Threshold behavior and non-quasiconvergent solutions with localized initial data for bistable reaction-diffusion equations

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Dedicated to John Mallet-Paret
on the occasion of his 60th birthday

Abstract

Abstract. We consider bounded solutions of the semilinear heat equation $u_t = u_{xx} + f(u)$ on R, where f is of the unbalanced bistable type. We examine the ω-limit sets of bounded solutions with respect to the locally uniform convergence. Our goal is to show that even for solutions whose initial data vanish at $x = \pm\infty$, the ω-limit sets may contain functions which are not steady states. Previously, such examples were known for balanced bistable nonlinearities. The novelty of the present result is that it applies to a robust class of nonlinearities. Our proof is based on an analysis of threshold solutions for ordered families of initial data whose limits at infinity are not necessarily zeros of f.

Key words: bistable reaction-diffusion equation, localized initial data, asymptotic behavior, nonconvergent solutions, threshold solutions

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1 Introduction

Consider the Cauchy problem

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$
$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where $f \in C^1(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$.

Problem (1.1), (1.2) has a unique solution $u(\cdot, t, u_0)$ defined on a maximal time interval $(0, T(u_0))$. By a solution we mean the mild solution, as defined in [20, 21], for example. The solution is classical in $\mathbb{R} \times (0, T(u_0))$ and if u_0 is uniformly continuous, then also $u(\cdot, t, u_0) \to u_0$ in $L^\infty(\mathbb{R})$, as $t \to 0$. If u is bounded on $\mathbb{R} \times [0, T(u_0))$, then necessarily $T(u_0) = \infty$, that is, the solution is global, and, by standard parabolic regularity estimates, the trajectory $\{u(\cdot, t, u_0) : t \geq 1\}$ is relatively compact in $L^\infty_{loc}(\mathbb{R})$. In this paper we are concerned with the large-time behavior of bounded solutions in a localized topology. We thus introduce the ω-limit set of such a bounded solution u, denoted by $\omega(u)$ or by $\omega(u_0)$ if the initial datum of u is given, as follows:

$$\omega(u) := \{\varphi : u(\cdot, t_n) \to \varphi \text{ for some sequence } t_n \to \infty\}. \tag{1.3}$$

Here the convergence is in $L^\infty_{loc}(\mathbb{R})$ (the locally uniform convergence). Thus we consider the behavior of u, as $t \to \infty$, on arbitrarily large compact sets. The set $\omega(u)$ is nonempty, compact and connected in $L^\infty_{loc}(\mathbb{R})$, and it attracts the solution in the following sense:

$$\operatorname{dist}_{L^\infty_{loc}(\mathbb{R})}(u(\cdot, t), \omega(u)) \to 0 \text{ as } t \to \infty. \tag{1.4}$$

Here we view $L^\infty_{loc}(\mathbb{R})$ as a metrizable locally convex space with the topology and metric derived in the usual way from the countable system of seminorms

$$p_N := \|\cdot\|_{L^\infty(-N,N)}, \quad N = 1, 2, \ldots .$$

By a theorem of [18] (see also [19]), the ω-limit set of any bounded solution *contains* an equilibrium, or a steady state, of (1.1). For more specific initial data a stronger result is available stating that the ω-limit set *consists of* a single equilibrium. For periodic initial data, such a convergence theorem can be found in [7]. In [9], the convergence of bounded nonnegative solutions with compact initial support was proved, assuming that f is locally Lipschitz and $f(0) = 0$. See also [10] for an improvement and extension of this result; earlier convergence results under more restrictive conditions can be found in [12, 14, 15, 27]). Other convergence theorems can be found in [22]. In [22] it is also proved that all nonnegative bounded solutions with initial data in $C_0(\mathbb{R})$ are *quasiconvergent*: their ω-limit set consists of equilibria.

