On bounded radial solutions of parabolic equations on $\mathbb{R}^N$: Quasiconvergence for initial data with a stable limit at infinity

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Dedicated to Yihong Du
on the occasion of his 60th birthday

Abstract

We consider the Cauchy problem for the nonlinear heat equation

$$u_t = \Delta u + f(u), \quad x \in \mathbb{R}^N,\ t > 0,$$

where $N \geq 2$ and f is a C^1 function satisfying minor nondegeneracy conditions. Our goal is to describe the large-time behavior of bounded solutions whose initial data are radially symmetric and have a finite limit ζ as $|x| \to \infty$. In the present paper, we examine the following two cases: $f(\zeta) \neq 0$, or $f(\zeta) = 0$ and ζ is a stable equilibrium of the equation $\dot{\xi} = f(\xi)$. We prove that bounded solutions with such initial data are quasiconvergent: as $t \to \infty$, they approach a set of steady states in the topology of $L^\infty_{loc}(\mathbb{R}^N)$.

Key words: semilinear parabolic equations, Cauchy problem, radial solutions, convergence, quasiconvergence

AMS Classification: 35K15, 35K57, 35B40

*Supported in part by the NSF Grant DMS–1856491

Contents

1 Introduction and statement of the main results

We consider the Cauchy problem

$$u_t = \Delta u + f(u), \qquad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$
$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where Δ in the Laplace operator in the spatial variable $x = (x_1, \dots, x_N)$, f is a C^1 function on $\mathbb{R}$, and $u_0 \in L^\infty(\mathbb{R}^N)$. We will mainly focus on solutions which are radially symmetric in x (radial solutions, for short), so at some point below we make the assumption that u_0 is radial.

We denote by $u(\cdot, t, u_0)$, or simply $u(\cdot, t)$ if there is no danger of confusion, the unique classical solution of (1.1), (1.2) and by $T(u_0) \in (0, +\infty]$ its maximal existence time. (To avoid ambiguities, we include the requirement that $u(\cdot, t)$ is bounded for all $t \in (0, T(u_0))$ in the definition of a classical solution.) If u is bounded on $\mathbb{R}^N \times [0, T(u_0))$, then necessarily $T(u_0) = \infty$, that is, the solution is global. In this paper, we only deal with bounded global solutions and our main concern is the behavior of such solutions as $t \to \infty$. More specifically, we examine the limit profiles with respect to the locally uniform convergence of bounded solutions as $t \to \infty$. For that purpose, we define the ω-limit sets of a bounded solution u as follows:

$$\omega(u) := \left\{\varphi \in C(\mathbb{R}^N) :\ u(\cdot, t_n) \to \varphi \text{ for some sequence } t_n \to \infty\right\}. \tag{1.3}$$

Here, the convergence is in the topology of $L^\infty_{loc}(\mathbb{R}^N)$, that is, the locally uniform convergence. Due to this relatively weak convergence requirement,

$\omega(u)$ is nonempty for any bounded solution u (see Section 2.3 for a summary of well-known properties of $\omega(u)$). If the $\omega-$limit set reduces to a single element φ, then u is *convergent:* $u(\cdot,t) \to \varphi$ in $L^\infty_{loc}(\mathbb{R}^N)$ as $t \to \infty$. Necessarily, φ is a steady state of (1.1) in this case. We say that the solution u is *quasiconvergent* if all elements $\varphi \in \omega(u)$ are steady states of (1.1).

If equation (1.1) is considered on a bounded domain, instead of $\mathbb{R}^N$, and coupled with some common boundary condition, then standard energy estimates imply that all bounded solutions are quasiconvergent (and one can take the uniform convergence in the definition of the ω-limit set). If $N = 1$, they are even convergent [32, 53, 54]. For equation (1.1) on $\mathbb{R}^N$, there are many interesting convergence and quasiconvergence results (see [3, 9, 13, 14, 17, 27, 28, 34, 35, 42, 38, 47, 55] and references therein for results with $N = 1$ and [2, 6, 7, 8, 11, 15, 16, 21, 22, 18, 44, 45] for results with $N \geq 1$; a 2017 overview was given in [41]). However, bounded solutions of (1.1) are not quasiconvergent in general. Examples of non-quasiconvergent bounded solutions can be found in [39, 40, 46]; earlier, the existence of such solution was indicated by the analysis of [12]. Moreover, as shown in [40], bounded non-quasiconvergent solutions of (1.1) exist rather frequently; namely, they exist whenever f is bistable in some interval. We also mention a result of [19] which shows that if $N \leq 2$, the ω-limit set of any bounded solution contains at least one steady state (see also [20]). Whether the same is true or not in any dimension is an open problem.

In view of the existence of non-quasiconvergent solutions, one naturally wonders whether quasiconvergence can be guaranteed by requiring the initial data to belong to a more specific, but still reasonably general, class of functions. Several such classes have been identified in some of the references mentioned above. For example, when $f(0) = 0$, these include nonnegative functions u_0 with compact support, and in this case one even gets the convergence of the corresponding bounded solutions; see [9] for the proof for $N = 1$, and [11] for the proof for any N under a nondegeneracy condition on f. A related quasiconvergence theorem for bounded solutions with compactly supported initial data was proved in [27] in a general setting of quasilinear equations on bounded or unbounded intervals, possibly depending on time. For equations (1.1) with $N = 1$ and $f(0) = 0$, the quasiconvergence was also proved for bounded solutions with initial data satisfying $u_0 \geq 0$ and $u_0(x) \to 0$ as $|x| \to \infty$, see [34]. Interestingly, this quasiconvergence result is not valid if $N \geq 3$, even when u_0 is radial, see [46].

In a recent project, joint with A. Pauthier, we examined equations on $\mathbb{R}$

with general convergent initial data. Specifically, we took any $u_0 \in C(\mathbb{R})$ such that the limits $u_0(-\infty)$ and $u_0(\infty)$ both exist and are finite. We proved that the corresponding solution of (1.1), (1.2) is quasiconvergent, if bounded, in the following two cases:

(C1) $u_0(-\infty) \neq u_0(\infty)$ (see [36]).

