On asymptotically symmetric parabolic equations

Juraj Földes
Institute for Mathematics and its Applicaitons, University of Minnesota
Minneapolis, MN 55455

P. Poláčik*
School of Mathematics, University of Minnesota
Minneapolis, MN 55455

December 4, 2012

Dedicated to Professor Hiroshi Matano on the occasion of his 60th birthday

Abstract

We consider global bounded solutions of fully nonlinear parabolic equations on bounded reflectionally symmetric domains, under nonhomogeneous Dirichlet boundary condition. We assume that, as $t \to \infty$, the equation is asymptotically symmetric, the boundary condition is asymptotically homogeneous, and the solution is asymptotically strictly positive in the sense that all its limit profiles are strictly positive. Our main theorem states that all the limit profiles are reflectionally symmetric and decreasing on one side of the symmetry hyperplane in the direction perpendicular to the hyperplane. We also illustrate by example that, unlike for equations which are symmetric at all finite times, the result does not hold under a relaxed positivity condition requiring merely that at least one limit profile of the solution be strictly positive.

*Supported in part by NSF Grant DMS–0900947

Contents

1 Introduction

In this paper, we continue our study of symmetry properties of positive solutions of nonlinear parabolic equations. In the previous papers [19, 11], we considered the problem

$$\begin{aligned} \partial_t u &= F(t, x, u, Du, D^2u), \qquad && (x,t) \in \Omega \times (0, \infty), \\ u &= 0, && (x,t) \in \partial\Omega \times (0, \infty), \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, which is reflectionally symmetric about a hyperplane and convex in the direction orthogonal to that hyperplane, and F is a Lipschitz function satisfying uniform ellipticity conditions. Under suitable symmetry hypotheses on the nonlinearity F, we established the asymptotic reflectional symmetry of bounded positive solutions. These results are in the spirit of earlier symmetry results for elliptic equations as proved in [13, 17, 10, 6, 9, 21] and many other papers (surveys can be found in [5, 16, 18]).

When dealing with the Cauchy-Dirichlet problem for parabolic equations, solutions cannot be spatially symmetric, unless they emanate from a symmetric initial condition. Thus the asymptotic symmetry, that is, the symmetry of all limit profiles of a solution as time approaches ∞, is a natural concept for the study of symmetry. First asymptotic symmetry results for parabolic equations were given in [2, 3, 14], more general results can be found in the later papers [4, 19, 11] (see also the survey [20] and the recent paper [23]).

While there are many similarities between the results in parabolic and elliptic equations, in particular the method of moving hyperplanes [1, 24] usually plays an important role in both, there are significant differences as well. For example, a solution of (1.1), even though positive, may converge to zero along a sequence of times and to some positive functions along different sequences. This causes major technical difficulties when one wants to establish the asymptotic symmetry of such solutions and new techniques had to be developed for this purpose [19].

When considering the asymptotic symmetry of solutions, several natural questions come to mind. For example, can the asymptotic symmetry be proved if the equation itself is not symmetric, but is merely asymptotically symmetric as $t \to \infty$? Then one can start thinking about relaxing other conditions: assuming the solutions in question to be asymptotically positive, rather than positive, or replacing the homogeneous Dirichlet boundary condition with an asymptotically homogeneous one. Our goal in this paper is to examine to what extend the asymptotic symmetry results remain valid for such asymptotically symmetric problems. To discuss our present contributions in a simpler setting, consider first the following semilinear problem

$$\begin{aligned} \partial_t u &= \Delta u + f(t,u) + g_1(x,t), && (x,t) \in \Omega \times (0,\infty), \\ u &= g_2(x,t), && (x,t) \in \partial\Omega \times (0,\infty). \end{aligned} \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain and $f : [0,\infty)\times\mathbb{R} \to \mathbb{R}$, $g_1 : \bar{\Omega}\times[0,\infty) \to \mathbb{R}$, $g_2 : \partial\Omega \times [0,\infty) \to \mathbb{R}$ are continuous functions such that the following conditions are satisfied.

(D) Ω is convex in x_1 and symmetric with respect to the hyperplane

$$H_0 := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}.$$

(f) f if Lipschitz in u: there is $\beta > 0$ such that

$$\sup_{t\geq 0} |f(t,u) - f(t,\tilde{u})| \leq \beta|u - \tilde{u}| \quad (t \geq 0,\, u, \tilde{u} \in \mathbb{R}). \tag{1.3}$$

(g) For $i = 1, 2$ one has

$$\lim_{t\to\infty} \sup_{x} |g_i(x,t)| = 0, \tag{1.4}$$

where the supremum is taken over $x \in \Omega$ if $i = 1$ and over $x \in \partial\Omega$ if $i = 2$.

Let us make a few comments on these hypotheses. Without g_1, the first equation in (1.2) is symmetric: it is invariant under reflections. With g_1 added, the equation is no longer symmetric, but the nonsymmetric perturbation diminishes as $t \to \infty$ and in this sense the equation is asymptotically symmetric. This setting is general enough to apply to a larger class of nonsymmetric perturbations. For example, instead of g_1 one could add to f another nonlinearity $g = g(t, x, u)$ converging to 0 as $t \to \infty$ uniformly in x and u. Since we only consider properties of individual solutions, we can always write this more general equation in the form (1.2) by setting $g_1(x, t) = g(t, x, u(x, t))$, where u is a solution under investigation. A similar remark applies to the asymptotically homogeneous Dirichlet boundary condition.

As indicated above, problems like (1.2) come about naturally when one thinks about the robustness of the symmetry results for (1.1). Asymptotically symmetric equations can also arise in studies of parabolic systems. Assume, for example, that a parabolic system for the unknown vector function (u, v) is considered in which the first equation has the form $u_t = \Delta u + f(t, u) + g(t, u, v)$. This equation can also be put in the form (1.2) by setting $g_1(x, t) = g(t, u(x, t), v(x, t))$, where $(u(x, t), v(x, t))$ is a solution to be examined. In this situation, the decay of g_1 as $t \to \infty$ might come from explicit decay assumptions on the function g or it can be forced by the behavior of v. The latter occurs when g has the form $g(t, u, v) = v\tilde{g}(t, u, v)$ and one can establish the decay of v (a specific example of a reaction-diffusion system that is reduced this way to an asymptotically symmetric autonomous scalar equation can be found in [15]).

By the *asymptotic symmetry* of a global solution u of (1.2) we mean the property

$$\lim_{t\to\infty} (u(-x_1, x', t) - u(x_1, x', t)) = 0 \quad (x = (x_1, x') \in \Omega). \tag{1.5}$$

Alongside (1.5), we consider the *asymptotic monotonicity* of u:

$$\limsup_{t\to\infty} u_{x_1}(x_1, x', t) \le 0 \quad (x \in \Omega_0 := \{x \in \Omega : x_1 > 0\}). \tag{1.6}$$

If $\{u(\cdot, t) : t \ge 0\}$ is relatively compact in $C(\bar{\Omega})$, these properties can be expressed in terms of the limit profiles of u, that is, the elements of its ω-limit set,

$$\omega(u) := \{\phi : \phi = \lim u(\cdot, t_n) \text{ for some } t_n \to \infty\},$$

where the convergence is in $C(\bar{\Omega})$ (with the supremum norm). The asymptotic symmetry and monotonicity of u mean that each $\phi \in \omega(u)$ is symmetric (even) in x_1 and monotone nonincreasing in x_1 on Ω_0.

In a theorem of [19], the asymptotic symmetry and monotonicity is established for bounded positive solutions of problem (1.2) with $g_i \equiv 0$, $i = 1, 2$. Two extra conditions on u, in addition to boundedness and positivity, are assumed in that theorem. One is an equicontinuity condition, which gives compactness of the orbit $\{u(\cdot, t) : t \geq 0\}$; it can be removed under minor regularity conditions on Ω, such as the exterior cone condition. The other condition requires that *at least one element of* $\omega(u)$ be strictly positive on Ω. It was also shown in [19] that without this strict positivity condition, the result is not valid in general, even if u itself is strictly positive ([11] contains sufficient conditions, in terms of the domain and the nonlinearity f, under which the strict positivity condition can be omitted).

