# Zeros of complex caloric functions and singularities of complex viscous Burgers equation

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**Abstract**

We show that the viscous Burgers equation $u_t + uu_x = u_{xx}$ considered for complex valued functions $u$ develops finite-time singularities from compactly supported smooth data. By means of the Cole-Hopf transformation, the singularities of $u$ are related to zeros of complex-valued solutions $v$ of the heat equation $v_t = v_{xx}$. We prove that such zeros are isolated if they are not present in the initial data.

## 1 Introduction

In a recent paper Sinai and Li [8] consider the initial-value problem for the three-dimensional incompressible Navier-Stokes equation, allowing the velocity field and the pressure to be complex-valued. They prove that, in this setting, there exist well-behaved (complex-valued) initial data for which the solution blows up in finite time. In this note we consider a similar problem for the viscous 1D Burgers equation

$$u_t + uu_x = u_{xx} \tag{1.1}$$

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in $\mathbb{R} \times (0, \infty)$ with initial condition $u(x,0) = u_0(x)$, where we allow $u_0$ to be complex-valued. A well-known fact about equation (1.1) is that the transformation $u = -2v_x/v$, called the Cole-Hopf transformation, leads to standard heat equation $v_t = v_{xx}$ for $v$. The singularities of $u$ correspond to the zeros of $v$. For real valued functions, $v$ cannot have zeros if they are not present in $v(x,0)$ and one sees immediately that for $u_0$ real and "sufficiently regular" the initial value problem for equation (1.1) has a unique smooth global solution (in some natural classes of functions), see [5]. This can, of course, also be seen without the use of the Cole-Hopf transformation, in a number of ways, since the equation (1.1) has a maximum principle and an energy estimate with respect to which it is subcritical. (The non-trivial scaling invariance of equation (1.1) is the same as for Navier-Stokes: $u(x,t) \to \lambda u(\lambda x, \lambda^2 t)$.)

The maximum principle and the energy estimates are lost when we pass to complex-valued functions. At the same time, existence proofs based on perturbation theory and Picard iteration, such as in [6] or [7], work also in the complex case, and it is therefore natural to expect that the proofs of local well-posedness for Navier-Stokes in critical (i. e. scale-invariant) spaces work also for equation (1.1), without using its "complete integrability". One can therefore expect local well-posedness of complex-valued equation (1.1) in $L^1$ (by analogy with [6]) and, in fact, in $(BMO)^{-1}$ (by analogy with [7]). With the Cole-Hopf transformation, the local $L^1$ well-posedness becomes completely transparent (see below). It it not quite so with the local $(BMO)^{-1}$ well-posedness, which shows the subtle nature of the well-posedness result in [7]. Our focus will be on global well-posedness, and therefore we will work with the $L^1$ space which is very simple and - as we will see - completely adequate for the problems we will consider.

Since the zeros of $v$ produce singularities of $u$, it is easy to find compactly supported smooth (complex-valued) initial data $u_0$ for which the solution of equation (1.1) blows up in finite time, see Proposition 2.2. The continuity argument used in the proof of this proposition allows us to formulate more general sufficient conditions on $u_0$ for the solution $u$ to blow up, see Remark 2.3(i). On the other hand, in Proposition 2.1 we give a sufficient condition on $u_0$ for the solution to converge to zero. Using these results, we can then explicitly describe the boundary, in some subsets of the space of initial data $L^1$, between the basin of attraction of zero and the region from which the solutions blow-up, see Remark 2.5(iii). The behavior of solutions with the initial conditions on this boundary is then naturally of interest. These solutions are global and bounded and we describe their asymptotics as $t \to \infty$, see Proposition 2.4 and Remark 2.5(iii).

Once we know that a solution can develop singularities, we can ask about the nature of the singular set. We will prove that, roughly speaking, if there are no singularities present in the initial data, then the set of singularities of the function $u$ defined by the Cole-Hopf transformation from $v$ (that is, $u = -2v_x/v$) is always discrete in $\mathbb{R} \times (0, \infty)$. This follows from a theorem about zeros of complex-valued solutions of 1d heat equation (Theorem 3.3). In a "typical situation" the number of singularities of such a solution $u$ will be finite. However, as we show in Section 5, certain regular initial data yield solutions with infinitely many (isolated) singularities. We will also briefly address the question what "right-hand side" the singularities produce in a suitable weak formulation of the equation (see Section 4).

The solution of equation (1.1) defined by $u = -2v_x/v$ is analytic outside a discrete set. This is not a typical behavior of solutions of non-linear parabolic equations with singularities. In fact, it is reasonable to expect that, for many equations, analyticity in the time variable will be destroyed in the whole time level $\{(x,t), t = t_0\}$ if we have a singularity at time $t_0$ at some point $x_0$. This conjecture is based on the study of singularities of the Complex Ginzburg-Landau equation in [9]. As far as we know, the issue has not been much studied.

In the case of the dispersive regularization of Burgers equation, which is the KdV equation $u_t + uu_x = u_{xxx}$, the singularities for complex-valued solutions are studied in [2]. Viscous Burgers equation with complex viscosity is studied by means of the Cole-Hopf transformation in [10] and [11] for a particular real initial condition, with the main focus on the behavior of singularities arising in complex time.

# 2 Cole-Hopf transformation and singularities

For a complex-valued $u \in L^1(\mathbb{R})$ we define $U(x) = \int_{-\infty}^{x} u(\xi)\, d\xi$ and $v = \exp(-U/2)$. Vice-versa, given a complex-valued $v \in W_0^{1,1}(\mathbb{R})$ (the space of all absolutely continuous functions that have the derivative in $L^1(\mathbb{R})$) with $v(x) \neq 0$ in $\mathbb{R}$ and $v(x) \to 1$ as $x \to -\infty$, we let $u = -2v_x/v$. For time-dependent functions on $\mathbb{R}$ we apply the above transformations at each time level.