(C2) $u_0(-\infty) = u_0(\infty) =: \zeta$ and either $f(\zeta) \neq 0$ or $f(\zeta) = 0$ and ζ is a stable equilibrium of the equation $\dot{\xi} = f(\xi)$ (see [37]).

In the case (C2), we assumed that all zeros of f are nondegenerate: $f'(\eta) \neq 0$ whenever $f(\eta) = 0$. This case, with an analogous global nondegeneracy condition, was independently treated in [47] in a more general setting of gradient systems. In the remaining case,

(C3) $u_0(-\infty) = u_0(\infty) =: \zeta$ and ζ is an unstable equilibrium of the equation $\dot{\xi} = f(\xi)$,

under the same nondegeneracy condition on f, the quasiconvergence conclusion is not valid in general (see [40]). In this case, we proved the quasiconvergence of the corresponding bounded solutions under additional conditions (see [38]). For example, it is sufficient that u_0 is of class C^1 and has only finitely many critical points, or, more generally, that u_0 is continuous and there is a positive number a such that u_0 is nonconstant and monotone in each of the intervals $(-\infty, -a)$, (a, ∞).

The results concerning bounded solutions of one-dimensional equations with convergent initial data motivated the research documented in this paper. Our goal is to address the question whether similar quasiconvergence theorems can be proved for radial solutions in higher dimensions. Obviously, for radial functions u_0 there is just one limit at infinity to deal with, so case (C1) has no analog in the radial setting. Presently, we consider the analog of case (C2). It turns out that similar quasiconvergence theorems as in [37] can indeed be proved, even under weaker nondegeneracy conditions on f (see conditions (ND), (ND2) below), and the proofs are much simpler.

We now give precise formulation of our main theorems; first stating their hypotheses. From now on, we assume that $N \geq 2$ and u_0 satisfies the following condition:

(IC) u_0 is a continuous radial function on $\mathbb{R}^N$ with $\zeta := \lim_{r\to\infty} u_0(r) \in \mathbb{R}$.

In (IC) and at many other places below, we often abuse the notation slightly and view radial functions on $\mathbb{R}^N$ as functions of the real variable $r = |x|$.

Since equation (1.1) is invariant under rotations, assumption (IC) implies that the solution $u(\cdot, t, u_0)$ is radial. Viewed as a function of t and r, it satisfies the following equation and boundary condition at $r = 0$:

$$u_t = u_{rr} + \frac{N-1}{r} u_r + f(u), \qquad r > 0,\ t > 0, \tag{1.4}$$

$$u_r(0, t) = 0, \qquad t > 0. \tag{1.5}$$

As above, the function f is assumed to be of class C^1. In our main theorems, we also assume the following nondegeneracy conditions:

(ND) For each $\eta \in f^{-1}\{0\}$ there is $\epsilon > 0$ such that the function f is monotone (not necessarily strictly) in each of the intervals $(\eta - \epsilon, \eta]$, $[\eta, \eta + \epsilon)$.

(ND2) For $N = 2$ only: $f^{-1}\{0\}$ does not contain any (nonempty) open interval.

Obviously, conditions (ND), (ND2) are satisfied if all zeros of f are nondegenerate: $f'(\eta) \neq 0$ for each $\eta \in f^{-1}\{0\}$; but they are much weaker than the latter condition. Essentially, (ND) just requires that f' not be oscillating in the one-sided neighborhoods of η. We make some comments on the role of the nondegeneracy conditions in our analysis at the end of the introduction.

Our first theorem concerns the case $f(\zeta) \neq 0$.

Theorem 1.1. *Assume* (IC), (ND), (ND2). *If $f(\zeta) \neq 0$ and the solution u of* (1.1) *is bounded, then it is convergent: as $t \to \infty$, $u(\cdot, t) \to \varphi$ in $L^\infty_{loc}(\mathbb{R}^N)$, where φ is a (radial) steady state of* (1.1).

Next we state a quasiconvergence theorem assuming that $f(\zeta) = 0$ and $f' \leq 0$ on a neighborhood of ζ. Note that the latter condition is equivalent to the stability of ζ as an equilibrium of the equation $\dot{\xi} = f(\xi)$ when condition (ND) is in effect.

Theorem 1.2. *Assume* (IC), (ND), (ND2). *Further assume that $f(\zeta) = 0$ and there is $\delta > 0$ such that $f'(u) \leq 0$ for all $u \in (\zeta - \delta, \zeta + \delta)$. If the solution u of* (1.1) *is bounded, then it is quasiconvergent: $\omega(u)$ consists of (radial) steady states of* (1.1).

Additional information on the steady states in $\omega(u)$ is given in Proposition 3.5 below. We show there that if $\omega(u)$ does not consist of a single steady state, then it contains a continuum of radial steady states which have the limit ζ, same as u_0, as $|x| \to \infty$ (note that we are not claiming that all steady states in $\omega(u)$ have the same limit at infinity). Thus, if the existence of such continuum can be ruled out for a given equation (1.1) and a given limit ζ, then Theorem 1.2 yields the convergence of the solution u. We emphasize that ζ is supposed to be a *stable* equilibrium of the equation $\dot{\xi} = f(\xi)$. Whether a continuum of steady states of (1.1) sharing the limit ζ may exist or not is therefore a nontrivial question. We suspect that for some nonlinearities it does exist, but for generic nonlinearities, in suitable C^k topologies, it does not.[1] It is also an interesting question whether, regardless of the existence of such continua, the convergence to single steady state can be proved under some explicit additional nondegeneracy conditions on f. Such questions will be addressed elsewhere.

The proofs of the above theorems are given in Section 3. The main technical tools in these proofs are the zero number and intersection comparison principles, along with some asymptotic properties of radial steady states of (1.1) (see the preliminary Section 2).

