Estimate (6.3) is a special case of an estimate given in [30, Lemma 3.5].

Since G is a connected component of Ω_λ , we have $\partial G \subset \partial\Omega_\lambda$. Therefore, by (3.10), $V_\lambda U \geq 0$ on $\partial G \times \mathbb{R}$. Moreover, since $V_\lambda U(\cdot, -t_n) \rightarrow V_\lambda z_0 \geq 0$, uniformly in Ω_λ , the last term in (6.3) approaches 0 as $n \rightarrow \infty$. Using this and (6.2), we obtain, upon passing to the limit in (6.3), that $V_\lambda z_1 \geq \kappa_1 r_1 > 0$ on D , as desired.

We have thus shown that $V_\lambda z_1 > 0$ in any connected component of Ω_λ , hence $V_\lambda z_1 > 0$ in Ω_λ . Since $\lambda \in [\lambda_0, \ell) \setminus \{0\}$ was arbitrary, at the same time we have verified that $\Lambda(z_1) \leq \lambda_0$. The proof of the lemma is complete. \square

7 Appendix B: Proof of (4.6)

Assume that (D1), (D2), (A), (F1), and (F2) hold and let U be an arbitrary entire solution of (2.4). We verify that the function $\tilde{U} : t \mapsto U(\cdot, t)$ satisfies

$$\tilde{U} \in C(\mathbb{R}, H_0^1(\Omega)) \cap L_{\text{loc}}^2(\mathbb{R}, D(\Delta)) \quad (7.1)$$

(see Remark 4.6 for the meaning of $D(\Delta)$).

First we rewrite (2.4) as a linear nonhomogeneous problem (cp. (4.2)):

$$U_t = L(x, t)U + f(x, 0, 0), \quad (x, t) \in \Omega \times \mathbb{R}, \quad (7.2)$$

$$U = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}, \quad (7.3)$$

where $L \in \mathcal{E}(\beta_0, \Omega \times \mathbb{R})$. Note that since the principal part of L is the Laplacian, (7.2) can be considered as an equation in the divergence form or nondivergence form, as desired. We claim that U is a weak solution of (7.2), (7.3). This is not completely obvious, even though U is a classical solution, due to the lack of regularity of Ω . The nontrivial part of the claim is that $\tilde{U} \in L_{\text{loc}}^2(\mathbb{R}, H_0^1(\Omega))$. We verify this by an approximation procedure. Take a sequence of smooth domains $\Omega_n \subset \Omega$ such that $\Omega_n \subset \bar{\Omega}_n \subset \Omega_{n+1}$ ($n = 1, 2, \dots$) and $\partial\Omega_n$ approaches $\partial\Omega$ in the Hausdorff distance. Also, let $\eta_n : \mathbb{R}^N \rightarrow [0, 1]$, $n = 1, 2, \dots$, be smooth functions such that for $n = 2, 3, \dots$, one has $\eta_n \equiv 0$ on $\mathbb{R}^N \setminus \Omega_n$ and $\eta_n \equiv 1$ on Ω_{n-1} .

Fix any $T \in (0, \infty)$. On $\Omega_n \times (-T, T)$, we solve the following initial-boundary value problem:

$$\begin{aligned} U_t^n &= L(x, t)U^n + f(x, 0, 0), & (x, t) &\in \Omega_n \times (-T, T), \\ U^n(x, t) &= 0, & (x, t) &\in \partial\Omega_n \times (-T, T), \\ U^n(x, -T) &= \eta_n(x)U(x, -T), & x &\in \Omega_n. \end{aligned} \quad (7.4)$$

There is a unique weak solution U_n of (7.4) and, as Ω_n is smooth, it coincides with the unique strong solution (see [25, 27] for these concepts and results). Now, $U^n - U$ is a strong solution of the linear equation

$$V_t = L(x, t)V, \quad (x, t) \in \Omega_n \times (-T, T).$$

By the maximum principle for strong solutions, there is a constant independent of C such that

$$\|U - U^n\|_{L^\infty(\Omega_n \times (-T, T))} \leq C \|U^n - U\|_{L^\infty(\partial_P(\Omega_n \times (-T, T)))}, \quad (7.5)$$

where ∂_P stands for the parabolic boundary (cp. (6.4)). Since $U \in C(\bar{\Omega} \times [-T, T])$ and it satisfies the Dirichlet boundary condition, the right hand side of (7.5) converges to zero as $n \rightarrow \infty$. Thus, extending U^n by zero outside $\Omega_n \times [-T, T)$, we have $U - U^n \rightarrow 0$ in $L^\infty(\Omega \times (-T, T))$. At the same time, since $0 \leq U^n \leq U$ on the parabolic boundary of $\Omega_n \times (-T, T)$, one can estimate the $L^2((-T, T), H_0^1(\Omega_n))$ -norm of the function $\tilde{U}_n : t \mapsto U_n(\cdot, t)$ by a constant independent of n (see for example [27, Theorem 6.1]). Obviously, the $L^2((-T, T), H_0^1(\Omega_n))$ -norm coincide with the $L^2((-T, T), H_0^1(\Omega))$ -norm for the extended function. Thus, passing to a subsequence if necessary, we obtain that $\tilde{U}^n \rightarrow \tilde{U}$ weakly in $L^2((-T, T), H_0^1(\Omega))$. Since T was arbitrary, this gives us the desired conclusion that $\tilde{U} \in L_{\text{loc}}^2(\mathbb{R}, H_0^1(\Omega))$.

The Lipschitz continuity of $f(x, u, p)$ in (u, p) (and the continuity in x) now implies that the function $\phi(x, t) := f(x, U(x, t), \nabla U(x, t))$ belongs to $L^2(\Omega \times (-T, T))$ for any $T > 0$. Also, $\tilde{U}(\tau) \in H_0^1(\Omega)$ for almost all τ . Pick any such $\tau \in (-\infty, 0)$ and set $T := -\tau$. The problem

$$\begin{aligned} V_t &= \Delta V + \phi(x, t), & (x, t) &\in \Omega \times (-T, T), \\ V(x, t) &= 0, & (x, t) &\in \partial\Omega \times (-T, T), \\ V(x, -T) &= U(x, -T), & x &\in \Omega \end{aligned}$$

has a unique weak solution V , and, as $U(\cdot, -T) = U(\cdot, \tau) \in H_0^1(\Omega)$ and $\phi \in L^2(\Omega \times (-T, T))$, the function $\tilde{V} : t \mapsto V(\cdot, t)$ satisfies

$$\tilde{V} \in C([-T, T), H_0^1(\Omega)) \cap L^2((-T, T), D(\Delta)) \quad (7.6)$$

(see for example [37, Section II.3]). Using the fact that $\tilde{U} \in L^2((-T, T), H_0^1(\Omega))$ one shows easily that U has to coincide with the weak solution V . Since $T = -\tau$ can be taken arbitrarily large, we have verified that (7.1) holds.

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