A well known simple calculation shows that $u$ satisfies equation (1.1) if and only if $v$ satisfies the standard heat equation $v_t = v_{xx}$, see for example

[5]. (If one does not impose the normalization $v(x) \to 1$ as $x \to -\infty$, the function $v$ is only determined up to a multiplicative factor depending on time, and the heat equation for $v$ needs an extra term which would account for this, see [5].)

We can now solve the initial value problem for equation (1.1) with a complex valued $u_0 \in L^1(\mathbb{R})$ as follows. Set $v_0(x) = \exp\{-\frac{1}{2}\int_{-\infty}^{x} u_0(\xi)\,d\xi\}$ and let $v$ be the bounded solution of the heat equation with initial data $v_0$. It is easy to check that there is $T > 0$ such that $|v(x,t)| \geq \varepsilon > 0$ in $\mathbb{R} \times (0,T)$ and hence $u = -2v_x/v$ is a well-defined local-in-time solution of equation (1.1) with $u(x,0) = u_0(x)$.

**Proposition 2.1.** *In the notation above, assume that $u_0 \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} |\operatorname{Im} u_0| \leq 2\pi$. Then equation (1.1) has a global smooth solution $u$ with $u(x,0) = u_0(x)$. If in addition $|\int_{\mathbb{R}} \operatorname{Im} u_0| < 2\pi$, then $\sup_x |u(x,t)| \to 0$ as $t \to \infty$.*

**Proof.** When $\int_{\mathbb{R}} |\operatorname{Im} u_0| \leq 2\pi$, the function $v_0(x) = \exp\{-\frac{1}{2}\int_{-\infty}^{x} u_0(\xi)\,d\xi\}$ takes values in a convex sector of the form $\{z,\ \alpha \leq \arg(z) \leq \beta\}$ with $\beta - \alpha \leq \pi$ and it has finite nonzero limits as $x \to \pm\infty$. Thus for suitable real $\theta$ we have $\operatorname{Re}(e^{i\theta} v_0(x)) \geq 0$, $x \in \mathbb{R}$, with the strict inequality at some $x$. Applying the strong maximum principle to $\operatorname{Re}(e^{i\theta} v)$ (which solves the heat equation), we see that $\operatorname{Re}(e^{i\theta} v) > 0$. Therefore $v$ cannot vanish at any point $(x,t) \in \mathbb{R} \times (0,\infty)$, proving the first statement. If $|\int_{\mathbb{R}} \operatorname{Im} u_0| < 2\pi$, we can choose $\theta$ so that $\operatorname{Re}(e^{i\theta} v_0(\pm\infty)) \geq \varepsilon_1 > 0$, and therefore for all large $t$ we will have $|v(x,t)| > \varepsilon_1/2$. Moreover, since $v_x$ also solves the heat equation and $v_{0,x} \in L^1(\mathbb{R})$, we have $\sup_x |v_x(x,t)| \to 0$ as $t \to \infty$. These properties imply the second statement. $\square$

**Proposition 2.2.** *For each $\delta > 0$ there exists a smooth, compactly supported (complex-valued) $u_0$ with $\int_{\mathbb{R}} |u_0| < 2\pi + \delta$ such that the solution of equation (1.1) with initial condition $u_0$ blows up in finite time.*

**Proof.** We choose a smooth compactly supported non-negative $\varphi$ with $\int_{\mathbb{R}} \varphi = 2\pi + \delta/2$. Set $u_0 = -i\varphi$ and let $v_0$ be the Cole-Hopf transformation of $u_0$. The function $v_0$ satisfies:

- $v_0(x) = 1$ for large negative $x$,
- $v_0(x) = \exp(i(\pi + \delta/4))$ for large positive $x$,
- $0 \leq \arg(v_0(x)) \leq \pi + \delta/4$ for $x \in \mathbb{R}$.

Let $v$ be the solution of the heat equation with initial data $v_0$. For $\theta \in \mathbb{R}$ the function $e^{-i\theta}v$ also solves the heat equation and choosing $\theta > 0$ small enough we achieve that

$$\lim_{x\to\pm\infty} \operatorname{Im}(e^{-i\theta}v_0(x)) < 0.$$

It then follows that for a sufficiently large $t_0 > 0$ we have $\operatorname{Im}(e^{-i\theta}v(x,t_0)) < 0$ for all $x \in \mathbb{R}$. Since the limits of $e^{-i\theta}v(x,t)$ as $x \to \pm\infty$ are independent of $t$, comparing the trajectories of $x \mapsto e^{-i\theta}v(x,t)$ for $t = 0$ and $t = t_0$ we conclude that $v$ has to vanish at some point $(x_1,t_1)$ with $t_1 \in (0,t_0)$. Consequently $u$ has a singularity at $(x_1,t_1)$. $\square$

**Remarks 2.3.** (i) The above continuity argument can also be used to show that $u$ develops a singularity whenever $u_0 \in L^1(\mathbb{R})$ satisfies $|\int_{\mathbb{R}} \operatorname{Im} u_0| > 2\pi$ and $\int_{\mathbb{R}} \operatorname{Im} u_0$ is *not* of the form $2\pi + 4k\pi$, where $k$ is an integer.