We now make some remarks on the role of the nondegeneracy conditions (ND), (ND2) in our proofs. Doing so, we also explain a key difference between one-dimensional problems and radial problems in higher dimensions which makes the proofs of Theorems 1.1, 1.2 much simpler compared to the proofs of similar results for $N = 1$, as given in [37].

The only purpose of the nondegeneracy conditions (ND), (ND2) is to guarantee that each bounded radial steady state ψ of (1.1) is convergent: $\psi(r)$ has a limit as $r \to \infty$. This is not true in general (see [30] for counterexamples with $N = 2$). If it is true for some specific equation, the above theorems are valid for such an equation with hypotheses (ND), (ND2) removed. The fact that conditions (ND), (ND2) imply the convergence follows, as we demonstrate in Lemma 2.2 below, from a result of [24] (see also [26, 31]). Weaker, but more complicated, conditions for the convergence than (ND), (ND2) can be found in or derived from [24, 26]; if desired, any such conditions can safely be used in place (ND), (ND2) in all our results.

[1] The fact that no such continua exist for generic nonlinearities has already been proven. It is a consequence of recent results of [49] concerning generic gradient systems of reaction diffusion equations.

The convergence property of the steady states makes them very useful for various intersection-comparison arguments involving radial solutions of (1.1) with convergent initial data. In contrast, steady states of the one-dimensional equation (1.1) correspond to solutions of a planar Hamiltonian system and convergent solutions of that system are rather exceptional (typical solutions are periodic), regardless of any nondegeneracy conditions on f. This is the main reason why some relatively simple arguments that we use in this paper do not apply in the one-dimensional case.

To conclude the introduction, we mention recent papers [10, 48] which also examine the large-time behavior of bounded radial solutions of (1.1), but from a different perspective. The main focus of these papers is on the structure of the solutions for large r, which is in a sense complementary to our results on convergence and quasiconvergence in $L^\infty_{loc}(\mathbb{R}^N)$.[2] In the main results of [10, 48], the asymptotic shape (for large t and large $r = |x|$) of a class of solutions of (1.1) is described in term of propagating terraces of one-dimensional equations $u_t = u_{rr} + f(u)$ (in [10], a class of nonradial solutions of (1.1) is considered as well, and the setting in [48] is more general with the scalar equation replaced by a gradient system).

In the rest of the paper, we assume as our **standing hypotheses** that $N \geq 2$, u_0 satisfies condition (IC) above, and f satisfies the following condition

(F) $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function with bounded derivative.

The boundedness of f', or the global Lipschitz continuity of f, which is assumed in (F) in addition to the previously assumed condition $f \in C^1$, is a convenience assumption made at no cost to generality. Since our main theorems concern individual bounded solutions, we can always modify f outside a large interval $(-R, R)$ containing the range of the solution in question so as to achieve the global Lipschitz continuity. We may clearly choose the modification such that the new function f is monotone in each of the intervals $(-\infty, R) \cup (R, \infty)$. This ensures that conditions (ND), (ND2) continue to be satisfied if they were satisfied by the original nonlinearity.

[2] A recent update to the preprint [48] does include a result describing the large-time behavior of solutions in bounded spatial intervals. It shows that for generic nonlinearities, radial bounded solutions converge to a steady state in $L^\infty_{loc}(\mathbb{R}^N)$ (see also [50] for a related result for nonradial solutions). The author is thankful to Emmanuel Risler for bringing these new results to his attention.

2 Preliminaries

This section contains preliminary results concerning radial steady states of equation (1.1), zero number of solutions of linear parabolic equations, and entire solutions of (1.1) in ω-limit sets.

2.1 Steady states

Radial steady states of (1.1) are solutions of the equation

$$\psi_{rr} + \frac{N-1}{r}\psi_r + f(\psi) = 0, \qquad r > 0, \tag{2.1}$$

and they also satisfy the condition $\psi_r(0) = 0$. In our analysis, we also use solutions of (2.1) which are not necessarily steady states of equation (1.1); they may well be unbounded as $r \searrow 0$.

Note that (2.1) is a regular ordinary differential equation on $(0, \infty)$, and our standing assumption (F) implies that for any $(a, b) \in \mathbb{R}^2$ and $r_0 > 0$ the solution of (2.1) satisfying the initial conditions

$$\psi(r_0) = a, \quad \psi_r(r_0) = b, \tag{2.2}$$

is defined globally on $(0, \infty)$. It is also well known that for $r_0 = 0$, $b = 0$ the initial value problem (2.1), (2.2) is well posed: it has a unique solution $\psi(\cdot, a) \in C^1[0, \infty) \cap C^2(0, \infty)$, and for any $R > 0$ the $C^1[0, R]$-valued map $a \mapsto \psi(\cdot, a)$ is continuously differentiable.

The proof of the following simple lemma can be found in [43].

Lemma 2.1. *Let ψ be a solution of (2.1). Then either $\psi(r)$ is unbounded as $r \to 0+$ or else the limit $a := \lim_{r\to 0+} \psi(r)$ exists and, after setting $\psi(0) = a$, ψ is the solution of (2.1), (2.2) with $r_0 = 0$ and $b = 0$.*

We next state the convergence property of solutions of (2.1).

Lemma 2.2. *Assume (ND), (ND2). If ψ is a solution of (2.1) which is bounded in $[1, \infty)$, then*

$$\lim_{r\to\infty} (\psi(r), \psi'(r)) = (\eta, 0), \quad \text{where } \eta \in f^{-1}\{0\}. \tag{2.3}$$

Proof. Assume ψ be a solution of (2.1) which is bounded in $[1, \infty)$. By standard estimates, the convergence of $\psi'(r)$ to zero follows from (2.1) once the convergence of $\psi(r)$ to $\eta \in f^{-1}\{0\}$ is proved. The proof of the latter is given in [24], although the result is not stated there in exactly the same way. We give some explanations.