Let us now consider a bounded positive solution of the asymptotically symmetric problem (1.2), assuming the same conditions on u as in [19]. It might be surprising at the first glance that, even with $g_2 \equiv 0$ and $g_1(\cdot, t)$ decaying to zero exponentially, one cannot prove the symmetry result in the same form as for $g_1 \equiv 0$. We show in Example 2.3 below that the asymptotic monotonicity fails in general. We do not know whether the asymptotic symmetry can be established without the asymptotic monotonicity. This might be an interesting problem to tackle (see [22] for a discussion of related issues in the context of elliptic equations).

As Example 2.3 demonstrates, stronger assumptions are needed for the symmetry result to hold for (1.2). In this paper, we prove that (1.5), (1.6) hold if the strict positivity assumption is strengthened so as to require *all elements of* $\omega(u)$ to be positive in Ω. We prove this statements in the setting of fully nonlinear equations, see Theorem 2.2 in the next section. The strict positivity condition can be equivalently stated as

$$\liminf_{t \to \infty} u(x, t) > 0 \quad (x \in \Omega). \tag{1.7}$$

This condition requires u to be asymptotically strictly positive; whether $u(\cdot, t)$ is positive or not at finite times is irrelevant. In a remark following Example 2.3 in Section 2, we mention an alternative condition, a sufficiently fast decay of the g_i, under which the symmetry theorem can also be proved.

With our strict positivity assumption, two different approaches to the symmetry problem are possible. One is based on symmetry results for entire solutions (that is, solutions defined for all $t \in \mathbb{R}$) of symmetric equations. Similarly as in [4], the idea is to view $\omega(u)$ as a set of entire solutions, positive by assumption, of a suitable "limit equation." Since the perturbation terms g_i disappear at $t = \infty$, the limit equation is symmetric. Thus known symmetry results for positive entire solutions [2, 4] can be used to establish the symmetry of the elements of $\omega(u)$. This approach requires stronger regularity assumptions on the solution u and the nonlinearity in the equation. We use a different approach, similar to that in [19]. Since it is based on direct Harnack-type estimates of the solution and does not depend on any limit equation, no extra regularity assumptions are needed. Such an approach was also used in [12], where asymptotically symmetric quasilinear equations on $\mathbb{R}^N$ were considered.

We have organized the exposition as follows. Our main symmetry result is stated in the next section and proved in Section 4. Estimates of solutions of linear nonhomogeneous equations that facilitate the proof are collected in Section 3. Section 5 contains the computations needed for Example 2.3.

To conclude the introduction, we add another point to the discussion of the robustness of the symmetry properties. In (1.2), the domain Ω is assumed to be symmetric. One can make the problem more general by allowing Ω to vary in time in such a way that it approaches, in a suitable sense, a symmetric domain, as $t \to \infty$. While we do not explicitly consider this generalization, our results cover it to some extent. Indeed, if the variable domain is sufficiently smooth, then using a time dependent change of coordinates one can transform it to a fixed symmetric domain. This changes the equation in an asymptotically symmetric way, although the transformed equation is no longer of the form (1.2). Nonetheless, our results on fully nonlinear asymptotically symmetric equations, as given in the next section, can be applied to the transformed problem.

2 Main results

Our main results concern parabolic problems of the form

$$\left.\begin{aligned} \partial_t u &= F(t,x,u,Du,D^2u) + G_1(x,t), && (x,t) \in \Omega \times (0,\infty), \\ u &= G_2(x,t), && (x,t) \in \partial\Omega \times (0,\infty). \end{aligned}\right\} \quad (2.1)$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ satisfying condition (D) from the introduction. The real valued function F is defined on $[0,\infty) \times \bar{\Omega} \times \mathcal{O}$, where $\mathcal{O}$

is an open convex subset of $\mathbb{R}^{1+N+N^2}$, invariant under the transformation

$$Q : (u, p, q) \mapsto (u, -p_1, p_2, \cdots, p_N, \tilde{q}),$$
$$\tilde{q}_{ij} = \begin{cases} -q_{ij} & \text{if exactly one of } i, j \text{ equals } 1, \\ q_{ij} & \text{otherwise}. \end{cases}$$

The assumptions on F are as follows.

(N1) *Regularity.* The function F is continuous, differentiable with respect to q, and Lipschitz continuous in (u, p, q) uniformly with respect to $(x, t) \in \bar{\Omega} \times \mathbb{R}^+$. This means that there is $\beta > 0$ such that

$$\sup_{x\in\bar{\Omega}, t\geq 0} |F(t, x, u, p, q) - F(t, x, \tilde{u}, \tilde{p}, \tilde{q})| \leq \beta|(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q})|$$
$$((u, p, q), (\tilde{u}, \tilde{p}, \tilde{q}) \in \mathcal{O}). \quad (2.2)$$

(N2) *Ellipticity.* There is a positive constant α_0 such that for each $\xi \in \mathbb{R}^N$ and $(t, x, u, p, q) \in [0, \infty) \times \bar{\Omega} \times \mathcal{O}$ one has

$$\frac{\partial F}{\partial q_{ij}}(t, x, u, p, q)\xi_i\xi_j \geq \alpha_0|\xi|^2.$$

Here and below we use the summation convention (summation over repeated indices); for example, in the above formula the left hand side represents the sum over $i, j = 1, \ldots, N$.

(N3) *Symmetry and monotonicity.* For all $(x_1, x'), (\tilde{x}_1, x') \in \Omega$ with $\tilde{x}_1 > x_1 \geq 0$ and for all $(t, u, p, q) \in [0, \infty) \times \mathcal{O}$ one has

$$F(t, \pm x_1, x', Q(u, p, q)) = F(t, x_1, x', u, p, q) \geq F(t, \tilde{x}_1, x', u, p, q).$$

The functions G_1 and G_2 are defined on $\Omega \times [0, \infty)$ and $\partial\Omega \times [0, \infty)$, respectively, and are assumed to satisfy the following conditions.

(G) $G_1 \in L^{N+1}(\Omega \times (0, T))$ for each $T \in (0, \infty)$, $G_2 \in C(\partial\Omega \times (0, \infty))$, and

$$\lim_{t\to\infty} \max\left\{\|G_1\|_{L^{N+1}(\Omega\times(t,t+1))}, \|G_2(\cdot, t)\|_{L^\infty(\partial\Omega)}\right\} = 0. \quad (2.3)$$

Remark 2.1. Some remarks on our hypotheses are in order.

(i) Although we are not assuming any regularity of the perturbation term G_1 (other than that contained in (G)), the reader may notice that when working with classical solutions, as we do in this paper, G_1 must be continuous for the first equation in (2.1) to be satisfied. However, the continuity is never used in our proofs. If one wishes to consider more specific semilinear or quasilinear equations with a weaker notion of solutions (as long as the solutions are in $W^{2,1}_{N+1}(\Omega \times (0,T))$ for each $T \in (0,\infty)$, as needed in our estimates of linearized problems), then it might be reasonable to consider functions G_1 which are not necessarily continuous on $\Omega \times (0,\infty)$.

(ii) As discussed in the introduction in the context of semilinear equations, the form of problem (2.1) is general enough to cover equations with nonlinear nonsymmetric perturbation terms when individual solutions are considered. For example, if $G_1(x,t)$ in the first equation is replaced with $\tilde{G}_1(x,t,u,Du,D^2u)$, where $\tilde{G}_1(x,t,u,p,q)$ is a function defined on $\bar{\Omega} \times [0,\infty) \times \mathbb{R}^{1+N+N^2}$, then, given a solution u of the modified equation, we set

$$G_1(x,t) = \tilde{G}_1(x,t,u,Du,D^2u)$$

to bring the equation to the form (2.1). The results of our paper are then applicable, provided $\tilde{G}_1$ satisfies a suitable decay assumption so that the resulting function G_1 satisfies (G) for any global solution u one wishes to consider.