(ii) It is perhaps worth pointing out a very non-local behavior implied by Propositions 2.1, 2.2. Consider a compactly supported $u_0$ with $\operatorname{Im} u_0 \geq 0$ and $\int_{\mathbb{R}} \operatorname{Im} u_0 = \pi + \delta$ for some small $\delta > 0$. Then the solution of equation (1.1) with initial condition $u_0$ exists for all time and converges to zero. Consider now the initial condition $\tilde{u}_0^a(x) = u_0(x-a)+u_0(x+a)$. With initial condition $\tilde{u}_0^a$, the solution of (1.1) will blow up, no matter how large $a$ is. If we take $a$ very large, the solution will become very small in $L^\infty$ before it starts growing again and blows up. (In fact, it is not hard to see that one can replace $L^\infty$ by $L^p$ for a fixed $p > 1$ in the last sentence.)

**Proposition 2.4.** *Assume $u_0 \in L^1(\mathbb{R})$ is compactly supported, with $|\int_{\mathbb{R}} \operatorname{Im} u_0| = \int_{\mathbb{R}} |\operatorname{Im} u_0| = 2\pi$. Then there exist a real $y_\alpha$ and a complex $\beta$ with $\operatorname{Im}\beta \neq 0$ such that the solution $u$ of equation* (1.1) *with $u(x,0) = u_0(x)$ satisfies*

$$u(x,t) = \frac{-2}{(x - y_\alpha\sqrt{2t}) + \beta} + O(\frac{1}{\sqrt{t}}), \quad (t \to \infty) \tag{2.1}$$

*uniformly in $x \in \mathbb{R}$.*

**Proof.** By Proposition 2.1, the solution $u$ is global. Let $U_0(x) = \int_{-\infty}^x u_0(\xi)\,d\xi$ and $v_0(x) = \exp(-U_0(x)/2)$. Let $L > 0$ be such that $u_0$ vanishes outside $[-L,L]$. We have $v_0(x) = 1$ on $(-\infty,-L)$ and $v_0(x) = -\exp(-I/2)$ on $(L,\infty)$, where $I = \int_R \operatorname{Re} u_0$. We set $F(x) = (2\pi)^{-1/2}\int_0^x \exp(-\xi^2/2)\,d\xi$. (In terms of the erf function used in probability we can write $F(x) = \operatorname{erf}(x) - 1/2$.) We note that the function $F(\frac{x}{\sqrt{2t}})$ solves the heat equation in $\mathbb{R}\times(0,\infty)$ with the initial data $\frac{1}{2}\operatorname{sign}(x)$. The fundamental solution $\Gamma(x,t)$

of the heat equation can be written as $\Gamma(x,t) = \frac{1}{\sqrt{2t}} F'(\frac{x}{\sqrt{2t}})$. We write the initial condition $v_0$ in the form

$$v_0(x) = -a\,\frac{\mathrm{sign}\,(x)}{2} + b + w_0(x) \tag{2.2}$$

where $a, b$ are chosen so that $w_0$ be compactly supported. This gives $a = 1+\exp(-I/2)$ and $b = (1-\exp(-I/2))/2$. Also note that $\mathrm{Im}\, w_0$ is continuous and does not change sign. Let $w(x,t)$ be the solution of the heat equation in $\mathbb{R}\times(0,\infty)$ with initial condition $w_0$. The solution with the initial condition $v_0$ is then

$$v(x,t) = -aF(\frac{x}{\sqrt{2t}}) + b + w(x,t)\,. \tag{2.3}$$

From the representation formula $w(x,t) = \int_{\mathbb{R}} w_0(y)\Gamma(x-y,t)\,dy$ we see that

$$|w(x,t)| + \sqrt{t}\,|w_x(x,t)| = O(\frac{1}{\sqrt{t}}), \qquad (t\to\infty) \tag{2.4}$$

uniformly in $x$. To get further estimates for $w$ in a simple way, we will use Appell's transformation and write

$$w(x,t) = \Gamma(x,t)\,\tilde{w}(\frac{x}{t}, -\frac{1}{t})\,, \tag{2.5}$$

where $\tilde{w}$ is a function of $\tilde{x} = x/t$ and $\tilde{t} = -1/t$ (defined for $(\tilde{x},\tilde{t}) \in \mathbb{R}\times(-\infty,0)$) which again satisfies the heat equation in $\tilde{x}$ and $\tilde{t}$. We have

$$\tilde{w}(\tilde{x},\tilde{t}) = \int_{\mathbb{R}} w_0(y)\frac{\Gamma(x-y,t)}{\Gamma(x,t)}\,dy = \int_{\mathbb{R}} w_0(y)\exp(-\frac{\tilde{x}y}{2} + \frac{\tilde{t}y^2}{4})\,dy\,, \tag{2.6}$$

which shows that $\tilde{w}$ can be analytically extended to $\mathbb{R}\times\mathbb{R}$. Letting $c = \int_{\mathbb{R}} w_0$, we see from (2.6) that $\tilde{w}(0,0) = c$. Setting $z = \tilde{w} - c$, we can write

$$v(x,t) = -a\,F(\frac{x}{\sqrt{2t}}) + b + c\,\Gamma(x,t) + z(\frac{x}{t}, -\frac{1}{t})\,\Gamma(x,t)\,, \tag{2.7}$$

where $z$ is smooth in $\mathbb{R}\times(-\infty,0]$ and $z(0,0) = 0$. We now set $\alpha = b/a$ and $\beta = -c/a$. Observe that $\alpha \in (-1/2, 1/2)$ and $\mathrm{Im}\,\beta \neq 0$. From (2.7) we get