First we note that, although it is assumed in [24] that ψ satisfies (2.1), (2.2) with $r_0 = 0$, $b = 0$ (and some $a \in \mathbb{R}$), the arguments in [24] work the same way for solutions satisfying (2.1), (2.2) with $b = 0$ and some $r_0 > 0$—and if $\psi'(r) \neq 0$ for all $r > 0$ the convergence is trivial. Now, as proved in [24] (see the Main Theorem and formulas (3.5), (3.6) in [24]), if ψ is not convergent as $r \to \infty$, then there exist $c_1 < c_2$ with the following properties:

(i) $f(c_1) \leq 0 \leq f(c_2)$ and one of the values $f(c_1)$, $f(c_2)$ is zero,

(ii) there exist sequences $\{m_k\}_k$ of local minimum points of ψ and $\{M_k\}_k$ of local maximum points of ψ such that $m_k \to \infty$, $M_k \to \infty$, and

$$\psi(m_k) \nearrow c_1, \ \psi(M_k) \searrow c_2, \ f(\psi(m_k)) < 0 < f(\psi(M_k)). \tag{2.4}$$

If $f(c_1) = 0$, then (2.4) in conjunction with condition (ND) imply that f is monotone nondecreasing in an interval $(c_1 - \epsilon, c_1)$ with $\epsilon > 0$. Likewise, if $f(c_2) = 0$, then (2.4) and (ND) imply that f is monotone nondecreasing in $(c_2, c_2 + \epsilon)$ for some $\epsilon > 0$. In this situation, by [24, Main Theorem], necessarily $N = 2$ and $f \equiv 0$ on $[c_1, c_2]$. But this possibility is ruled out by condition (ND2). Thus ψ must be convergent. □

2.2 Zero number

If u, $\tilde{u}$ are two radial solutions of (1.1) on an open time interval J, the function $v := u - \tilde{u}$ solves the linear equation

$$v_t = \Delta v + c(x,t)v, \qquad |x| < r_1, \ t \in J, \tag{2.5}$$

where $r_1 = \infty$, and c is a continuous bounded radial function given by

$$c(x,t) = \int_0^1 f'(\tilde{u}(x,t) + s(u(x,t) - \tilde{u}(x,t))\, ds. \tag{2.6}$$

In the variables t and r, the equation for $v = v(r,t)$ takes the form

$$v_t = v_{rr} + \frac{N-1}{r} v_r + c(r,t)v, \qquad r_0 < r < r_1, \ t \in J, \tag{2.7}$$

with $r_0 = 0$, $r_1 = \infty$, and we also have $v_r(0,t) = 0$ for $t \in J$. For $\tilde{u}$ we usually take a radial steady state of (1.1), but we also consider differences $v := u - \psi$, where ψ is just a solution (possibly unbounded as $r \to 0+$) of equation (2.1). In this case, we take $r_0 > 0$ in (2.7); occasionally, it will also be convenient to take $r_1 < \infty$. When referring to equation (2.5), we always assume that c is a continuous bounded radial function on $\{x \in \mathbb{R}^N : |x| < r_1\} \times J$. Similarly, in equation (2.7), c is always assumed to be a continuous bounded function on $(r_0, r_1) \times J$.

Note that for $r_0 > 0$ (2.7) is a nonsingular parabolic equation.

If $I \subset [0, \infty)$ is an interval and $g : I \to \mathbb{R}$ is a continuous function, we denote by $z_I(g)$ the number of zeros of g in I. If $I = [0, \infty)$, we often omit the subscript I: $z(g) = z_{[0,\infty)} g$.

Lemma 2.3. *Let $0 \leq r_0 < r_1 \leq \infty$; and $I := [r_0, r_1]$ if $r_1 < \infty$, $I = [r_0, \infty)$ if $r_1 = \infty$. Assume that either $r_0 = 0$ and $v(r,t)$ is a nontrivial bounded radial solution of* (2.5)*, or $r_0 > 0$ and $v(r,t)$ is a nontrivial bounded solution of* (2.7) *such that $v \in C(I \times J)$ and $v(r_0, t) \neq 0$ for all $t \in J$. Finally, if $r_1 < \infty$ assume also that $v(r_1, t) \neq 0$ for all $t \in J$. Then the following statements are valid:*

(i) For each $t \in J$, the zeros of $v(\cdot, t)$ in I are isolated. In particular, if $r_1 < \infty$, then $z_I(v(\cdot, t)) < \infty$ for all $t \in J$.

(ii) The function $t \mapsto z_I(v(\cdot, t))$ is monotone nonincreasing.

(iii) If for some $t_0 \in J$ the function $v(\cdot, t_0)$ has a multiple zero ρ_0 in I (that is, $v(\rho_0, t_0) = v_\rho(\rho_0, t_0) = 0$) and $z_I(v(\cdot, t_0)) < \infty$, then for any $t_1, t_2 \in J$ with $t_1 < t_0 < t_2$, one has $z_I(v(\cdot, t_1)) > z_I(v(\cdot, t_2))$.

Proof. For $r_0 = 0$ the lemma is proved in [5]; for $r_0 > 0$, proofs can be found in [1, 4]. □

Corollary 2.4. *Assuming the hypotheses of Lemma 2.3, consider the set M of all $t \in J$ such that the function $v(\cdot, t)$ has a multiple zero.*

(i) If $r_1 < \infty$, then the set M is discrete and its only possible accumulation point is $\inf J$.

(ii) If $r_1 = \infty$, then the set M is at most countable.