By a solution of (2.1) we mean a classical solution, that is, a function $u \in C^{2,1}(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$, such that (u, Du, D^2u) takes values in $\mathcal{O}$ and all relations in (2.1) are satisfied everywhere. We shall consider solutions such that

$$\sup_{t \in [0,\infty)} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty \tag{2.4}$$

and the family of functions $u(\cdot, \cdot + s)$, $s \geq 1$, is equicontinuous on $\bar{\Omega} \times [0,1]$, that is,

$$\lim_{h \to 0} \sup_{\substack{x,\bar{x} \in \bar{\Omega}, t, \bar{t} \in [0,1], \\ |t-\bar{t}|, |x-\bar{x}| < h \\ s \geq 1}} |u(x,t+s) - u(\bar{x}, \bar{t}+s)| = 0\,. \tag{2.5}$$

We remark that using [19, Proposition 2.7], one can give sufficient conditions for (2.5) to hold for any bounded solution of (2.1). This is true, for example, if $G_2 \equiv 0$, the function $F(t,x,0,0,0) + G_1(t,x)$ is bounded, and $\partial\Omega$

satisfies the exterior cone condition. The proof given in [19, Proposition 2.7] for $G_1 \equiv 0$ applies here with just a notational change.

For a solution u satisfying (2.4), (2.5), the set $\{u(\cdot,t) : t \geq 1\}$ is relatively compact in the space $C(\bar{\Omega})$. Consequently, the ω-limit set of u in $C(\bar{\Omega})$,

$$\omega(u) := \{\phi : \phi = \lim u(\cdot, t_n) \text{ for some } t_n \to \infty\},$$

is nonempty and compact. Moreover,

$$\operatorname{dist}(u(\cdot,t), \omega(u)) \to 0 \text{ in } C(\bar{\Omega}) \text{ as } t \to \infty. \tag{2.6}$$

We are ready to formulate our main symmetry result.

Theorem 2.2. *Assume (D), (N1) – (N3), (G), and let u be a global solution of* (2.1) *satisfying* (2.4), (2.5), *and* (1.7). *Then for each $z \in \omega(u)$*

$$z(-x_1, x') = z(x_1, x') \quad ((x_1, x') \in \Omega)$$

and z is strictly decreasing in x_1 in $\Omega_0 = \{x \in \Omega : x_1 > 0\}$. The latter holds in the form $z_{x_1} < 0$ provided $z_{x_1} \in C(\Omega_0)$.

Note that without extra conditions, like boundedness of spatial derivatives of u, we cannot in general assume that elements of $\omega(u)$ are differentiable.

By (2.6) and the compactness of $\omega(u)$ in $C(\bar{\Omega})$, condition (1.7) is equivalent to the following condition:

$$\text{for each } z \in \omega(u), \text{ one has } z > 0 \text{ on } \Omega. \tag{2.7}$$

One can give several sufficient conditions for (2.7). For example, assume that the functions $u(\cdot, 0)$, G_2, and $F(\cdot,\cdot,0,0,0)+G_1$ are all nonnegative, and there exist positive constant γ, t_0 and a ball $B \subset \Omega$ such that

$$F(t,x,0,0,0) + G_1(x,t) \geq \gamma \quad (x \in B, t > t_0). \tag{2.8}$$

Let us indicate how (2.7) is derived from these conditions. First one uses the strong comparison principle to show that $u > 0$ (note that $u \equiv 0$ is not a solution by (2.8)). Next one shows that if x_0 is the center of B, then $u(x_0, t)$ stays above a positive constant as $t \to \infty$. This is done by constructing a suitable subsolution: choose a smooth function φ with a compact support contained in B such that $\varphi(x_0) = 1$. By (2.8) and (N1) there are positive constants ϵ_0 and t_0 such that

$$F(t,x,\epsilon\varphi(x),\epsilon D\varphi(x),\epsilon D^2\varphi(x)) + G_1(x,t) \geq 0 \quad ((x,t) \in B \times [t_0,\infty)),$$

whenever $\epsilon \in (0, \epsilon_0)$. If $\epsilon \in (0, \epsilon_0)$ is chosen such that $\epsilon\varphi < u(\cdot, t_0)$ in B, then a comparison argument gives $\epsilon\varphi < u(\cdot, t)$ in B for all $t \geq t_0$. In particular $u(x_0, t) \geq \epsilon$ for all $t \geq t_0$, hence $z(x_0) > 0$ for each $z \in \omega(u)$. The proof of (2.7) is now completed by a Harnack-type estimate for which we refer to [19, Proof of Theorem 2.5].

The following example shows that in general one cannot relax the strict positivity condition to merely require that at least one $z \in \omega(u)$ be positive in Ω, even if the nonsymmetric perturbation terms decay exponentially. This contrasts with the result of [19] which says that if $G_1 \equiv 0$, $G_2 \equiv 0$, then the relaxed positivity condition is sufficient for the asymptotic symmetry and monotonicity result.

Example 2.3. Let $I = (-2\pi, 2\pi)$, $\Omega = I \times I$, and fix $\beta > 1$. There is a continuous function $f : I \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$, piecewise linear in the last variable with Lipschitz constant $\beta + 2$, and a continuous function $R : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$ satisfying

$$\|R(\cdot, t)\|_{L^\infty(\Omega)} \leq Ce^{-\frac{\beta}{25}t} \qquad (t \geq 0)$$

for some $C = C(\beta) > 0$ such that the problem

$$\begin{aligned} &u_t = \Delta u + f(t, y, u) + R(x, y, t), && (x, y, t) \in \Omega \times (0, \infty), \\ &u = 0, && (x, y, t) \in \partial\Omega \times (0, \infty), \\ &u > 0, && (x, y, t) \in \Omega \times (0, \infty), \end{aligned}$$

has a global, bounded solution u with the following properties. There exist $z, w \in \omega(u)$ such that $z > 0$ in Ω, $w > 0$ in $(0, 2\pi) \times I$, and $w(0, y) = 0$ for every $y \in I$. In particular, since w satisfies the Dirichlet boundary condition, it is not monotone in x on $(0, 2\pi) \times I$.

See Section 5 for the detailed construction. Similar examples can be given with $G_1 \equiv 0$, and with $\|G_2(\cdot, t)\|_{L^\infty(\partial\Omega)}$ decaying exponentially.

In Example 2.3, we emphasize the relation between the exponential decay of the function R and the Lipschitz constants of f. In fact, R cannot have an arbitrarily fast exponential decay rate. In general, it can be proved, that if the nonsymmetric perturbation functions G_1, G_2 decay to zero with sufficiently fast exponential rate, relative to the Lipschitz constant of the nonlinearity F, then Theorem 2.2 holds if the strict positivity assumption (1.7) is relaxed to the weaker assumption requiring the existence of just one positive function in $\omega(u)$. This result was mentioned in the survey [20], with reference to the present paper. However, since this statement requires a substantially different and rather involved proof, we decided not to include it in this paper.

Our final remark in this section concerns problem (2.1), where Ω is a ball centered at the origin and F satisfies the radial symmetry assumptions as in [19]. The assumptions essentially say that condition (N3) holds after an arbitrary rotation of the coordinate system. Then, assuming also the other hypotheses of Theorem 2.2, one obtains that all elements of $\omega(u)$ are radially symmetric, that is, they are functions of $|x|$ only. Since this result is deduced in a standard way from the reflectional symmetry in any direction, we omit the details.

3 Estimates for linear equations

In this section, we state several estimates of solutions of linear parabolic equations to which we will refer when using the method of moving hyperplanes.

For an open set $D \subset \mathbb{R}^N$ and for $t < T$, we denote by $\partial_P(D \times (t,T))$ the parabolic boundary of $D \times (t,T)$: $\partial_P(D \times (t,T)) := (D \times \{t\}) \cup (\partial D \times [t,T])$. For bounded sets U, U_1 in $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$, the notation $U_1 \subset\subset U$ means $\bar{U}_1 \subset U$, $\operatorname{diam} U$ stands for the diameter of U, and $|U|$ for its Lebesgue measure (if it is measurable). The open ball in $\mathbb{R}^N$ centered at x with radius r is denoted by $B(x,r)$. Symbols f^+ and f^- denote the positive and negative parts of a function f: $f^\pm := (|f| \pm f)/2 \geq 0$.