$$u = -2\frac{v_x}{v} = -2\frac{1 + \beta\frac{\Gamma_x}{\Gamma} - \frac{z}{a}\frac{\Gamma_x}{\Gamma} - \frac{z_{\tilde{x}}}{a}\frac{1}{t}}{\frac{F-\alpha}{\Gamma} + \beta - \frac{z}{a}}\,, \tag{2.8}$$

where $F$ is evaluated at $x/\sqrt{2t}$. Given any fixed $R > 0$ and $\tilde{t}_0 < 0$ we can see that for $|\tilde{x}| < R$, $\tilde{t} > \tilde{t}_0$ we have $|z_{\tilde{x}}| \le c_1$, $|z| \le c_2|\tilde{x}| + c_3|\tilde{t}|$. Together with (2.8), and after taking into account that $\operatorname{Im}\beta \neq 0$, this gives

$$u = -2\frac{1 + O(\frac{1}{\sqrt{t}})}{\frac{F-\alpha}{\Gamma} + \beta + O(\frac{1}{\sqrt{t}})} = \frac{-2}{\frac{F-\alpha}{\Gamma} + \beta} + O(\frac{1}{\sqrt{t}}), \quad (t \to \infty) \tag{2.9}$$

uniformly in regions $\{(x,t);\ |x|/\sqrt{t} \le R\}$, where $F$ is again evaluated at $x/\sqrt{2t}$.

Let $y_\alpha$ be the unique root of the equation $F(y) = \alpha$ and let us fix a (small) $\delta_1 > 0$. We note that when $|x/\sqrt{2t} - y_\alpha| \ge \delta_1$, then $|aF(x/\sqrt{2t}) - b| \ge \varepsilon_1 > 0$, and from (2.3), (2.4) we see that, in the region $\{(x,t);\ |x/\sqrt{2t}\ - y_\alpha| \ge \delta_1\}$, one has

$$u = -2\frac{v_x}{v} = O(\frac{1}{\sqrt{t}}), \quad (t \to \infty),$$

uniformly in $x$ (for $(x,t)$ in the above region). Taking into account (2.9), we see that it only remains to show that

$$\frac{-2}{\frac{F-\alpha}{\Gamma} + \beta} = \frac{-2}{(x - y_\alpha\sqrt{2t}) + \beta} + O(\frac{1}{\sqrt{t}}), \quad (t \to \infty) \tag{2.10}$$

uniformly in $\{(x,t);\ |x/\sqrt{2t} - y_\alpha| \le \delta_1\}$. (As above, $F$ is evaluated at $x/\sqrt{2t}$.) For $y \in \mathbb{R}$ we set

$$\kappa(y) = \frac{F(y) - F(y_\alpha)}{F'(y)(y - y_\alpha)}$$

with the understanding that $\kappa(y_\alpha) = 1$. Since $\Gamma(x,t) = \frac{1}{\sqrt{2t}}F'(\frac{x}{\sqrt{2t}})$, we have

$$\frac{-2}{\frac{F(\frac{x}{\sqrt{2t}})-\alpha}{\Gamma(x,t)} + \beta} = \frac{-2}{\sqrt{2t}\frac{F(\frac{x}{\sqrt{2t}})-F(y_\alpha)}{F'(\frac{x}{\sqrt{2t}})} + \beta} = \frac{-2}{\kappa(\frac{x}{\sqrt{2t}})(x - y_\alpha\sqrt{2t}) + \beta} \tag{2.11}$$

Clearly

$$\kappa(y) = 1 + O(|y - y_\alpha|), \quad (y \to y_\alpha). \tag{2.12}$$

Using the elementary formula

$$\frac{1}{\kappa\xi + \beta} - \frac{1}{\xi + \beta} = \int_0^1 \frac{\xi(1-\kappa)}{(\xi[(1-s) + s\kappa] + \beta)^2}\, ds$$

we see that (2.10) follows from (2.11) and (2.12). $\square$

**Remarks 2.5.** (i) We note that, with the notation used in the proof, we have $\operatorname{Im}\beta = \frac{1}{a}\int_{\mathbb{R}} \operatorname{Im} w_0$. Under the assumptions of Proposition 2.4 the function $\operatorname{Im} w_0$ does not change sign and is integrable. If we do not assume that $u_0$ is compactly supported, it can easily happen that $|\int_{\mathbb{R}} \operatorname{Im} w_0| = +\infty$. We then have $|\operatorname{Im}\beta| = +\infty$ and in view of (2.1) it is natural to expect that in that case $u(x,t) \to 0$ as $t \to \infty$ uniformly in $x$. This can indeed be proved. If $u_0$ is not compactly supported, but $\int \operatorname{Im} w_0$ is finite, we expect that the asymptotics of $u(x,t)$ will be similar to (2.1), with perhaps a slower rate of convergence.

(ii) The constant $y_\alpha$ is given by the equation $F(y_\alpha) = \frac{1}{2}\tanh(\frac{I}{4})$, with $I = \int_{\mathbb{R}} \operatorname{Re} u_0$. In particular, $y_\alpha = 0$ if and only if $\int_{\mathbb{R}} \operatorname{Re} u_0 = 0$. We note that $\frac{-2}{x+\beta}$ is a steady-state solution of equation (1.1). For $I \neq 0$ we have $\frac{d}{dt}(y_\alpha\sqrt{2t}) = \frac{y_\alpha}{\sqrt{2t}} \neq 0$, and we can interpret formula (2.1) as an "almost steady state solution", which is slowly drifting to $\pm\infty$, at speed $\frac{y_\alpha}{\sqrt{2t}}$.