Proof. Assume first that $r_1 < \infty$. If some $\bar{t} \in J$ with $\bar{t} > \inf J$ were an accumulation point of M, then, picking $t_1 \in J$ with $t_1 < \bar{t}$, the finite zero number $z_I(v(\cdot,t))$ would have to drop infinitely many times in $[t_1,\infty) \cap J$, which is impossible by the monotonicity. Now let $r_1 = \infty$. To prove statement (ii), it is sufficient to prove that for $n = 1, 2, \dots$ the set

$$M_n := \{t \in J : \text{ the function } v(\cdot,t) \text{ has a multiple zero in } [0,n]\}$$

is discrete, hence countable. Fix an arbitrary $t_0 \in M_n$ (if there is any). Since the zeros of $v(\cdot,t_0)$ are isolated, there is $\bar{r}_1 > n$ such that $v(\bar{r}_1,t_0) \neq 0$. By continuity, $v(\bar{r}_1,t) \neq 0$ for all $t \in (t_0-\epsilon, t_0+\epsilon)$, where ϵ is a sufficiently small positive number. Therefore, statement (i) applies with r_1 replaced by $\bar{r}_1$ and J replaced by $(t_0-\epsilon,t_0+\epsilon)$. Consequently, since $r_1 > n$, making $\epsilon > 0$ smaller if necessary, we obtain that $M_n \cap (t_0-\epsilon,t_0+\epsilon) = \{t_0\}$, as desired. $\square$

We will also use the following robustness property.

Lemma 2.5. *Let $w_n(r,t)$ be a sequence of functions converging to $v(r,t)$ in $C^1(\bar{I} \times [s,T])$, where $s < T$ are numbers in J, and either $I = (r_0,r_1)$ for some $0 < r_0 < r_1 < \infty$ or $I = [0,r_1)$ for some $r_1 \in (0,\infty)$. If $I = (r_0,r_1)$, assume that v is a solution of (2.7); and if $I = [0,r_1)$, assume that v is a radial solution of (2.5) and $\partial_r w_n(0,t) = 0$ for all $t \in [s,T]$ and $n = 1,2,\dots$. Finally, assume that $v \not\equiv 0$ and $v(\cdot,t_0)$ has a multiple zero $\rho_0 \in I$ for some $t_0 \in (s,T)$. Then there exist sequences $\{r_n\}$ in I and $\{t_n\}$ in J such that $r_n \to \rho_0$, $t_n \to t_0$, and for all sufficiently large n the function $w_n(\cdot,t_n)$ has a multiple zero at r_n: $w_n(r_n,t_n) = \partial_r w_n(r_n,t_n) = 0$.*

Proof. In the case $I = (r_0,r_1)$ with $0 < r_0 < r_1 < \infty$, the lemma is a reformulation of [9, Lemma 2.6]. The proof for $r_0 = 0$ uses similar arguments to those in [9]; see [43] for the details. $\square$

2.3 Limit sets and entire solutions

We summarize here some well-know properties of the ω-limit set.

By standard parabolic estimates, the orbit $\{u(\cdot,t),\ t \geq 1\}$ of a bounded solution u of (1.1) is relatively compact in $L^\infty_{loc}(\mathbb{R}^N)$. This implies that the omega limit set of u, as defined in (1.3), is nonempty, compact, and connected in $L^\infty_{loc}(\mathbb{R}^N)$, and it attracts the solution in (the metric space) $L^\infty_{loc}(\mathbb{R}^N)$:

$$\mathrm{dist}_{L^\infty_{loc}(\mathbb{R}^N)}\left(u(\cdot,t),\omega(u)\right) \underset{t\to\infty}{\longrightarrow} 0.$$

It is also well-known that for each $\varphi \in \omega(u)$ there exists a unique entire solution $U(x,t)$ of (1.1)

$$U(\cdot,0) = \varphi, \qquad U(\cdot,t) \in \omega(u) \quad (t \in \mathbb{R}). \tag{2.8}$$

The *entire solution* refers to a solution defined for all $t \in \mathbb{R}$. It will be useful to recall how such an entire solution U is found. Let $t_n \to \infty$ be as in the definition of $\omega(u)$: $u(\cdot,t_n) \to \varphi$ in $L^\infty_{loc}(\mathbb{R}^N)$. The boundedness of u and parabolic regularity estimates imply that all first-order spatial and temporal partial derivatives, and all second-order spatial derivatives of u are bounded on $\mathbb{R}^N \times [1,\infty)$ and are globally $\alpha-$Hölder continuous for any $\alpha \in (0,1)$. Therefore, denoting $u_n(x,t) := u(x,t+t_n)$, $n = 1,2\ldots$, and passing to a subsequence if necessary, we obtain that $u_n \to U$ in $C^1_{loc}(\mathbb{R}^N \times \mathbb{R})$ for some function U; this function U is then easily shown to be an entire solution of (1.1). By definition, U satisfies (2.8). The uniqueness of this entire solution follows from the uniqueness and backward uniqueness for the Cauchy problem (1.1), (1.2).

Obviously, if the solution u is radial, then all functions in $\omega(u)$ are radial.

3 Some general results and the proofs of Theorems 1.1, 1.2

In this section, we first prove some useful general results and then give proofs of Theorems 1.1, 1.2. By default, in the following statements we only assume the standing hypotheses (F) and (IC); the nondegeneracy conditions (ND), (ND2) are assumed only where indicated.

We start with the following characterization of convergence of bounded solutions of equation (1.1) (we are not involving (1.2) here, so condition (IC) is not relevant in this result).

Proposition 3.1. *Let u be any bounded radial solution of* (1.1)*. Then $u(\cdot,t)$ is convergent if and only if $u(0,t)$ is convergent as $t \to \infty$.*

Recall that the convergence of $u(\cdot,t)$ refers to the convergence (to a steady state) in $L^\infty_{loc}(\mathbb{R}^N)$.

Proof of Proposition 3.1. If $u(\cdot,t)$ is convergent, then, trivially $u(0,t)$ is convergent. To prove the converse, assume that $u(0,t) \to a \in \mathbb{R}$ as $t \to \infty$.

Then each $\varphi \in \omega(u)$ has $\varphi(0) = a$. Pick any $\varphi \in \omega(u)$ and let U be the radial entire solution of (1.1) with $U(\cdot, 0) = \varphi$, $U(\cdot, t) \in \omega(u)$ for all $t \in \mathbb{R}$ (cp. Section 2.3). Then $U(0, t) = a$ and so $U_t(0, t) = 0$ for all $t \in \mathbb{R}$. The function $v := U_t$ is a radial solution of a linear equation (2.5), and in particular $v_r(0, t) = 0$ for all t. Thus, $r = 0$ is a multiple zero of $v(\cdot, t)$ for all t. By Corollary 2.4, this is only possible if $v \equiv 0$, hence φ is a steady state of (1.1).