Unlike on bounded intervals, the quasiconvergence result for (1.1), (1.2) is not a consequence of the formal gradient structure of equation (1.1); the usual energy functional may not even be defined along a general bounded solution. In fact, it is known that not all bounded solutions are quasiconvergent. This is illustrated by a construction of [11] (see also [25]) with $f(u) = u(1-u^2)$. If the initial function u_0 oscillates between the stable constants -1 and 1, being identical or close to one of them on larger and larger intervals, then $\omega(u_0)$ contains the two constant equilibria as well as some functions which are not equilibria. (See also [8] where such initial data are considered for the *linear* heat equation; the corresponding solutions approach a continuum of constant equilibria in that case). A similar construction works for any nonlinearity f which is of the *balanced* bistable type: there are two zeros $\alpha < \gamma$ of f such that $f'(\alpha) < 0$, $f'(\gamma) < 0$, and the function $F(u) = \int_0^u f(s)\,ds$ satisfies

$$F(u) < F(\alpha) = F(\gamma) \quad (u \in (\alpha, \gamma)). \tag{1.5}$$

In [25], we provided further examples of bounded solutions which are not quasiconvergent. We showed in particular that in the balanced bistable case there are examples of such solutions with initial data in $C_0(\mathbb{R})$. Of course, these initial data have to change sign, otherwise the quasiconvergence theorem of [22] applies.

Another example in [25] deals with unbalanced bistable nonlinearities:

(BU) For some $\alpha < 0 < \gamma$ one has $f(\alpha) = f(0) = f(\gamma) = 0$, $f'(\alpha) < 0$, $f'(\gamma) < 0$, $f < 0$ in $(\alpha, 0)$, $f > 0$ in $(0, \gamma)$, and

$$F(\gamma) > F(\alpha). \tag{1.6}$$

As above,

$$F(u) = \int_0^u f(s)\, ds.$$

Note that, unlike (1.5), condition (1.6) is robust. Consequently, if f satisfies (BU) and in addition $f'(0) > 0$, then any small perturbation $\tilde{f}$ of f in the class of C^1-functions vanishing at 0 also satisfies (BU) with some perturbed zeros $\tilde{\alpha} \approx \alpha$, $\tilde{\gamma} \approx \gamma$. Thus the example of [25] shows that the existence of bounded solutions which are not quasiconvergent is not limited to a meager class of nonlinearities. Of course, here we fix the middle zero at 0 just for convenience. If we do not fix it, then the perturbation will have a middle zero at some $\beta \approx 0$ and the construction of [25] still applies.

The solution found in [25] for the unbalanced bistable nonlinearity does not have limits at $x = \pm\infty$; its initial value u_0 satisfies

$$\limsup_{x \to \pm\infty} u_0(x) \geq 0, \quad \liminf_{|x| \to \pm\infty} u_0(x) = \alpha.$$

Our main goal in the present paper is to give another example in the unbalanced case, one with $u_0 \in C_0(\mathbb{R})$. Thus we show that even for a robust class nonlinearities, the decay of u_0 at $\pm\infty$ alone is not sufficient for the quasiconvergence.

To formulate our main theorem, recall that if f satisfies (BU), then the equation

$$v_{xx} + f(v) = 0, \quad x \in \mathbb{R}, \tag{1.7}$$

has a solution v such that $v > \alpha$ and $v - \alpha \in C_0(\mathbb{R})$. We refer to any such solution as a *ground state* of (1.7). More precisely, it is a ground state at level α, but we do not consider any other ground states in this paper. The ground state is unique up to translations [2] and, if its point of maximum is placed at the origin, it is even in x and decreasing with increasing $|x|$.

Theorem 1.1. *Let f be a C^1 function satisfying* (BU). *Then there exists $u_0 \in C_0(\mathbb{R})$ with $\alpha \leq u_0 \leq \gamma$ such that $\omega(u_0)$ contains the constant equilibrium α, a ground state of* (1.7), *and some functions which are not equilibria of* (1.1).