(iii) Consider a complex valued $L^1$ function $\tilde{u}_0$ supported in $[-L, L]$, with $\int_{\mathbb{R}} |\operatorname{Im}\tilde{u}_0| = \int_{\mathbb{R}} \operatorname{Im}\tilde{u}_0 = 2\pi$. Let $\mathcal{O}(\tilde{u}_0, L, \varepsilon) = \{u_0 \in L^1(\mathbb{R}) : ||u_0 - \tilde{u}_0||_{L^1} < \varepsilon,\ u_0 \text{ is supported in } [-L, L]\}$. From the above one can see that for sufficiently small $\varepsilon$, one has, in the set $\mathcal{O}(\tilde{u}_0, L, \varepsilon)$, an explicit description of the boundary between the basin of attraction of the zero solution of equation (1.1) and the region from which the solutions of equation (1.1) blow up is finite time: The boundary (in $\mathcal{O}(\tilde{u}_0, L, \varepsilon)$) is given by the equation $\int_{\mathbb{R}} \operatorname{Im} u_0 = 2\pi$. (To be precise, for the proof of this one needs to augment the above propositions by a slightly modified version of Proposition 2.1, in which we assume that $u_0$ is in $\mathcal{O}(\tilde{u}_0, L, \varepsilon)$, $\int_{\mathbb{R}} \operatorname{Im} u_0 < 2\pi$, and we allow $\int_{\mathbb{R}} |\operatorname{Im} u_0| \leq 2\pi + \delta$, where $\delta = \delta(L, \varepsilon, \tilde{u}_0) > 0$ is sufficiently small. The restriction on the support of $u_0$ is crucial in this step. We leave the details to the reader.) It is not hard to check that the large-time asymptotics of the solutions starting at the boundary (in $\mathcal{O}(\tilde{u}_0, L, \varepsilon)$) of the basin of attraction of the zero solution is given by the solutions described in Proposition 2.4. If we replace $\mathcal{O}(\tilde{u}_0, L, \varepsilon)$ by $\mathcal{O}(\tilde{u}_0, \infty, \varepsilon)$ (i. e. we remove the restriction on the support of the perturbed function), the situation changes and the boundary is no longer described in a simple way. In addition, even when $\varepsilon$ is small, we expect that for $\mathcal{O}(\tilde{u}_0, \infty, \varepsilon)$ some solutions at the boundary of the basin of attraction of zero will have more complicated behavior, such as slow oscillations with large amplitude.

## 3 Nodal sets of caloric functions

Let $u$ be a bounded real-valued nontrivial solution of the heat equation $u_t = u_{xx}$ in $\mathbb{R} \times (0, \infty)$. We define

$$\begin{array}{lcl} Z & = & \{(x,t) \in \mathbb{R} \times (0,\infty);\ u(x,t) = 0\}\,, \\ Z_{\mathrm{reg}} & = & \{(x,t) \in Z;\ u_t^2 + u_x^2 \neq 0\}\,, \quad \text{and} \\ Z_{\mathrm{sing}} & = & Z \setminus Z_{\mathrm{reg}}\,. \end{array}$$

The analyticity of $u$ implies that $Z_{\mathrm{sing}}$ is discrete and that $Z$ is locally a regular real analytic curve in a neighborhood of each point $(x_0, t_0)$ in $Z_{\mathrm{reg}}$. By a regular (real) analytic curve $C$ in an open set $U \subset \mathbb{R} \times (0, \infty)$ we mean a one-dimensional analytic (imbedded) submanifold of $U$ with $\bar{C} \setminus C \subset \partial U$.

**Lemma 3.1.** *In the notation introduced above, the regular analytic curves describing $Z$ in a neighborhood of $(x_0, t_0) \in Z_{\mathrm{reg}}$ can be analytically continued through the points of $Z_{\mathrm{sing}}$. In other words, $Z$ is a (locally finite) union of regular analytic curves in $\mathbb{R} \times (0, \infty)$.*

**Proof.** We first recall some facts about caloric polynomials. As usual in the parabolic setting we say that a function $f(x,t)$ is parabolically $m-$homogeneous if $f(\lambda x, \lambda^2 t) = \lambda^m f(x,t)$ for $\lambda > 0$. The $m-$th caloric polynomial is a parabolically $m-$homogeneous polynomial satisfying the heat equation. It is unique, modulo a multiplicative factor, and can be given for example by

$$P_m(x,t) = \sum_{k=0}^{[m/2]} \frac{m!}{k!(m-2k)!}\, x^{m-2k} t^k. \tag{3.1}$$

The polynomial $P_m(x, -1)$ is the $m-$th Hermite polynomial. For $m$ even, $m = 2k$, the polynomial $P_m$ is of the form

$$P_m(x,t) = (x^2 + a_1 t) \cdots (x^2 + a_k t), \tag{3.2}$$

with $0 < a_1 < \cdots < a_k$. For $m$ odd, $m = 2k+1$, $P_m$ is of the form

$$P_m(x,t) = x(x^2 + a_1 t) \cdots (x^2 + a_k t), \tag{3.3}$$

with $0 < a_1 < \cdots < a_k$. (The $a_j$'s may be different for different $m$, of course.)