We have proved that all elements φ of $\omega(u)$ are steady states. Since they are all radial, the relation $\varphi(0) = a$ implies that there is just one of them. This shows that u is indeed convergent. □

Proposition 3.2. *Let u be the radial solution of* (1.1)*,* (1.2) *(with u_0 satisfying the standing hypotheses* (IC)*.) Then for any $t \in [0, T(u_0))$ the limit $\xi(t) := \lim_{r\to\infty} u(r, t)$ exists, and the function $\xi(t)$ is the solution of the differential equation $\dot{\xi} = f(\xi)$ with $\xi(0) = \zeta$.*

Proof. For one-dimensional equations, the proof of a (more general) version of this result can be found in [52]. This proof, consisting essentially in taking limits in Picard iterates in the variation of constants formula, works, after straightforward modifications, for equations (1.1) in any dimension.

Alternatively, one can use the following argument. If $f(0) = 0$, then it is well-known (see [23, 29]) that equation (1.1) is well-posed on $C_0(\mathbb{R}^n)$, the space of all continuous functions whose limit as $|x| \to \infty$ is 0. In other words the space $C_0(\mathbb{R}^n)$ is invariant under (1.1) (which is well-posed on larger spaces as well). The same is true if $f(u)$ is replaced by a t-dependent function $f(t, u)$ satisfying suitable regularity assumptions and the condition $f(t, 0) = 0$ for all t. One can obtain the conclusion of Proposition 3.2 from the previous observation by considering the function $\tilde{u}(x, t) := u(x, t) - \xi(t)$, where $\xi(t)$ is the solution of the equation $\dot{\xi} = f(\xi)$ with $\xi(0) = \zeta$. Indeed, $\tilde{u}$ is a solution of (1.1) with $f(u)$ replaced by the function $\tilde{f}(t, u) = f(u + \xi(t)) - f(\xi(t))$ satisfying $\tilde{f}(t, 0) = 0$. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. Assume that (ND), (ND2) and the other hypotheses of the theorem are satisfied: (IC) holds with $f(\zeta) \neq 0$ and the solution u of (1.1) is bounded. For a contradiction, assume that u is not convergent. Then, by Proposition 3.1,

$$a_1 := \liminf_{t\to 0} u(0, t) < a_2 := \limsup_{t\to 0} u(0, t).$$

Pick any $a \in (a_1, a_2)$ and let ψ be the solution of (2.2) with $r_0 = 0$, $b = 0$. Then there is a sequence $t_k \to \infty$ such that $u(0, t_k) = a$, which means, by symmetry, that $r = 0$ is a multiple zero of the function $u(\cdot, t_k) - \psi$. We now distinguish the cases when ψ is bounded and when it is unbounded.

If ψ is bounded, then, by Lemma 2.2, the limit $\eta := \psi(\infty)$ exists and $f(\eta) = 0$. In particular, $\eta \neq \zeta = u_0(\infty)$. By Proposition 3.2, the limit $u(\infty, t)$ exists and is different from η for any $t > 0$. This and the fact that the zeros of $u(\cdot, t) - \psi$ are isolated (see Lemma 2.3) imply that $z(u(\cdot, t) - \psi)$ is finite. By the monotonicity of the zero number, $z(u(\cdot, t) - \psi)$ can drop only finitely many times as t increases in $[1, \infty)$. On the other hand, by Lemma 2.3(iii), $z(u(\cdot, t) - \psi)$ drops at $t = t_1, t_2, \dots$, and we have a contradiction.

If ψ is unbounded, then, since u is bounded, there exists r_1 such that $u(r_1, t) - \psi(r_1) \neq 0$ for all $t \geq 0$. Applying the zero number arguments as in the previous paragraph, considering this time the zero number in $[0, r_1]$ rather than $[0, \infty)$, we obtain a contradiction.

Thus the assumption that u is not convergent always leads to a contradiction, which proves the convergence of u. $\square$

With u_0 and ζ as is as in (IC), we next assume that $f(\zeta) = 0$ and give a sufficient condition for the quasiconvergence of the solution of (1.1), (1.2). The sufficient condition involves the stable set of ζ with respect to the ordinary differential equation (2.1), which we now define. For any $r_0 > 0$ and $(a, b) \in \mathbb{R}^2$, denote by $\psi(r; r_0, a, b)$ the solution of (2.1), (2.2). The stable set of ζ (with the initial "time" r_0) is defined as follows:

$$S(\zeta, r_0) := \{(a, b) \in \mathbb{R}^2 : \lim_{r \to \infty} \psi(r; r_0, a, b) = \zeta\}. \tag{3.1}$$

Note that

$$S(\zeta, r_0) = \{(a, b) \in \mathbb{R}^2 : \lim_{r \to \infty} (\psi(r; r_0, a, b), \psi_r(r; r_0, a, b)) = (\zeta, 0)\}, \tag{3.2}$$

for the convergence $\psi_r(r; r_0, a, b) \to 0$ follows automatically from the convergence of the solution $\psi(r; r_0, a, b)$.

Proposition 3.3. *Assume* (ND), (ND2). *Further assume that* (IC) *holds with $f(\zeta) = 0$ and for some $r_0 > 0$ the set $S(\zeta, r_0) \subset \mathbb{R}^2$ has empty interior. Then, if the solution u of* (1.1) *is bounded, it is quasiconvergent: $\omega(u)$ consists of (radial) steady states of* (1.1).

We remark, that if $S(\zeta, r_0)$ has empty interior for some r_0, then it has empty interior for all $r_0 > 0$. This follows from standard properties of the planar map $(a, b) \mapsto (\psi(r; r_0, a, b), \psi_r(r; r_0, a, b))$ (see the proof of Lemma 3.4 below).