We consider time dependent elliptic operators L of the form

$$L(x,t) = a_{km}(x,t)\frac{\partial^2}{\partial x_k \partial x_m} + b_k(x,t)\frac{\partial}{\partial x_k} + c(x,t)\,. \tag{3.1}$$

Definition 3.1. Given an open set $U \subset \mathbb{R}^N$, an interval I, and positive numbers α_0, β, we say that an operator L of the form (3.1) belongs to $E(\alpha_0, \beta, U, I)$ if its coefficients a_{km}, b_k, c are measurable functions defined on $U \times I$ and they satisfy

$$\begin{aligned}
&|a_{km}|, |b_k|, |c| \leq \beta \quad (k, m = 1, \ldots, N),\\
&a_{km}(x,t)\xi_k\xi_m \geq \alpha_0|\xi|^2 \quad ((x,t) \in U \times I,\ \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N).
\end{aligned}$$

Let us briefly recall how linear equations arise when the method of moving hyperplanes is applied to (2.1). For more details and explicit expressions using Hadamard's formulas see [19]. For any $\lambda \in \mathbb{R}$, we set

$$\begin{aligned} H_\lambda &:= \{x \in \mathbb{R}^N : x_1 = \lambda\}, \\ \Omega_\lambda &:= \{x \in \Omega : x_1 > \lambda\}, \\ \ell &:= \sup\{x_1 \in \mathbb{R} : (x_1, x') \in \Omega \text{ for some } x' \in \mathbb{R}^{N-1}\}. \end{aligned} \tag{3.2}$$

Further, let P_λ stand for the reflection in the hyperplane H_λ and for $x \in \bar{\Omega}$ let $x^\lambda := P_\lambda x$.

Assume that Ω satisfies hypothesis (D), the nonlinearity F satisfies (N1) – (N3) and the functions G_i, $i = 1, 2$ satisfy (G). Condition (D) in particular implies that $P_\lambda(\Omega_\lambda) \subset \Omega$ for each $\lambda \in [0, \ell)$. Let u be a global solution of (2.1) satisfying (2.4), (2.5), and (1.7). By (N3),

$$F(t, x^\lambda, Q(u, p, q)) \geq F(t, x, u, p, q)$$

for any $(t, u, p, q) \in [0, \infty) \times \mathcal{O}$, $\lambda > 0$, and any $(x_1, x') \in \Omega_\lambda$. If $u^\lambda(x, t) := u(x^\lambda, t)$, we obtain

$$\partial_t u^\lambda \geq F(t, x, u^\lambda, Du^\lambda, D^2 u^\lambda) + G_1(x^\lambda, t), \quad (x, t) \in \Omega_\lambda \times (0, \infty)\,.$$

Hence, the function $w^\lambda : \bar{\Omega}_\lambda \times (0, \infty) \to \mathbb{R}$, $w^\lambda : (x, t) \mapsto u^\lambda(x, t) - u(x, t)$, $\lambda \in [0, \ell)$ satisfies

$$\begin{aligned} &\partial_t w^\lambda(x, t) \geq F(x, t, u(x^\lambda, t), Du(x^\lambda, t), D^2 u(x^\lambda, t)) \\ &-F(x, t, u(x, t), Du(x, t), D^2 u(x, t)) + G_1(x^\lambda, t) - G_1(x, t) \\ &= L^\lambda(x, t) w^\lambda + f^\lambda(x, t), \qquad (x, t) \in \Omega_\lambda \times (0, \infty), \end{aligned} \tag{3.3}$$

where $L^\lambda \in E(\alpha_0, \beta, \Omega_\lambda, (0, \infty))$ and f^λ is a measurable function satisfying

$$\lim_{t \to \infty} \|(f^\lambda)^-\|_{L^{N+1}(\Omega_\lambda \times (t, t+1))} = 0. \tag{3.4}$$

Also, w^λ satisfies the following boundary conditions

$$w^\lambda(x, t) \geq g^\lambda(x, t) := \begin{cases} u(x^\lambda, t) - G_2(x, t), & (x, t) \in (\partial\Omega_\lambda \setminus H_\lambda) \times (0, \infty), \\ 0, & (x, t) \in (\partial\Omega_\lambda \cap H_\lambda) \times (0, \infty)\,. \end{cases} \tag{3.5}$$

Note that (1.7) and the compactness of $\{u(\cdot,t) : t \geq 1\}$ in $C(\bar{\Omega})$ imply

$$\lim_{t\to\infty} \|u^-(\cdot,t)\|_{L^\infty(\partial\Omega_\lambda)} = 0. \tag{3.6}$$

This and (G) give

$$\lim_{t\to\infty} \|(g^\lambda)^-(\cdot,t)\|_{L^\infty(\partial\Omega_\lambda)} = 0\,. \tag{3.7}$$

In the rest of this section we consider a general class of linear problems including (3.3), (3.5). The symmetry of Ω plays no role in this consideration, thus one can assume that Ω is any fixed bounded domain in $\mathbb{R}^N$.

We fix positive constants β, α_0 and consider the problem

$$v_t \geq L(x,t)v + f(x,t)\,, \quad (x,t) \in U \times (\tau,T), \tag{3.8}$$

$$v \geq g(x,t)\,, \qquad (x,t) \in \partial U \times (\tau,T), \tag{3.9}$$

where $0 \leq \tau < T \leq \infty$, $U \subset \Omega$ is an open set, $L \in E(\alpha_0,\beta,U,(\tau,T))$, and f, g are bounded measurable functions.

We say that v is a solution of (3.8) (or that it satisfies (3.8)) if it is an element of the space $W^{2,1}_{N+1,loc}(U \times (\tau,T))$ and (3.8) is satisfied almost everywhere. By a solution of (3.8), (3.9), we mean a solution of (3.8) which is continuous on $\bar{U} \times (\tau,T)$ and satisfies (3.9) everywhere.

We now give estimates of solutions (3.8), (3.9) to be used in the next section. We start with a version of the maximum principle for small domains. The first such maximum principles were proved for elliptic equations [6, 7] (see also [8]). The following result is an extension to nonhomogeneous linear equations of Lemma 3.5 from [19] and it can be proved along similar lines. However, a more general result, [11, Lemma 3.5], is now available and we refer the reader to that paper for the proof.

Lemma 3.2. *Given any $k > 0$, there exists $\delta = \delta(\alpha_0,\beta,N,diam\,\Omega,k)$ such that for any open set $U \subset \Omega$ with $|U| < \delta$ and any $0 \leq \tau < T \leq \infty$ the following holds. If $v \in C(\bar{U} \times [\tau,T))$ is a solution of* (3.8), (3.9), *with $L \in E(\alpha_0,\beta,U,(\tau,T))$, then*

$$\begin{aligned} \|v^-(\cdot,t)\|_{L^\infty(U)} \leq 2\max\{&e^{-k(t-\tau)}\|v^-(\cdot,\tau)\|_{L^\infty(U)}, \|g^-\|_{L^\infty(\partial U\times(\tau,t))}\} \\ &+ C\|f^-\|_{L^{N+1}(U\times(\tau,t))} \qquad (t \in (\tau,T)), \end{aligned} \tag{3.10}$$

where C depends only on $N,\beta,\alpha_0, diam\,(\Omega)$.

Corollary 3.3. *There exists $\delta = \delta(\alpha_0, \beta, N, diam\,\Omega)$ such that for any open set $U \subset \Omega$ with $|U| < \delta$ and any $0 \le \tau_0 < \infty$ the following holds. If $v \in C(\bar{U} \times [\tau_0, \infty)) \cap L^\infty(\bar{U} \times [\tau_0, \infty))$ satisfies (3.8), (3.9) with $L \in E(\alpha_0, \beta, U, (\tau_0, \infty))$ and*

$$\lim_{t\to\infty} \|g^-(\cdot, t)\|_{L^\infty(U)} = \lim_{t\to\infty} \|f^-\|_{L^{N+1}(U\times(t,t+1))} = 0, \tag{3.11}$$

then

$$\lim_{t\to\infty} \|v^-(\cdot, t)\|_{L^\infty(U)} = 0\,.$$

Proof. Choose $\delta > 0$ such that the conclusion of Lemma 3.2 holds for $k = \ln 4$. If $t > \tau_0 + 1$, applying estimate (3.10) with $\tau = t - 1$, we obtain

$$\begin{aligned} \|v^-(\cdot, t)\|_{L^\infty(U)} \le 2 \max\{\frac{1}{4}\|v^-(\cdot, t-1)\|_{L^\infty(U)}, \|g^-\|_{L^\infty(\partial U\times(t-1,t))}\} \\ + C\|f^-\|_{L^{N+1}(U\times(t-1,t))} \qquad (t \in (\tau_0 + 1, \infty))\,, \end{aligned} \tag{3.12}$$

where C is independent of t. Take a sequence $t_n \to \infty$ such that

$$\lim \|v^-(\cdot, t_n)\|_{L^\infty(U)} = \sigma := \limsup_{t\to\infty} \|v^-(\cdot, t)\|_{L^\infty(U)}.$$

Then (3.12) and (3.11) give $\sigma \le \sigma/2$, hence $\sigma = 0$. □

If Q is an open bounded subset of $\mathbb{R}^{N+1}$, $u : Q \to \mathbb{R}$ is a bounded, continuous function, and $p > 0$, we set

$$[u]_{p,Q} := \left(\frac{1}{|Q|}\int_Q |u|^p\, dx\, dt\right)^{\frac{1}{p}}\,.$$

The following lemma is proved in [19, Lemma 3.5].