In a neighborhood of a point $(x_0, t_0) \in Z_{\mathrm{sing}}$ we can write $u$ as a convergent series

$$u(x,t) = a_m P_m(x - x_0, t - t_0) + a_{m+1} P_{m+1}(x - x_0, t - t_0) + \cdots \tag{3.4}$$

where $m \geq 3$, $a_m \neq 0$. In what follows we will assume that $m$ is odd, $m = 2k+1$. (The proof for $m$ even is similar and, in fact, easier.) We change coordinates so that $(x_0, t_0)$ corresponds to $(0,0)$ in the new coordinates, which we still denote $(x,t)$. We let $t = -y^2, a_1 = b_1^2, \cdots, a_k = b_k^2,\ b_j > 0$. The equation $u(x,t) = 0$ can be written as

$$u(x, -y^2) = x(x - b_1 y)(x + b_1 y) \cdots (x - b_k y)(x + b_k y) + R(x,y) = 0, \tag{3.5}$$

where $R(x,y)$ is analytic in a neighborhood of $(0,0)$ with vanishing derivatives of order $1, 2, \dots m$. Letting $b_0 = 0$, we will look for analytic curves $x = x(y)$ of the form $x(y) = by + y^2 f(y)$ (with $b = \pm b_j$, $j = 0, 1, \dots, k$) defined for small $y$ on which $u(x, -y^2)$ vanishes. Substituting the expression $x(y) = by + y^2 f(y)$ in equation (3.5), it is easy to check that we get an equation of the form

$$f(y) = F(y, f(y)) \tag{3.6}$$

where $F = F(y,f)$ is analytic in $f$ and depends on $f$ only through $yf$. Therefore $f_0 := F(0,f)$ is independent of $f$ and $\frac{\partial F}{\partial f}(0,f) = 0$. Applying the standard implicit function theorem one shows that equation (3.6) has an analytic solution $f$ defined on a neighborhood of 0 with $f(0) = f_0$.

Observe that with $y \neq 0$ fixed, the curves found above give $m$ different solutions of (3.5). Therefore, by the Weierstrass preparation theorem, they yield all solutions of (3.5) in a neighborhood of $(0,0)$. (One can also use the Malgrange preparation theorem.)

To finish the proof, we note that for $j \geq 1$, instead of writing $x = x(y)$ we can write $y = y(x)$ and the equation $t = -y^2 = -(y(x))^2$ then defines the analytic branch of $Z$ which has contact of the second order with the parabola $t = -x^2/a_j$. When $j = 0$ we note that the function in equation (3.6) is of the form $F(y,f) = \tilde{F}(-y^2, f)$, and hence the corresponding curve is of the form $x = x(t) = t\tilde{f}(t)$, with an analytic $\tilde{f}$. $\square$

**Remark 3.2.** It is clear that the proof of the lemma also works when the equation has lower-order terms and analytic coefficients. Although we did not find the precise statement of the lemma in the literature, we assume it is known to experts. For example, it follows easily from the analysis of nodal sets in [1], where the method of Newton polygons is used. Also, once the specific form of the caloric polynomials is taken into account, the lemma can be derived easily from general principles used in algebraic geometry for "desingularization". Nevertheless, we think that the elementary proof above is still of some interest and we have included it for completeness. We remark that even if we allow nonanalytic variable coefficients, $Z$ is still a finite union of regular $C^1$ curves in a neighborhood of any point in $Z_{\text{sing}}$, see [3].

**Theorem 3.3.** *Let $v$ be a bounded complex-valued solution of the heat equation in $\mathbb{R} \times (0, \infty)$. Assume $v$ has no zeros in some neighborhood of $\mathbb{R} \times \{0\}$. Then all zeros of $v$ in $\mathbb{R} \times (0, \infty)$ are isolated.*

**Proof.** Let $v = v_1 + iv_2$, $Z_1 = \{v_1 = 0\}$, $Z_2 = \{v_2 = 0\}$. Assume $Z_1 \cap Z_2$ has an accumulation point $(x_0, t_0)$ inside $\mathbb{R} \times (0, \infty)$. By Lemma 3.1 we know that $Z_1$ is a locally finite union of regular analytic curves in $\mathbb{R} \times (0, \infty)$. Consider the curves passing through $(x_0, t_0)$. Clearly $v_2$ has infinitely many zeros accumulating at $(x_0, t_0)$ on one of the curves, let's call it $C$. By analyticity, $v_2$ vanishes on $C$. The curve $C$ cannot be closed, for otherwise the maximum principle would imply that both $v_1$ and $v_2$ vanish in the interior of $C$, which is impossible by our assumption and analyticity. Hence we can parametrize $C$ by a parameter $s \in (-\infty, \infty)$. Also, since $C$ is a regular analytic curve, we have $\bar{C} \setminus C \subset \mathbb{R} \times \{0\}$, thus our assumption implies that $\bar{C} \setminus C = \emptyset$. Now either the time coordinate $t$ has a strict local minimum on $C$ or we can choose the parametrization so that $t(s)$ is monotone nonincreasing for large $s$ and $x(s)$ approaches $\infty$ or $-\infty$ as $s \to \infty$. In either case, we find a (bounded or unbounded) domain in $\mathbb{R} \times (0, \infty)$ such that both functions $v_1$ and $v_2$ vanish on its parabolic boundary. Since they are bounded, the maximum principle [4] implies that they vanish on a nonempty open set, hence on $\mathbb{R} \times (0, \infty)$, and we again have a contradiction to our assumption.

□

It is clear that the proof of Theorem 3.3 works without much change also for complex-valued harmonic functions in a half-plane.