Proof of Proposition 3.3. Fix an arbitrary $\varphi \in \omega(u)$. We show by contradiction that φ is a steady state. Assume it is not.

With $r_0 > 0$ as in the assumptions, let ψ be the solution of (2.1) with

$$\psi(r_0) = \varphi(r_0), \quad \psi_r(r_0) = \varphi_r(r_0). \tag{3.3}$$

Let U be the radial entire solution of (1.1) with $U(\cdot, 0) = \varphi$, $U(\cdot, t) \in \omega(u)$ for all $t \in \mathbb{R}$ (cp. Section 2.3). Relations (3.3) imply that $U(\cdot, 0) - \psi$ has a multiple zero at $r = r_0$. We have $U - \psi \not\equiv 0$, as φ is not a steady state. Since $\operatorname{int} S(\zeta, r_0) = \emptyset$, there is a sequence $(a_n, b_n) \in \mathbb{R}^2 \setminus S(\zeta, r_0)$ converging to $(\varphi(r_0), \varphi_r(r_0))$. By the continuity with respect to initial data, $\psi(\cdot; r_0, a_n, b_n) \to \psi$ in $C^1_{loc}(0, \infty)$. Therefore, by Lemma 2.5, for a sufficiently large n, which we henceforth fix, the function $U(\cdot, \tau) - \psi(\cdot; r_0, a_n, b_n)$ has a multiple zero ρ_0 near r_0 for some $\tau \approx 0$. As noted in Section 2.3, there is a sequence $t_k \to \infty$ such that $u(\cdot, t_k + \cdot) \to U$ in $C^1_{loc}(0, \infty)$. Applying Lemma 2.5 again, we find a sequences $\tau_k \to \tau$ and $\rho_k \to \rho_0$ such that for all large enough k the function $u(\cdot, t_k + \tau_k) - \psi(\cdot; r_0, a_n, b_n)$ has a multiple zero at the point ρ_k.

To simplify the notation, denote $\tilde{\psi} := \psi(\cdot; r_0, a_n, b_n)$ and $\tilde{t}_k := t_k + \tau_k$. Thus, $\tilde{\psi}$ is a solution of (2.1), $\tilde{t}_k \to \infty$, and there is a compact interval $[\rho_0 - \delta, \rho_0 + \delta] \subset (0, \infty)$ such that for each k the function $u(\cdot, \tilde{t}_k) - \tilde{\psi}$ has a multiple zero in $[\rho_0 - \delta, \rho_0 + \delta]$.

If $\tilde{\psi}(r)$ is bounded as $r \to \infty$ set $r_1 := \infty$. Note that in this case the limit $\eta := \tilde{\psi}(\infty)$ exists (cp. Lemma 2.2) and $\eta \neq \zeta$ since $(a_n, b_n) \notin S(\zeta, r_0)$. If $\tilde{\psi}(r)$ is not bounded as $r \to \infty$, pick $r_1 > \rho_0 + \delta$ such that $u(r_1, t) - \tilde{\psi}(r_1) \neq 0$ for all $t > 0$ (such r_1 exists as u is bounded).

Similarly, if $\tilde{\psi}(r)$ is bounded as $r \to 0$ set $\bar{r}_0 := 0$. Note that $\tilde{\psi}$ is then the solution of (2.1), (2.2) with $r_0 = 0$, $b = 0$, and $a := \tilde{\psi}(0+)$ (cp. Lemma 2.1). If $\tilde{\psi}(r)$ is not bounded as $r \to 0$, pick $\bar{r}_0 < \rho_0 - \delta$ such that $u(\bar{r}_0, t) - \psi(\bar{r}_0) \neq 0$ for all $t > 0$.

Let $I := [r_0, r_1]$ if $r_1 < \infty$ and $I := [r_0, \infty)$ if $r_1 = \infty$.

Observe that the zero number $z_I(u(\cdot, t) - \psi)$ is finite for $t > 0$. This is obvious if $r_1 < \infty$ (cp. Lemma 2.3(i)); in the case $r_1 = \infty$, it follows from Lemma 2.3(i) and the facts that the limits $\zeta = u(\infty, t)$ and $\eta = \psi(\infty)$

are different. By the monotonicity, $z_I(u(\cdot,t)-\psi)$ can drop only finitely many times as t increases in $[1,\infty)$; but, on the other hand, Lemma 2.3(iii) implies that it does drop at $t=\tilde t_1,\tilde t_2,\dots$. Thus, we have obtained a desired contradiction, showing that each $\varphi\in\omega(u)$ is a steady state. $\square$

To prove Theorem 1.2, it is now sufficient to show that $S(\zeta,r_0)$ has empty interior if $f'(u)\le 0$ on a neighborhood of ζ. In the following lemma, we derive this result from the maximum principle. We remark that under the stronger assumption $f'(\zeta)<0$, one can also use results of [25, 51] to obtain a more precise description (a manifold structure) of $S(\zeta,r_0)$.

Lemma 3.4. *Assume that $f(\zeta)=0$ and there is $\delta>0$ such that $f'(u)\le 0$ for all $u\in[\zeta-\delta,\zeta+\delta]$. Given any $r_0>0$, the set $S(\zeta,r_0)\subset\mathbb{R}^2$ has empty interior.*

Proof. We show that the set $S(\zeta,r_0)$ is covered by a countable union of nowhere dense sets. The desired conclusion then follows immediately from the Baire category theorem.

To simplify the notation, we assume that $\zeta=0$. This is at no cost to generality, just replace $f(u)$ by $f(u+\zeta)$.