The proof of this theorem is given in Section 4. It uses similar ingredients as constructions in [25]: intersection comparison properties, continuous dependence in $L^\infty_{loc}(\mathbb{R})$ of solutions on their initial data, and a class of solutions with specific large-time behavior. The specific solutions employed in the present paper are the so-called threshold solutions for a suitable one-parameter family initial data. The family is chosen such that for small values of the parameter μ the corresponding solutions converge to the constant α, whereas for large values of μ they converge to γ. By definition, the threshold solutions are solutions which do not exhibit either of these behaviors. Threshold solutions for reaction diffusion equations on $\mathbb{R}$ have been studied quite extensively by several authors [5, 12, 13, 14, 15, 9, 23, 27]. In all these studies the initial data in the considered families have their limits at infinity equal to a zero of f and are bounded below by these limits. Usually they also coincide with the limits outside a compact set, or other strong monotonicity and symmetry conditions are imposed on them. In contrast, our method of proof of Theorem 1.1 necessitates that we deal with more general families of initial data: their limit at infinity is not a zero of f and, moreover, they are not necessarily bounded from below or from above by the limits. This is the main difference from a construction in [25, Section 6], where results on threshold solutions from [9] are employed. The new results on threshold solutions are formulated and proved in Section 3 and these are of independent interest. We remark that threshold solutions were also used in a different setting, but for a similar purpose, in [26].

In the remainder of the paper we assume (BU) to be satisfied.

We use the following notation. The support of a function $g \in L^\infty(\mathbb{R})$ is denoted by $\mathrm{spt}(g)$ (this is the minimal closed set K such that $g = 0$, a.e., in $\mathbb{R} \setminus K$). For two functions g, $\tilde{g}$, the relations $g \le \tilde{g}$ and $g < \tilde{g}$ are understood in the pointwise sense; specifically, the latter means that $g(x) < \tilde{g}(x)$ for all $x \in \mathbb{R}$.

The symbol $\mathcal{B}$ stands for the space all continuous functions on $\mathbb{R}$ taking values in $[\alpha, \gamma]$. We equip $\mathcal{B}$ with the metric given by the weighted sup-norm

$$\|v\|_w \equiv \sup_{x\in\mathbb{R}} w(x)|v(x)|, \tag{1.8}$$

where $w(x) := 1/(1 + |x|^2)$. The topology on $\mathcal{B}$ generated by this metric is the same as the topology induced from $L^\infty_{loc}(\mathbb{R})$.

2 Preliminaries

We start this preliminary section with a brief discussion of the steady states of (1.1), then we summarize some consequences of the intersection comparison principle, and recall a useful lemma on the continuity in $L^\infty_{loc}(\mathbb{R})$ of solutions with respect to their initial data.

2.1 Steady states

The steady states of (1.1) are solutions of the equation

$$v_{xx} + f(v) = 0, \quad x \in \mathbb{R}. \tag{2.1}$$

The first-order system associated with (2.1),

$$v_x = w, \quad w_x = -f(v), \tag{2.2}$$

is a Hamiltonian system with respect to the energy

$$H(v, w) := w^2/2 + F(v), \quad F(u) = \int_0^u f(s)\, ds.$$

Thus the trajectories of (2.2) are contained in the level sets of H. Note that these level sets are symmetric about the v axis. The following results are all well known and easily proved by phase-plane analysis of system (2.2) (cp. Figure 1), therefore we include them without proofs.

Lemma 2.1. *Assume* (BU)*.*

(i) Let v be a solution of (2.1) *with $\alpha < v(0) < \gamma$. Then exactly one of the following possibilities occurs:*

(a) $v \in \mathcal{B}$;

(b) there are constants $a < 0 < b$ such that $v(a) = v(b) = \alpha$ and $\alpha < v < \gamma$ on (a, b),

(c) there are values $a \in [-\infty, 0)$ and $b \in (0, \infty]$, at least one of them finite, such that $\alpha < v(x) < \gamma$, $|v'(x)| > 0$ for all $x \in (a, b)$, and either $v(a) = \alpha$, $v(b) = \gamma$ with $a > -\infty$, or $v(a) = \gamma$, $v(b) = \alpha$ with $b < \infty$.

(ii) *If $v \in \mathcal{B}$ is a nonconstant periodic solution, then v has infinitely many zeros (all of these zeros are simple due the uniqueness for the initial value problem).*

(iii) *Let ϕ be the ground state with $\phi_x(0) = 0$ and v another solution of (2.1) with $v_x(0) = 0$. Then $v \in \mathcal{B}$ if and only if $\alpha \leq v(0) \leq \phi(0)$ or $v(0) = \gamma$. If $\alpha < v(0) < \phi(0)$, then v is a nonconstant periodic solution. If $\phi(0) < v(0) < \gamma$, then v is even, and for some $a > 0$ one has $v(\pm a) = \alpha$ and $v_x < 0$ in $(0, a]$.*