Lemma 3.4. *Given $d > 0$, $\varepsilon > 0$, $\theta > 0$ there are positive constants κ, κ_1, p determined only by N, $diam\,\Omega, \alpha_0, \beta, d, \varepsilon$, and θ with the following properties. Let D and U be domains in Ω with $D \subset\subset U$, $dist(\bar{D}, \partial U) \ge d$, $|D| > \varepsilon$, and let $L \in E(\alpha_0, \beta, U, (\tau, \tau + 4\theta))$, $f \in L^{N+1}(U \times (\tau, \tau + 4\theta))$. If $v \in C(\bar{U} \times [\tau, \tau + 4\theta])$ satisfies (3.8) with $T = \tau + 4\theta$, then*

$$\begin{aligned} \inf_{D\times(\tau+3\theta,\tau+4\theta)} v(x, t) \ge \kappa[v^+]_{p,D\times(\tau+\theta,\tau+2\theta)} - e^{4\beta\theta} \sup_{\partial_P(U\times(\tau,\tau+4\theta))} v^- \\ - \kappa_1\|f^-\|_{L^{N+1}(U\times(\tau,\tau+4\theta))}\,. \end{aligned} \tag{3.13}$$

4 Proofs of the symmetry results

Throughout this section, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain satisfying (D), F satisfies (N1) – (N3), and G_1, G_2 satisfy (G). Also we assume that u is a solution of (2.1) satisfying (2.4), (2.5), and (1.7).

We use the notation from Section 3 (see (3.2)). For any function $g : \Omega \to \mathbb{R}$, and any $\lambda \in [0, \ell)$ we denote

$$V_\lambda g(x) := g(x^\lambda) - g(x) \quad (x \in \Omega_\lambda).$$

Further, for the solution u, we let

$$w^\lambda(x,t) := V_\lambda u(x,t) = u(x^\lambda, t) - u(x,t) \quad (x \in \Omega_\lambda,\ t > 0)\,.$$

As shown in Section 3, the function w^λ solves a linear problem (3.3), (3.5), with $L \in E(\alpha_0, \beta, \Omega_\lambda, (0,\infty))$, and with measurable functions f^λ, g^λ satisfying (3.4), (3.7), respectively. Hence the estimates from Section 3 are applicable to w^λ. We use this observation below, usually without notice.

We carry out the process of moving hyperplanes in the following way. Starting from $\lambda = \ell$, we move λ to the left as long as the following property is satisfied

$$\lim_{t\to\infty} \|(w^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = 0. \tag{4.1}$$

In Lemma 4.2 below we show that (4.1) holds for all $\lambda < \ell$ close to ℓ. Defining

$$\lambda_0 := \inf\{\mu > 0 : \lim_{t\to\infty} \|(w^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = 0 \text{ for each } \lambda \in [\mu, \ell)\}, \tag{4.2}$$

our goal will be to prove that $\lambda_0 = 0$.

Remark 4.1. Note that, by compactness of $\{u(\cdot,t) : t \geq 0\}$ in $C(\bar\Omega)$, (4.1) is equivalent to $V_\lambda z \geq 0$ in Ω_λ for each $z \in \omega(u)$. Thus, by continuity,

$$V_\lambda z(x) \geq 0 \quad (x \in \Omega_\lambda,\ z \in \omega(u),\ \lambda \in [\lambda_0, \ell)). \tag{4.3}$$

This implies that $z \in \omega(u)$ is nonincreasing in x_1 in Ω_{λ_0}. Indeed, if (x_1, x'), $(\tilde x_1, x')$ are points in Ω_{λ_0} with $x_1 > \tilde x_1$, then $V_\lambda z \geq 0$ with $\lambda = (x_1 + \tilde x_1)/2 > \lambda_0$ gives $z(x_1, x') \geq z(\tilde x_1, x')$.

Lemma 4.2. *If $\delta = \delta(\alpha_0, \beta, N, diam\,\Omega) > 0$ is as in Corollary 3.3, then* (4.1) *holds whenever $|\Omega_\lambda| < \delta$. Consequently, $\lambda_0 < \ell$.*

Proof. Corollary 3.3 applied to $v = w^\lambda$ implies (4.1), whenever $|\Omega_\lambda| < \delta$. Since this is true for all $\lambda < \ell$ sufficiently close to ℓ, we have $\lambda_0 < \ell$. $\square$

Lemma 4.3. *For any $\lambda > 0$ with $\lambda_0 \le \lambda < \ell$ and any $z \in \omega(u)$, we have $V_\lambda z > 0$ in Ω_λ.*

Proof. Fix arbitrary $\lambda > 0$ with $\lambda_0 \le \lambda < \ell$ and $z \in \omega(u)$. Let U_λ be any connected component of Ω_λ. We have $z > 0$ in Ω by (2.7) and $z = 0$ on $\partial\Omega$. Consequently, $V_\lambda z \not\equiv 0$ in U_λ. By Remark 4.1, $V_\lambda z$ is a nonnegative continuous function, thus there exist an open ball $B_0 \subset\subset U_\lambda$ and $d_0 > 0$ such that

$$V_\lambda z(x) > 4d_0 \qquad (x \in B_0)\,.$$

Choose an increasing sequence $(t_k)_{k\in\mathbb{N}}$ converging to ∞ such that $u(\cdot, t_k) \to z$ in $C(\bar{\Omega})$. Then $w^\lambda(\cdot, t_k) \to V_\lambda z$, and therefore $w^\lambda(\cdot, t_k) > 2d_0$ in $\bar{B}_0$ for all $k > k_0$, if k_0 is large enough. By the equicontinuity property (2.5), there is $\vartheta > 0$ independent of k, such that

$$w^\lambda(x, t) > d_0 \qquad ((x, t) \in \bar{B}_0 \times [t_k - 4\vartheta, t_k], k > k_0)\,.$$

Since $\lambda_0 \le \lambda$, one has $\|(w^\lambda)^-(\cdot, t)\|_{L^\infty(U_\lambda)} \to 0$ as $t \to \infty$. Fix an arbitrary domain $D \subset\subset U_\lambda$ with $B_0 \subset\subset D$. An application of Lemma 3.4 with $(v, \tau, \theta, f) = (w^\lambda, t_k, \vartheta, f^\lambda)$ yields

$$\begin{aligned} w^\lambda(x,t) \ge \kappa[(w^\lambda)^+]_{p, D\times(t_k-3\vartheta, t_k-2\vartheta)} - \sup_{U_\lambda\times(t_k-4\vartheta,t_k)} e^{4\beta\vartheta}(w^\lambda)^- \\ - \kappa_1 \|(f^\lambda)^-\|_{L^{N+1}(U_\lambda\times(t_k-4\vartheta,t_k))} \qquad ((x,t) \in D \times [t_k - \vartheta, t_k]), \end{aligned}$$

where κ, κ_1, and p do not depend on k. Since the last two terms converge to zero as $k \to \infty$ and the first term stays bounded from below by a positive constant (independent of k), there are $d_1 > 0$ and $k_1 \ge k_0$ (depending on D) such that

$$w^\lambda(x, t) \ge d_1 \qquad ((x, t) \in \bar{D} \times [t_k - \vartheta, t_k], k \ge k_1)\,.$$

Choose $t = t_k$ and let $k \to \infty$ to obtain

$$V_\lambda z(x) > 0 \qquad (x \in D).$$

Since D was an arbitrary domain with $B_0 \subset\subset D \subset\subset U_\lambda$, $V_\lambda z > 0$ in U_λ. $\square$

In our last preliminary lemma, we establish a strict monotonicity property of the functions in $\omega(u)$. Note that the requirement that Ω_{λ_0} be connected will be verified by condition (D), once we prove that $\lambda_0 = 0$.