With $\delta>0$ as in the assumption on f and $n=1,2,\dots$, define

$$T_n:=\{(\alpha,\beta)\in S(0,n):|(\psi(r;n,\alpha,\beta),\psi_r(r;n,\alpha,\beta))|\le\delta\quad(r\ge n)\}.$$

Clearly, if $(a_0,b_0)\in S(0,r_0)$, then for sufficiently large n we have

$$(\psi(n;r_0,a_0,b_0),\psi_r(n;r_0,a_0,b_0))\in T_n.$$

This means that (a_0,b_0) belongs to the preimage, denoted by Q_n, of the set T_n under the map

$$(a,b)\mapsto(\psi(n;r_0,a,b),\psi_r(n;r_0,a,b)):\mathbb{R}^2\to\mathbb{R}^2. \tag{3.4}$$

Note that this is the time $n-r_0$ map (with the initial time r_0) of the first order system corresponding to equation (2.1). By well-known results in ordinary differential equations, this maps is a diffeomorphism. Therefore Q_n is nowhere dense if T_n is such. We will prove the latter momentarily. Since the sets Q_n, $n=1,2,\dots$ cover $S(0,r_0)$, we will then be done.

We now show that for each n the set T_n is nowhere dense, that is, its closure $\bar T_n$ has empty interior. This clearly follows from the following claim: if (α,β), $(\alpha,\tilde\beta)$ are elements of $\bar T_n$ (with the same α), then $\beta=\tilde\beta$.

To prove the claim, let $\{(\alpha_k, \beta_k)\}$, $\{(\tilde{\alpha}_k, \tilde{\beta}_k)\}$ be sequences in T_n converging to (α, β), $(\alpha, \tilde{\beta})$, respectively. For any k, consider the solutions

$$\psi_k(r) := \psi(r; n, \alpha_k, \beta_k), \quad \tilde{\psi}_k(r) := \psi(r; n, \tilde{\alpha}_k, \tilde{\beta}_k) \tag{3.5}$$

of (2.1). The definition of T_n yields

$$\psi_k(\infty) = 0 = \tilde{\psi}_k(\infty), \qquad |\psi_k(r)|,\ |\tilde{\psi}_k(r)| \leq \delta \quad (r \geq n). \tag{3.6}$$

The difference $v := \psi_k - \tilde{\psi}_k$ satisfies the linear equation

$$v_{rr} + \frac{N-1}{r} v_r + c(r) v = 0, \quad r > 0, \tag{3.7}$$

where

$$c(r) = \int_0^1 f'(s\psi_k(r) + (1-s)\tilde{\psi}_k(r))\, ds.$$

Note that $c(r) \leq 0$ for all $r \in [n, \infty)$ due to (3.6) and the assumption on f. Since $v(\infty) = 0$, we deduce from the maximum principle that

$$|\psi_k(r) - \tilde{\psi}_k(r)| \leq |\psi_k(n) - \tilde{\psi}_k(n)| = |\alpha_k - \tilde{\alpha}_k| \quad (r \geq n).$$

Since the sequences $\{\alpha_k\}$, $\{\tilde{\alpha}_k\}$ both converge to α, we obtain

$$|\psi_k(r) - \tilde{\psi}_k(r)| \to 0 \quad (r \geq n). \tag{3.8}$$

On the other hand, by the continuity of solutions with respect to initial data,

$$\begin{aligned} \psi_k(r) &= \psi(r; n, \alpha_k, \beta_k) \to \psi(r; n, \alpha, \beta), \\ \tilde{\psi}_k(r) &= \psi(r; n, \tilde{\alpha}_k, \tilde{\beta}_k) \to \psi(r; n, \alpha, \tilde{\beta}). \end{aligned}$$

By (3.8), these limits are equal, which gives $\beta = \tilde{\beta}$, proving our claim. $\square$

The proof of of Theorem 1.2 can now be completed.

Proof of Theorem 1.2. The theorem follows directly from Proposition 3.3 and Lemma 3.4. $\square$

By Theorem 1.2, $\omega(u)$ is a connected and compact set of radial steady states (the connectedness and compactness refer to the topology of $L^\infty_{loc}(\mathbb{R}^N)$).

We conclude this section with an additional information on these steady states. Let
$$\mathcal{T} := \{\psi(0) : \psi \in \omega(u)\}.$$
Denoting by $\psi(r, a)$ the solution of (2.1), (2.2) with $r_0 = 0$ and $b = 0$, we have
$$\omega(u) = \{\psi(\cdot, a) : a \in \mathcal{T}\}. \tag{3.9}$$
Since $\mathcal{T}$ is clearly compact and connected, it is a singleton or a nontrivial compact interval. The following proposition concerns the latter.

Proposition 3.5. *If $a \in \operatorname{int} \mathcal{T}$, then the following relations are valid*
$$z(u(\cdot, t) - \psi(\cdot, a)) = \infty \quad (t > 0), \tag{3.10}$$
$$\psi(\infty, a) := \lim_{r\to\infty} \psi(r, a) = \zeta. \tag{3.11}$$

As already elucidated in the introduction, if for some f and ζ one can rule out the existence of an interval of values $a \in \mathbb{R}$ for which (3.11) holds, $\mathcal{T}$ and $\omega(u)$ consist of a single element. Theorem 1.2 then becomes a convergence theorem.

Proof of Proposition 3.5. By the definition of $\mathcal{T}$, from $a \in \operatorname{int} \mathcal{T}$ it follows that there is an increasing sequence $\bar{t}_n$ in $(0, \infty)$ such that $\bar{t}_n \to \infty$ and $u(0, \bar{t}_n) = a$. Consider the function $v(r, t) := u(r, t) - \psi(r, a)$, which is a nontrivial radial solution of a linear equation (2.5) with $r_1 = \infty$. For each n, $r = 0$ is a zero of $v(\cdot, \bar{t}_n)$, automatically a multiple zero due to the symmetry. By the monotonicity and diminishing properties of the zero number (see Lemma 2.3), the existence of such an infinite sequence $\bar{t}_n$ is possible only if (3.10) holds.

Now, being an element of $\omega(u)$, the function $\psi(\cdot, a)$ is bounded. By Lemma 2.2, the limit $\psi(\infty, a) \in \mathbb{R}$ exists. Since the zeros of $u(\cdot, t) - \psi(\cdot, a)$ are isolated and $u(\infty, t) = \zeta$ (cp. Proposition 3.2), (3.10) implies (3.11). □

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