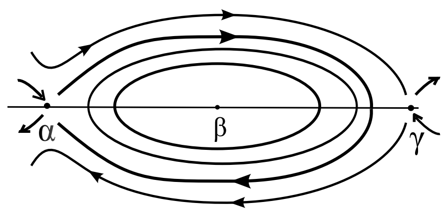

Figure 1: Trajectories of system (2.2)

2.2 Zero number

If $v = u - \tilde{u}$ or $v = u_x$, where u, $\tilde{u}$ are global solutions of (1.1), then v is a solution of a linear equation

$$v_t = v_{xx} + c(x,t)v, \quad x \in \mathbb{R}, \; t > 0, \tag{2.3}$$

with a suitable coefficient c. Specifically,

$$c(x,t) = \int_0^1 f'(\tilde{u}(x,t) + s(\tilde{u}(x,t) - u(x,t)))\, ds$$

if $v = u - \tilde{u}$, and $c(x,t) = f'(u(x,t))$ if $v = u_x$. For an interval $I = (a,b)$, with $-\infty \leq a < b \leq \infty$, we define $z_I(v(\cdot,t))$ as the number of zeros, possibly infinite, of the function $x \to v(x,t)$ in I. If $I = \mathbb{R}$, we usually omit the subscript I:

$$z(v(\cdot,t)) := z_{\mathbb{R}}(v(\cdot,t)).$$

The following intersection-comparison principle holds (see [1, 6]).

Lemma 2.2. *Let $v \in C(\mathbb{R} \times [0, \infty))$ be a nontrivial solution of (2.3) and $I = (a, b)$, where $-\infty \le a < b \le \infty$. Assume that for some interval $[\tau, T) \subset [0, \infty)$ the following conditions are satisfied:*

(c1) *if $b < \infty$, then $v(b, t) \neq 0$ for all $t \in [\tau, T)$,*

(c2) *if $a > -\infty$, then $v(a, t) \neq 0$ for all $t \in [\tau, T)$.*

Then the following statements hold true:

(i) *For each $t \in (\tau, T)$, all zeros of $v(\cdot, t)$ are isolated. In particular, if $a > -\infty$ and $b < \infty$, then $z_I(v(\cdot, t)) < \infty$ for all $t \in (\tau, T)$.*

(ii) *$t \mapsto z_I(v(\cdot, t))$ is a monotone nonincreasing function on $[\tau, T)$ with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$.*

(iii) *If for some $t_0 \in (\tau, T)$, the function $v(\cdot, t_0)$ has a multiple zero in I and $z_I(v(\cdot, t_0)) < \infty$, then for any $t_1, t_2 \in [\tau, T)$ with $t_1 < t_0 < t_2$ one has*

$$z_I(v(\cdot, t_1)) > z_I(v(\cdot, t_0)) \ge z_I(v(\cdot, t_2)). \tag{2.4}$$

If (2.4) holds, we say that $z_I(v(\cdot, t))$ drops in the interval (t_1, t_2). If this holds for all t_1, t_2 with $t_1 < t_0 < t_2$, we also say that $z_I(v(\cdot, t))$ drops at t_0.

Corollary 2.3. *Under the assumption of Lemma 2.2, the following statements hold.*

(i) If $\xi \in (a, b)$, then the functions $v(\xi, \cdot)$ $v_x(\xi, \cdot)$ cannot both vanish identically on any open subinterval of (τ, T).

(ii) If $z_I(v(\cdot, \tau)) < \infty$, then $z_I(v(\cdot, t))$ can drop at most finitely many times in (τ, T) hence it is constant on an interval $[\tau_0, T)$. If $z_I(v(\cdot, t))$ is constant on a compact interval $[\tau_1, \tau_2] \subset (\tau, T)$, then $v(\cdot, t)$ has only simple zeros in I for each $t \in (\tau_1, \tau_2]$.

Proof. We start with the second statement. Clearly, the monotonicity of $t \to z_I(v(\cdot, t))$ implies that $z_I(v(\cdot, t))$ can drop at most finitely many times in (τ, T) if it is finite, hence it has to stay constant in $[\tau_0, T)$ for some $\tau_0 < T$. The second conclusion in statement (ii) follows directly from Lemma 2.2(iii).