Lemma 4.4. *If Ω_{λ_0} is connected, then each $z \in \omega(u)$ is strictly decreasing in x_1 in Ω_{λ_0}. If $z_{x_1} \in C(\Omega_{\lambda_0})$, then $z_{x_1} < 0$.*

Proof. Fix any $z \in \omega(u)$. For $h > 0$ let $\Omega^h_{\lambda_0} := \Omega_{\lambda_0} \cap \{x \in \Omega : x + he_1 \in \Omega\}$ and

$$d_h z(x) := \frac{z(x + he_1) - z(x)}{h} \qquad (x \in \Omega^h_{\lambda_0}).$$

By Remark 4.1, $d_h z \leq 0$ in $\Omega^h_{\lambda_0}$ for all $z \in \omega(u)$.

We claim that if $h > 0$ and U is a connected component of $\Omega^h_{\lambda_0}$, then either $d_h z \equiv 0$ in U or $d_h z < 0$ in U.

The proof of this statement is similar to the proof of Lemma 4.3. Assume $d_h z \not\equiv 0$ in U. Then there is a ball $B \subset\subset \Omega^h_{\lambda_0}$ and $\rho_0 > 0$ such that

$$d_h z(x) < -4\rho_0 \qquad (x \in B). \tag{4.4}$$

Set

$$d_h u(x,t) := \frac{u(x + he_1, t) - u(x,t)}{h} \qquad ((x,t) \in \Omega^h_{\lambda_0} \times (0,\infty)).$$

Similarly as with w^λ, hypotheses (N1)-(N3) and Hadamard's formulas (see [19]) imply that

$$(d_h u)_t \leq L(x,t)(d_h u) + f^h(x,t), \quad (x,t) \in \Omega^h_{\lambda_0} \times (0,\infty), \tag{4.5}$$

where $L \in E(\alpha_0, \beta, \Omega^h_{\lambda_0}, (0,\infty))$ and f^h is a measurable function with

$$\lim_{t\to\infty} \|f^h\|_{L^{N+1}(\Omega^h_{\lambda_0} \times (t,t+1))} = 0.$$

Let $(t_k)_{k\in\mathbb{N}}$ be an increasing sequence converging to ∞ such that $u(\cdot,t_k) \to z$ in $C(\bar\Omega)$. Then $d_h u(\cdot,t_k) \to d_h z$ in $C(\bar\Omega^h_{\lambda_0})$, and therefore there is k_0 such that $d_h u(\cdot,t_k) < -2\rho_0$ in B for all $k > k_0$. The equicontinuity assumption (2.5) yields $\vartheta > 0$, independent of k, such that $d_h u(x,t) < -\rho_0$ for all $x \in B$, $t \in [t_k - 4\vartheta, t_k]$, and $k > k_0$.

Since $d_h \tilde z \leq 0$ in $\Omega^h_{\lambda_0}$ for all $\tilde z \in \omega(u)$,

$$\lim_{t\to\infty} \|(d_h u)^+(\cdot,t)\|_{L^\infty(\Omega^h_{\lambda_0})} = 0. \tag{4.6}$$

Let $D \subset\subset U$ be any domain with $B \subset\subset D$. Applying Lemma 3.4 in a similar way as in the proof of Lemma 4.3, we obtain

$$d_h u(x, t_k) \le -\rho_1 < 0 \qquad (x \in \bar{D}, k \ge k_1), \tag{4.7}$$

where $\rho_1 = \rho_1(D) > 0$ is independent of k and $k_1 = k_1(D)$ is a sufficiently large integer. Passing to the limit, as $k \to \infty$, in (4.7), we obtain

$$d_h z(x) \le -\rho_1 \qquad (x \in \bar{D}). \tag{4.8}$$

In particular, $d_h z < 0$ in $\bar{D}$ and since the domain D with $B \subset\subset D \subset\subset U$ was arbitrary, the claim is proved.

Now, if $d_{h_0} z(\tilde{x}) = 0$ for some $h_0 > 0$ and $\tilde{x} \in \Omega^{h_0}_{\lambda_0}$, then from the monotonicity of z it follows that $d_h z(\tilde{x}) = 0$ for all $0 < h \le h_0$. Then the claim implies that for each $h \in (0, h_0)$ one has $d_h z \equiv 0$ in the connected component of $\Omega^h_{\lambda_0}$ containing $\tilde{x}$. Since Ω_{λ_0} is connected, this clearly implies that z is constant in x_1 in Ω_{λ_0}, hence, by the boundary condition, $z \equiv 0$ in Ω_{λ_0}. This would contradict (2.7), hence no such $h_0 > 0$ and $\tilde{x}$ can exist. Therefore $d_h z < 0$ in $\Omega^h_{\lambda_0}$ for all $h > 0$, and consequently z is strictly decreasing in x_1.

To prove the last conclusion assume that $z_{x_1} \in C(\Omega_{\lambda_0})$. Since $z_{x_1} \not\equiv 0$ in Ω_{λ_0}, we can choose a ball $B \subset\subset \Omega_{\lambda_0}$ and $\rho_0 > 0$, both independent of h, such that (4.4) holds for all sufficiently small $h > 0$. The connectedness of Ω_{λ_0} implies that given any $x \in \Omega_{\lambda_0}$, if $h > 0$ is small enough, then x and $\bar{B}$ lie in the same connected component of $\Omega^h_{\lambda_0}$. Hence there is a domain D containing both x and $\bar{B}$ such that $D \subset\subset \Omega^h_{\lambda_0}$ for each sufficiently small $h > 0$. Estimate (4.8) then holds with ρ_1 independent of h, which gives in particular $z_{x_1}(x) < 0$. □

Proof of Theorem 2.2. Let $\lambda_0 \ge 0$ be as in (4.2). We show that $\lambda_0 = 0$. Assume $\lambda_0 > 0$. For each $z \in \omega(u)$ we have $z > 0$ by assumption and this implies $V_{\lambda_0} z \not\equiv 0$ on $\partial\Omega_{\lambda_0} \setminus H_{\lambda_0}$. Thus by Lemma 4.3, $V_{\lambda_0} z > 0$ in Ω_{λ_0}. Choose δ as in Corollary 3.3 and fix a compact set $K \subset \Omega_{\lambda_0}$ with $|\Omega_{\lambda_0} \setminus K| \le \delta/2$. By the compactness of $\omega(u) \subset C(\bar{\Omega})$, there is $d_0 > 0$ such that

$$V_{\lambda_0} z(x) > 4d_0 \qquad (x \in K, z \in \omega(u)).$$

This implies that there exists $t_0 > 0$ such that

$$w^{\lambda_0}(x, t) > 2d_0 \qquad ((x, t) \in K \times (t_0, \infty)).$$

The equicontinuity assumption (2.5) implies that if λ is sufficiently close to λ_0, then

$$w^\lambda(x,t) > d_0 \qquad ((x,t) \in K \times (t_0,\infty)) . \tag{4.9}$$

Let $\lambda \in [0,\lambda_0]$ be close enough to λ_0 so that (4.9) holds together with $|\Omega_\lambda \setminus \Omega_{\lambda_0}| < \delta/2$. Then $|\Omega_\lambda \setminus K| < \delta$ and an application of Corollary 3.3 with $(U,\tau,v,f) = (\Omega_\lambda \setminus K, t_0, w^\lambda, f^\lambda)$ gives

$$\lim_{t\to\infty} \|(w^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = 0,$$

a contradiction to the definition of λ_0. This contradiction shows that $\lambda_0 = 0$. Hence $V_0 z \geq 0$ for all $z \in \omega(u)$.