# 4 Additional comments on the singularities

Given complex-valued initial data $u_0 \in L^1(\mathbb{R})$ for equation 1.1 and constructing the solution $u$ of the initial-value problem by means of the Cole-Hopf transformation as in Section 2 by setting $u = -2v_x/v$, we see from Theorem 3.3 that the singularities of $u$ are isolated. It is natural to ask if equation (1.1) is satisfied in some weak sense across the singularities, or if the singularities introduce a non-trivial "right-hand side", i. e. we want to calculate the distribution $f$ given by $u_t + uu_x - u_{xx} = f$, where the left-hand side requires a suitable interpretation. Clearly $f$ should be supported in the singular set. Even if we write the operator $u_t + uu_x - u_{xx}$ as $u_t + (u^2/2)_x - u_{xx}$ the definition of $f$ can still be somewhat ambiguous, since $u$ and $u^2$ are not locally integrable in a neighborhood of a singularity. We suggest one possible interpretation. For simplicity we will consider only the simplest case when the function $v$ defining $u$ has a simple zero at the singularity $(x_0, t_0)$, i. e. $v(x,t) = a(x-x_0) + b(t-t_0) + O((x-x_0)^2 + (t-t_0)^2)$, with $a, b$ complex and linearly independent over $\mathbb{R}$. We consider a smooth test function $\varphi = \varphi(x,t)$ supported in a small neighborhood of $(x_0, t_0)$, so that no other singularity is present in the support of $\varphi$. We want to define

$$I = \int_{\mathbb{R}\times(0,\infty)} (-u\varphi_t - u^2\varphi_x/2 - u\varphi_{xx})\, dx\, dt. \tag{4.1}$$

For $t \neq t_0$ we let

$$h(t) = \int_{\mathbb{R}} (-u(x,t)\varphi_t(x,t) - u^2(x,t)\varphi_x(x,t)/2 - u(x,t)\varphi_{xx}(x,t))\, dx \tag{4.2}$$

Using well-known facts about the behavior of the distributions $1/(x \pm i\varepsilon)$ and $1/(x \pm i\varepsilon)^2$ as $\varepsilon \to 0$, together with a change of variables $v(x,t) = y$ (for a fixed $t$) one can see that $h(t)$ has one-sided limits as $t \to t_0$. Therefore it seems to be natural to define integral $I$ in expression (4.1) as

$$I = \int_0^\infty h(t)\, dt = \lim_{\tau\to 0} \big( \int_0^{t_0-\tau} h(t)\, dt + \int_{t_0+\tau}^\infty h(t)\, dt. \big) \tag{4.3}$$

Integration by parts, the equation satisfied by $u$, and the specific form of the singularity of $u$ now give

(4.4)
$$I = \lim_{\tau\to 0} \int_{\mathbb{R}} \big[ u(x, t_0+\tau)\varphi(x, t_0+\tau) - u(x, t_0-\tau)\varphi(x, t_0-\tau) \big]\, dx = \pm 4\pi i \varphi(x_0),$$

where the correct sign is the same as the sign of the imaginary part of $\bar{a}b$. In other words, in a neighborhood of $(x_0, t_0)$ we have, in some sense,

$$u_t + uu_x - u_{xx} = \pm 4\pi i \delta_{(x_0,t_0)}, \tag{4.5}$$

where we use the usual notation $\delta_{(x_0,t_0)}$ for the Dirac distribution at the point $(x_0, t_0)$. Therefore we cannot interpret $u$ as a global weak solution of equation (1.1).

## 5 Infinitely many singularities

If is not hard to show that, typically, the solution $u$ will only have finitely many singularities. In fact, one can check easily that a sufficient condition for $u$ to have only finitely many singularities is that $\int_{\mathbb{R}} \operatorname{Im} u_0(x)\, dx$ is *not* of the form $2\pi + 4k\pi$ with $k$ an integer. We now show that, on the other hand, there are solutions with regular initial data having infinitely many singularities. This is an immediate consequence of Proposition 5.1 below. We recall that we denote by $W_0^{1,1}(\mathbb{R})$ the space of all functions on $\mathbb{R}$ which are absolutely continuous with the derivative in $L^1(\mathbb{R})$. (In particular, constant functions belong to $W_0^{1,1}$.)

**Proposition 5.1.** *There exists a smooth (complex-valued) function $v_0 \in W_0^{1,1}(\mathbb{R})$ such that $v_0(-\infty) = 1$, $|v_0(x)| \geq \varepsilon_0 > 0$ for any $x \in \mathbb{R}$, and the solution $v$ of the heat equation with $v(\cdot, 0) = v_0$ vanishes at infinitely many points $(0, \tau_k)$, with $\tau_k \to \infty$.*

**Proof.** First choose a smooth real-valued odd function $w \in W_0^{1,1}(\mathbb{R})$ such that $w(-\infty) = 1$ and $w > 0$ on $(-\infty, 0)$. The solution $v_1$ of the heat equation with $v_1(\cdot, 0) = w$ has a unique zero at $x = 0$ for each $t$. We shall next find a smooth real-valued function $z \in W_0^{1,1}(\mathbb{R})$ such that $z(\pm\infty) = 0$, $z(0) \neq 0$, and the solution $v_2$ of the heat equation with $v_2(\cdot, 0) = z$ vanishes at points $(0, \tau_k)$ with $\tau_k \to \infty$. From this the conclusion of the proposition follows upon setting $v_0 = w + iz$.

We first choose sequences $R_k > 0$, $\epsilon_k \in (0,1)$ with the following properties:

(a1) $\sum_{k=1}^{\infty} \epsilon_k < \infty$,

(a2) $R_{k+1} > R_k$ and $R_{k+1}\epsilon_{k+1} > \sum_{j=1}^{k} \epsilon_j (R_j + 1) \quad (k = 1, 2 \ldots)$.