Now, to prove the statement (i), assume that for some interval $(\tau_1, \tau_2) \subset (\tau, T)$ one has

$$v(\xi, \cdot) \equiv v_x(\xi, \cdot) \equiv 0 \text{ on } (\tau_1, \tau_2). \tag{2.5}$$

Using the fact that the zeros of $v(\cdot,t)$ are isolated (Lemma 2.2(i)) and the continuity of v, we find points $\tilde a < \tilde b$ in (a,b) and times $\tilde t_1 < \tilde t_2$ in (t_1,t_2) such that $v(\tilde a,t) \neq 0 \neq v(\tilde b,t)$ for each $t \in (\tilde t_1,\tilde t_2)$. Then, by Lemma 2.2, $z_{(\tilde a,\tilde b)}(v(\cdot,t)) < \infty$ for each $t \in (\tilde t_1,\tilde t_2)$, and relations (2.5) give a contradiction to statement (ii) just proved above. $\square$

2.3 Continuity with respect to initial data

At several places we use the continuous dependence of the solutions of (1.1) on their initial data. A standard result is the following one.

Lemma 2.4. *Given constants $T > \tau > 0$, $p \in [1,\infty]$, there is a constant $L(\tau,T,p)$ such that if u_0, $\tilde u_0 \in L^\infty(\mathbb{R})$, $\alpha \le u_0, \tilde u_0 \le \gamma$, then for each $t \in [\tau,T]$ one has*

$$\begin{aligned}\|u(\cdot,t,u_0) - u(\cdot,t,\tilde u_0)\|_{L^\infty(\mathbb{R})},\ \|u_x(\cdot,t,u_0) - u_x(\cdot,t,\tilde u_0)\|_{L^\infty(\mathbb{R})}& \\ \le L(\tau,T,p)\|u_0 - \tilde u_0\|_{L^p(\mathbb{R})}&.\end{aligned}$$

The estimate for $v := u(\cdot,t,u_0) - u(\cdot,t,\tilde u_0)$ is a standard $L^p - L^\infty$ estimate for the linear equation satisfied by v (see (2.3) and note that the coefficient c is bounded independently of u_0, $\tilde u_0$ since the solutions stay between α and γ). The estimate for the derivatives then follows, enlarging $L(\tau,T,p)$ if necessary, from parabolic regularity estimates.

The following lemma gives a continuity result with respect to the norm defined in (1.8).

Lemma 2.5. *Given any finite $T > 0$ there is a constant $L(T)$ such that for any u_0, $\tilde u_0 \in \mathcal{B}$, one has*

$$\|u(\cdot,t) - \tilde u(\cdot,t)\|_w \le L(T)\|u(\cdot,0) - \tilde u(\cdot,0)\|_w \quad (t \in [0,T]).$$

This continuity result is proved easily by considering the linear parabolic equation satisfied by $v(x,t) := w(x)(u(x,t) - \tilde u(x,t))$, see [15, Lemma 6.2].

Lemma 2.5 and standard parabolic estimates give the following result.

Corollary 2.6. *Given any $u_0 \in \mathcal{B}$, $T > t_0 > 0$, $R > 0$, and $\epsilon > 0$, there exist $\rho \ge R$ and $\delta > 0$ with the following property. If $\tilde u_0 \in \mathcal{B}$ satisfies*

$$\sup_{x\in[-\rho,\rho]} |u_0(x) - \tilde u_0(x)| < \delta,$$

then

$$\sup_{x\in[-R,R],\ t\in[0,T]} |u(x,t,u_0) - u(x,t,\tilde{u}_0)| < \epsilon,$$
$$\sup_{x\in[-R,R],\ t\in[t_0,T]} |u_x(x,t,u_0) - u_x(x,t,\tilde{u}_0)| < \epsilon.$$

3 Threshold solutions

In this section we examine threshold solutions of (1.1) for families of initial data which are identical to a constant outside a compact interval (the constant is not necessarily a zero of f). The main result is Proposition 3.2 below. In its formulation, the following is used.