Now, the problem (2.1) and the assumptions of the theorem are invariant under the transformation $x_1 \to -x_1$. Therefore, applying the above conclusion to the function $u(-x_1,x',t)$, we obtain $V_0 z \leq 0$ in Ω_0 for all $z \in \omega(u)$. Hence, $V_0 z \equiv 0$ in Ω_0, or equivalently

$$z(-x_1,x') = z(x_1,x') \quad ((x_1,x') \in \Omega, z \in \omega(u)) .$$

The remaining statements of the theorem follow from Lemma 4.4. □

5 Details for Example 2.3

Recall the notation $I = (-2\pi, 2\pi)$, $\Omega = I \times I$. Let β be an arbitrary fixed number in $(1,\infty)$.

The key part of our construction deals with the following one-dimensional problem

$$\begin{aligned} &u_t = u_{xx} + f(t,u) + R(x,t), && (x,t) \in I \times (0,\infty) , \\ &u = 0, && (x,t) \in \partial I \times (0,\infty) , \\ &u \geq 0, && (x,t) \in I \times (0,\infty) . \end{aligned} \tag{5.1}$$

Our goal is to find a continuous function $f : I \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$, which is piecewise linear in the second variable with Lipschitz constant $\beta + 1$, and a continuous function $R : \bar{I} \times [0,\infty) \to \mathbb{R}$ satisfying

$$\|R(\cdot,t)\|_{L^\infty(\Omega)} \leq Ce^{-\frac{\beta}{25}t} \qquad (t \geq 0)$$

for some $C = C(\beta) > 0$ such that (5.1) has a global, bounded solution u with the following properties. There exist $g, h \in \omega(u)$ (ω-limit set in $C[-2\pi, 2\pi]$)

such that $h > 0$ in I, $g > 0$ in $(0, 2\pi)$, and $g(0) = 0$. Once such functions have been found, the construction for Example 2.3 is completed in two simple steps as follows. First we modify the functions u and R to achieve $u > 0$ (note that in (5.1) we only require $u \geq 0$). This is done by adding to u a smooth function $v : \bar{I} \times [0, \infty)$ such that $v > 0$ in $I \times [0, \infty)$, $v(-2\pi, t) = v(2\pi, t) = 0$ for all $t \in [0, \infty)$, and $\|H(\cdot, t)\|_{L^\infty(I)} \leq e^{-\beta t}$, where H is any of the functions v, v_x, v_{xx}, or v_t. Then $\tilde{u} := u + v$ is a positive solution of (5.1), if $R(x, t)$ is replaced with the function

$$\begin{aligned}\tilde{R}(x,t) &:= \tilde{u}_t(x,t) - \tilde{u}_{xx}(x,t) - f(t, \tilde{u}(x,t)) \\ &= [f(t, u(x,t)) - f(t, \tilde{u}(x,t))] + v_t(x,t) - v_{xx}(x,t) + R(x,t).\end{aligned}$$

Since f is Lipschitz in u, the continuous function $\tilde{R}$ has the same exponential decay as R: $\|\tilde{R}(\cdot, t)\|_{L^\infty(I)} \leq Ce^{-t\beta/25}$, possibly with a larger constant C. Of course, u and $\tilde{u}$ have the same ω-limit sets, so we have a positive solution of the one-dimensional problem as desired.

The second step is to use this example and separation of variables to obtain a solution of the problem on Ω. This is done in much the same way as in [19, Example 2.3]. One takes $U(x, y, t) = \tilde{u}(x, t)\psi(y)$, with $\psi(y) = \cos(y/4)$. Then U is a positive solution of the problem

$$U_t = \Delta U + f^*(t, y, U) + R^*(x, y, t), \quad (x, y) \in \Omega, t > 0, \tag{5.2}$$

$$U = 0, \quad (x, y) \in \partial\Omega, t > 0, \tag{5.3}$$

where

$$f^*(t, y, U) = \begin{cases} f\left(t, \dfrac{U}{\psi(y)}\right)\psi(y) + \dfrac{1}{16}U & \text{if } |y| \neq 2\pi, \\ \dfrac{1}{16}U \quad \text{if } |y| = 2\pi \end{cases}$$

and $R^*(x, y, t) = \tilde{R}(x, t)\psi(y)$. Clearly f^* is continuous in all variables and it is Lipschitz continuous in U with Lipschitz constant $\operatorname{Lip}_U f^* = \operatorname{Lip}_u f + 1/16 \leq \beta + 2$. The functions $z := h\psi, w := g\psi$ are contained in $\omega(U)$ and have the properties as stated in Example 2.3.

Let us now return to the one-dimensional problem (5.1). Set

$$h(x) = \cos\left(\frac{x}{4}\right), \qquad g(x) = \frac{1 - \cos(x)}{2} \qquad (x \in I).$$

These are the functions we want to be contained in the ω-limit set of a solution u. Our strategy is to first find a nonlinearity $f(t, u)$ such that the equation $u_t = u_{xx} + f(t,u)$ has four distinctive regimes occurring, in succession, on disjoint time intervals: in the first regime, there is a solution which decreases from its initial condition h to a small multiple of h; in the second one, a solution increases from a small multiple of g to g; in the third one, a solution decreases from its initial condition g to another small multiple of g; and, finally, in the forth regime, a solution increases from a small multiple of h back to h. Using a suitable perturbation function $R(x,t)$, we then connect the solutions in these four regimes. Repeating the cycle infinitely many times, we produce a solution u of (5.1) such that $h, g \in \omega(u)$. Care is needed in the construction to guarantee that the function R has the indicated exponential decay.

We now give the details. To simplify the notation, for any function ζ of the variables $x \in I$, $u \in \mathbb{R}$, and $t \geq 0$, $\mathcal{S}_T\zeta$ stands for the time shift of ζ:

$$\mathcal{S}_T[\zeta(x,u,t)] := \zeta(x,u,t-T) \qquad (T < t)\,.$$

Let $s : [0,1] \to [0,1]$ and $m : [0,1] \to \mathbb{R}$ be smooth functions such that

$$\begin{gathered} s(0) = 1, \quad s(1) = 0, \quad s'(0) = s'(1) = -\frac{\beta}{2}, \quad |s'| < \beta, \\ m(0) = m(1) = 1, \quad m'(0) = -m'(1) = \beta, \quad m \geq 1, \quad |m'| \leq \beta \end{gathered}$$

(for the existence of s we invoke the condition $\beta > 1$). For $n = 1, 2, \dots$, let

$$b_n := \frac{5^n}{4} - 1, \quad a_n := e^{-\beta b_n}.$$

Define functions $u_n : \bar{I} \times [0, 4b_n + 4] \to \mathbb{R}$ and $f_n : [0, 4b_n + 4] \times \mathbb{R}$ as follows:

$$u_n(x,t) := \begin{cases} e^{-\beta t} h(x) & t \in [0, b_n), \\ \mathcal{S}_{b_n}[a_n(s^2(t)h(x) + (1-s(t))^2 g(x))] & t \in [b_n, b_n+1), \\ \mathcal{S}_{b_n+1}[a_n e^{\beta t} g(x)] & t \in [b_n+1, 2b_n+1), \\ \mathcal{S}_{2b_n+1}[m(t)g(x)] & t \in [2b_n+1, 2b_n+2), \\ \mathcal{S}_{2b_n+2}[e^{-\beta t} g(x)] & t \in [2b_n+2, 3b_n+2), \\ \mathcal{S}_{3b_n+2}[a_n(s^2(t)g(x) + (1-s(t))^2 h(x))] & t \in [3b_n+2, 3b_n+3), \\ \mathcal{S}_{3b_n+3}[a_n e^{\beta t} h(x)] & t \in [3b_n+3, 4b_n+3), \\ \mathcal{S}_{4b_n+3}[m(t)h(x)] & t \in [4b_n+3, 4b_n+4], \end{cases}$$

where $x \in \bar{I}$, and

$$f_n(t,u) :=$$
$$\begin{cases} (-\beta + \frac{1}{16})u & t \in [0, b_n), \\ \mathcal{S}_{b_n}[(1-t)(-\beta+\frac{1}{16})u + t((1+\beta)u - \frac{a_n}{2})] & t \in [b_n, b_n+1), \\ \mathcal{S}_{b_n+1}[(\beta+1)u - \frac{a_n}{2}e^{\beta t}] & t \in [b_n+1, 2b_n+1), \\ \mathcal{S}_{2b_n+1}[(\frac{m'(t)}{m(t)}+1)u - \frac{m(t)}{2}] & t \in [2b_n+1, 2b_n+2), \\ \mathcal{S}_{2b_n+2}[(1-\beta)u - \frac{1}{2}e^{-\beta t}] & t \in [2b_n+2, 3b_n+2), \\ \mathcal{S}_{3b_n+2}[(1-t)((1-\beta)u - \frac{a_n}{2}) + t(\beta+\frac{1}{16})u] & t \in [3b_n+2, 3b_n+3), \\ \mathcal{S}_{3b_n+3}[(\beta+\frac{1}{16})u] & t \in [3b_n+3, 4b_n+3), \\ \mathcal{S}_{4b_n+3}[(\frac{m'(t)}{m(t)}+\frac{1}{16})u] & t \in [4b_n+3, 4b_n+4]\,, \end{cases}$$