Next, for each $k$ we choose a smooth function $z_k$ such that

(a3) $z_k \equiv \epsilon_k$ on $[-R_k, R_k]$, $\quad z_k \equiv 0$ on $\mathbb{R} \setminus [-R_k - 1, R_k + 1]$,

(a4) $0 \leq z_k \leq \epsilon_k$ and $|z_k'| \leq 2\epsilon_k$ on $\mathbb{R}$.

We will show that if $x_k$ is a suitably chosen sequence, then the function

$$z(x) = \sum_{k=1}^{\infty} (-1)^{k+1} z_k(x - x_k) \tag{5.1}$$

has the desired properties.

The sequence $x_k$ will be constructed so that, in particular,

$$|x_j| > |x_{j-1}| + R_j + R_{j-1} + 2 \tag{5.2}$$

for $j = 2, 3 \dots$. This guarantees that the functions $z_k(\,\cdot - x_k)$ have non-overlapping supports, hence, by (a1), (a3), and (a4), $z$ is a smooth function in $W_0^{1,1}(\mathbb{R})$ satisfying $z(\pm\infty) = 0$. In addition to (5.2), we need to ensure that the function

$$v_2(0,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{\frac{-|y|^2}{4t}} z(y)\, dy \tag{5.3}$$

has infinitely many sign changes.

We shall recursively construct sequences $x_k$, $t_k$, $d_k$ such that $(-1)^{k+1}x_k$ $t_k$, $d_k$ are nonnegative and increasing with $k$ and the following statement is satisfied for each $k$:

($\mathrm{I}_k$) Relations (5.2) hold for $j = 2, \dots, k$ and, with any choice of $x_{k+1}, x_{k+2}, \dots$ satisfying (5.2) for $j = k+1, k+2, \dots$ and $|x_{k+1}| \geq d_k$, one has

$$(-1)^j \int_{\mathbb{R}} e^{\frac{-|y|^2}{4t_j}} z(y)\, dy < 0, \quad j = 1, \dots, k.$$

Take $x_1 = 0$, $t_1 = 1$. It is obvious that ($\mathrm{I}_1$) is satisfied if $d_1$ is sufficiently large. We fix such a $d_1$ satisfying also $d_1 \geq R_1 + R_2 + 2$.

Assume that $x_j$, $t_j$, $d_j$ have been constructed for $j = 1, \dots, k$. To define the next terms assume for definiteness that $k$ is odd (in case it is even, the construction is analogous). Set $x_{k+1} = -d_k$. Assuming $x_{k+2}, x_{k+3}, \dots,$ are any numbers satisfying (5.2) for $j = k+2, k+2, \dots$ and $|x_{k+2}| \geq d_{k+1}$, with $d_{k+1}$ to be specified below, we use (a3), (a4) to estimate

$$\begin{aligned}
\int_{\mathbb{R}} e^{\frac{-|y|^2}{4t}} z(y)\, dy &= \sum_{j=1}^{\infty} (-1)^{j+1} \int_{x_j - R_j - 1}^{x_j + R_j + 1} e^{\frac{-|y|^2}{4t}} z_j(y - x_j)\, dy \\
&\leq 2 \sum_{j=1}^{k} \epsilon_j (R_j + 1) - \epsilon_{k+1} \int_{x_{k+1} - R_{k+1}}^{x_{k+1} + R_{k+1}} e^{\frac{-|y|^2}{4t}}\, dy \qquad (5.4) \\
&\quad + \int_{\mathbb{R} \setminus [-d_{k+1} + R_{k+2} + 1\,,\, d_{k+1} - R_{k+2} - 1]} e^{\frac{-|y|^2}{4t}}\, dy. \qquad (5.5)
\end{aligned}$$

In view of (a2), if $t = t_{k+1} > t_k$ is large enough, the expression in (5.4) is negative. Fixing such a $t_{k+1}$ and subsequently choosing a large enough $d_{k+1} > d_k + R_{k+1} + R_k + 2$, we make the whole expression in (5.4), (5.5) negative. Hence $(\mathrm{I}_{k+1})$ is satisfied.

It is obvious that with sequences $x_k$ and $t_k$ resulting from the above construction, the function $z$ has all the desired properties. In particular, $v_2(0,t)$ has a zero $\tau_k$ in $(t_k, t_{k+1})$ for each $k$. $\square$

**Remarks 5.2.** (i) It is not difficult to check that with the initial data $z$ constructed above, the solution $v_2$ of the heat equation has a unique zero $x(t)$ for each $t > 0$ (and $x(t)$ changes sign infinitely many times as $t \to \infty$).

(ii) It is conceivable that if the initial condition $u_0$ is compactly supported, the solution $u$ of equation (1.1) given by the Cole-Hopf transformation will always have only finitely many singularities. For example, the construction above cannot be carried out if we demand that $z$ be compactly supported. This can be seen from the following observation: *If $z$ solves the heat equation in $\mathbb{R} \times (0,\infty)$ with compactly supported initial data and $t \to z(0,t)$ has infinitely many zeros, then $z(0,t) = 0$ for all $t > 0$.* To see this we recall that Appell's transformation $\tilde{z}$ of $z$ is defined in $\mathbb{R} \times (-\infty, 0)$ by $z(x,t) = \Gamma(x,t)\tilde{z}(x/t, -1/t)$. One can see from formula (2.6) applied to $z$ that $\tilde{z}$ has an analytic extension to $\mathbb{R} \times \mathbb{R}$ and the assumptions on $z$ imply that $\tilde{t} \to \tilde{z}(0,\tilde{t})$ has infinitely many roots accumulating at 0. Hence $\tilde{z}(0,\tilde{t}) = 0$ for all $\tilde{t}$, which implies our statement.

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