Lemma 3.1. *For each $\theta \in (0,\gamma]$, there exists $\ell = \ell(\theta)$ such that if $u_0 \in L^\infty(\mathbb{R})$, $\alpha \le u_0 \le \gamma$, and $u_0 \ge \theta$ on an interval of length ℓ, then $u(\cdot,t,u_0) \to \gamma$, as $t \to \infty$, in $L^\infty_{loc}(\mathbb{R})$.*

This result is proved in [16] (proofs and extensions can also be found in [9, 10, 15, 24]).

Proposition 3.2. *Let $\theta \in (0,\gamma)$ and $\eta \in (\alpha, 0)$ be arbitrary, and let $\ell = \ell(\theta)$ be as in Lemma 3.1. Assume that ψ_μ, $\mu \in [0,1]$, is a family of even functions in $\mathcal{B}$ with the following properties:*

(a1) *For each $\mu \in [0,1]$, $\psi_\mu - \eta$ has compact support, $\psi_1 \ge \theta$ on an interval of length ℓ, and there is $s_0 \ge 0$ such that*

$$u(\cdot, s_0, \psi_0) < 0. \tag{3.1}$$

(a2) *The function $\mu \to \psi_\mu : [0,1] \to L^1(\mathbb{R})$ is continuous and monotone increasing in the sense that if $\mu < \nu$, then $\psi_\mu \le \psi_\nu$ with the strict inequality on a nonempty (open) set.*

Then there is $\mu^* \in (0,1)$ with the following two properties:

(t1) *If $u_0 = \psi_\mu$ with $\mu \in (0,\mu^*)$, then $\lim_{t\to\infty} u(\cdot,t,u_0) = \alpha$ in $L^\infty(\mathbb{R})$.*

(t2) *If $u_0 = \psi_{\mu^*}$, then $\lim_{t\to\infty} u(\cdot,t,u_0) \to \phi_0$ in $L^\infty_{loc}(\mathbb{R})$, where ϕ_0 is the (unique) even ground state of (1.7).*

We assume that the functions ψ_μ are even for the sake of convenience and because it is sufficient for the proof of Theorem 1.1. The result appears to be valid, without the evenness of the limit ground state, if the evenness assumption on the ψ_μ is dropped, but would require extra technical work to prove the convergence for $\mu = \mu^*$. At the end of this section (see Remark 3.7), we shall discuss another issue commonly considered in connection with threshold solutions, the sharpness of the threshold value.

In the introduction we already mentioned several earlier papers concerning threshold solutions of (1.1). In particular, the case $\eta = \alpha$ is covered in greater generality in [9].

We prepare the proof of Proposition 3.2 by several lemmas. In the whole section, $y(t, \eta)$ denotes the solution of

$$\dot{y} = f(y), \quad y(0) = \eta.$$

Lemma 3.3. *Given any $\eta \in [\alpha, \gamma]$, $h \in \mathbb{R}$, assume that $u_0 \in L^\infty(\mathbb{R})$, $\alpha \leq u_0 \leq \gamma$, and $u_0 \equiv \eta$ on (h, ∞). Then for any positive constants $\delta < T$ one has*

$$\lim_{x \to \infty} u(x,t) = y(t, \eta) \text{ uniformly for } t \in [0, T], \tag{3.2}$$

$$\lim_{x \to \infty} u_x(x,t) = 0 \text{ uniformly for } t \in [\delta, T]. \tag{3.3}$$

An analogous conclusion holds, with the limits at $-\infty$ in place of the limits at ∞, if $u_0 \equiv \eta$ on $(-\infty, h)$.

Proof. We prove the first statement, the second statement then follows by reflection in x. If is sufficient to prove (3.2); relation (3.3) follows from (3.2) and standard parabolic estimates. Without loss of generality, we may also assume that u_0 satisfies one of the relations $u_0 \geq \eta$ or $u_0 \leq \eta$. Indeed, if (3.2) is proved for such initial data, then one obtains the general result from the comparison principle. We only treat the case $u_0 \geq \eta$, the other case is completely analogous.