where $u \in \mathbb{R}$. One easily verifies that that $u_n \in C^{2,1}(I \times [0, 4b_n+4)]$ and f_n is a continuous function on $[0, 4b_n+4)] \times \mathbb{R}$, which is piecewise linear in u with Lipschitz constant $\beta + 1$. Also the following relations are straightforward to verify:

$$u_n(0,t) = u_n(1,t) = 0 \quad (t \in [0, 4b_n+4]), \tag{5.4}$$
$$u_n(\cdot,0) = u_n(\cdot, 4b_n+4) = h, \tag{5.5}$$
$$(u_n)_t(\cdot,0) = (u_n)_t(\cdot, 4b_n+4) = -\beta h, \tag{5.6}$$
$$u_n(\cdot, 2b_n+1) = g, \tag{5.7}$$
$$f_n(0,u) = f_n(4b_n+4, u) = (-\beta + \frac{1}{16})u \quad (u \in \mathbb{R}), \tag{5.8}$$

$$(u_n)_t(x,t) - (u_n)_{xx}(x,t) - f_n(t, u_n(x,t)) = 0$$
$$(x \in I,\ t \in [0, 4b_n+4] \setminus ((b_n, b_n+1) \cup (3b_n+2, 3b_n+3))). \tag{5.9}$$

The four different regimes mentioned in the above outline are active on the "long" time intervals, that is, those intervals in the definitions of u_n and f_n that have the length b_n.

Next define R_n by

$$R_n(x,t) := (u_n)_t(x,t) - (u_n)_{xx}(x,t) - f_n(t, u_n(x,t)) \quad ((x,t) \in I \times [0, 4b_n+4])\,. \tag{5.10}$$

By (5.9), $R_n(\cdot,t) \neq 0$ only if $t \in (b_n, b_n+1) \cup (3b_n+2, 3b_n+3)$. An easy calculation shows that R_n is a continuous function with

$$\|R_n(\cdot,t)\|_{L^\infty(I)} \leq C a_n \quad (t \in [0, 4b_n+4]), \tag{5.11}$$

where C is independent of n (it depends on β).

Finally, to complete the construction, set

$$T_0 := 0, \quad T_n := \sum_{i=1}^{n} (4b_i + 4) = \frac{5}{4}(5^n - 1)$$

(recalling that $b_i := 5^i/4 - 1$) and

$$u(x,t) := \mathcal{S}_{T_n} u_n(x,t), \quad f(t,u) := \mathcal{S}_{T_n} f_n(t,u), \quad R(x,t) := \mathcal{S}_{T_n} R_n(x,t)$$
$$((x,u,t) \in I \times \mathbb{R} \times [T_n, T_{n+1}), n \in \mathbb{N} \cup \{0\}) .$$

It follows from (5.5), (5.6), and (5.8), that $u \in C^{2,1}(\bar{I} \times [0,\infty))$, f and R are continuous on $\bar{I} \times [0,\infty)$, and f is Lipschitz in u with Lipschitz constant $\beta + 1$. Clearly, u is bounded and $u \geq 0$ everywhere. By (5.10) and (5.4), u is a solution of (5.1), and by (5.5), (5.7), $h, g \in \omega(u)$.

It remains to show that R has the specified exponential decay. Given any $t > 0$, pick the integer n for which $5(5^n - 1)/4 \leq t < 5(5^{n+1} - 1)/4$. Then, by (5.11),

$$\|R(\cdot,t)\|_{L^\infty(I)} \leq C a_n = C e^{-\beta b_n} = C e^{-\beta(\frac{5^n}{4} - 1)} \leq C e^{-\beta(\frac{t}{25} - \frac{19}{20})} = \tilde{C} e^{-\frac{\beta}{25} t},$$

where $\tilde{C} = \tilde{C}(\beta)$ is a constant independent of t.

References

[1] A. D. Alexandrov, *A characteristic property of spheres*, Ann. Mat. Pura Appl. (4) **58** (1962), 303–315.

[2] A. V. Babin, *Symmetrization properties of parabolic equations in symmetric domains*, J. Dynam. Differential Equations **6** (1994), no. 4, 639–658.

[3] ______, *Symmetry of instabilities for scalar equations in symmetric domains*, J. Differential Equations **123** (1995), no. 1, 122–152.

[4] A. V. Babin and G. R. Sell, *Attractors of non-autonomous parabolic equations and their symmetry properties*, J. Differential Equations **160** (2000), no. 1, 1–50.

[5] H. Berestycki, *Qualitative properties of positive solutions of elliptic equations*, Partial differential equations (Praha, 1998), Chapman & Hall/CRC Res. Notes Math., vol. 406, Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 34–44.

[6] H. Berestycki and L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat. (N.S.) **22** (1991), no. 1, 1–37.

[7] H. Berestycki, L. Nirenberg, and S. R. S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains*, Comm. Pure Appl. Math. **47** (1994), 47–92.

[8] X. Cabré, *On the Alexandroff-Bakel′man-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math. **48** (1995), no. 5, 539–570.

[9] F. Da Lio and B. Sirakov, *Symmetry results for viscosity solutions of fully nonlinear uniformly elliptic equations*, J. European Math. Soc. **9** (2007), 317–330.

[10] E. N. Dancer, *Some notes on the method of moving planes*, Bull. Austral. Math. Soc. **46** (1992), no. 3, 425–434.

[11] J. Földes, *On symmetry properties of parabolic equations in bounded domains*, J. Differential Equations **250** (2011), no. 12, 4236–4261.

[12] ———, *Symmetry of positive solutions of asymptotically symmetric parabolic problems on* $\mathbb{R}^N$, J. Dynam. Differential Equations **23** (2011), no. 1, 45–69.

[13] B. Gidas, W. M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243.

[14] P. Hess and P. Poláčik, *Symmetry and convergence properties for nonnegative solutions of nonautonomous reaction-diffusion problems*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), no. 3, 573–587.

[15] J. Jiang and J Shi, *Dynamics of a reaction-diffusion system of autocatalytic chemical reaction*, Discrete Contin. Dynam. Systems **21** (2008), 245-258.

[16] B. Kawohl, *Symmetrization—or how to prove symmetry of solutions to a PDE*, Partial differential equations (Praha, 1998), Chapman & Hall/CRC Res. Notes Math., vol. 406, Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 214–229.

[17] C. Li, *Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on bounded domains*, Comm. Partial Differential Equations **16** (1991), no. 2-3, 491–526.

[18] W.-M. Ni, *Qualitative properties of solutions to elliptic problems*, Stationary partial differential equations. Vol. I, Handb. Differ. Equ., North-Holland, Amsterdam, 2004, pp. 157–233.

[19] P. Poláčik, *Estimates of solutions and asymptotic symmetry for parabolic equations on bounded domains*, Arch. Ration. Mech. Anal. **183** (2007), no. 1, 59–91. Addendum: www.math.umn.edu/ polacik/Publications

[20] ______, *Symmetry properties of positive solutions of parabolic equations: a survey*, Recent progress on reaction-diffusion systems and viscosity solutions, World Sci. Publ., Hackensack, NJ, 2009, pp. 170–208.

[21] ______, *Symmetry of nonnegative solutions of elliptic equations via a result of Serrin*, Comm. Partial Differential Equations **36** (2011), 657–669.

[22] ______, *On symmetry of nonnegative solutions of elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Lineaire **29** (2012), 1-19.

[23] A. Saldaña and T. Weth, *Asymptotic axial symmetry of solutions of parabolic equations in bounded radial domains*, J. Evol. Equ. (to appear).

[24] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304–318.