Consider the function $v(x,t) := u(x,t) - y(t, \eta)$. It is a solution of a linear equation (2.3) and $v(\cdot, 0) = u_0 - \eta \geq 0$. Hence, v is nonnegative. Since both u and $y(\cdot, \eta)$ are bounded, we have $|f(u(x,t)) - f(y(t, \eta))| \leq Mv(x,t)$ for some constant $M \geq 0$. Therefore, by comparison,

$$v(x,t) \leq e^{Mt} \bar{v}(x,t) \tag{3.4}$$

where $\bar{v}$ is the solution of $\bar{v}_t = \bar{v}_{xx}$ with $\bar{v}(\cdot,0) = v(\cdot,0) = u_0 - \eta$. Since $\bar{v}(\cdot,0) \equiv 0$ on (h,∞), for each $x \in \mathbb{R}$ and $t > 0$ one has

$$0 \le \bar{v}(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{h} e^{-\frac{|x-s|^2}{4t}} (u_0(s) - \eta)\, ds.$$

The substitution $r = (s-x)/\sqrt{4t}$ then yields

$$0 \le \bar{v}(x,t) \le \frac{\|u_0 - \eta\|_{L^\infty(\mathbb{R})}}{\sqrt{\pi}} \int_{-\infty}^{\frac{h-x}{\sqrt{4t}}} e^{-r^2}\, dr.$$

The last integral converges to zero, as $x \to \infty$, uniformly for $t \in (0,T]$, which gives (3.2). $\square$

Lemma 3.4. *Assume that $u_0 \in L^\infty(\mathbb{R})$, $\alpha \le u_0 \le \gamma$, and for some constants $\rho > 0$ and $\eta_1, \eta_2 \in [\alpha, 0)$ one has*

$$u_0 \equiv \eta_1 \text{ on } (-\infty,-\rho) \quad \text{and} \quad u_0 \equiv \eta_2 \text{ on } (\rho,\infty). \tag{3.5}$$

Then either

$$\lim_{t\to\infty} u(\cdot,t,u_0) = \gamma \text{ in } L^\infty_{loc}(\mathbb{R}), \tag{3.6}$$

or the following conclusion holds. Any sequence $\{(x_n,t_n)\}$ in $\mathbb{R}\times(0,\infty)$ with $t_n \to \infty$ can be replaced by a subsequence such that $u(\cdot + x_n, \cdot + t_n) \to \varphi$, where $\varphi \equiv \alpha$ or φ is a ground state of (1.1), and the convergence is in $L^\infty_{loc}(\mathbb{R}^2)$.

Saying that the $u(\cdot + x_n, \cdot + t_n)$ converges in $L^\infty_{loc}(\mathbb{R}^2)$ is a slight abuse of language, for the functions are not defined on $\mathbb{R}^2$. However, for each compact set $K \subset \mathbb{R}^2$, omitting a finite number of terms we obtain a sequence of functions defined on K and we require it to converge uniformly on K. Similarly we understand the convergence in the spaces $C^1_{loc}(\mathbb{R}^2)$, $C^{2,1}_{loc}(\mathbb{R}^2)$.

Proof of Lemma 3.4. Assume that (3.6) does not hold. Also we assume that u_0, as an element of $L^\infty(\mathbb{R})$, is not identical to α, otherwise the statement is trivial. Thus, by comparison principle, $\alpha < u(\cdot,t) < \gamma$ for all $t > 0$.

Let $\{(x_n,t_n)\}$ be any sequence in $\mathbb{R}\times(0,\infty)$ with $t_n \to \infty$. Using parabolic estimates and a standard diagonalization procedure (as for ω-limit sets) one shows that if $\{(x_n,t_n)\}$ is replaced by a subsequence, then $u(\cdot + x_n, \cdot + t_n, u_0) \to U$, where $U(x,t)$ is an entire solution of (1.1) and the convergence is in $L^\infty_{loc}(\mathbb{R}^2)$ as well as in $C^{2,1}_{loc}(\mathbb{R}^2)$. (Recall that an entire solution refers to a solution defined for all $t \in \mathbb{R}$).

Our goal is to prove that $U(x,0)$ is a steady state; ultimately, we want to show that it is either identical to a ground state or the constant α. To a large extent, we do this by adapting some arguments from [9] (extra difficulties come from the space translations). A prerequisite for these arguments is the following claim.

Claim. Let ψ be a solution of (2.1), different from any ground state, such that $\alpha < \psi(x_0) < \gamma$ for some x_0. Let J be the connected component of the set $\{x : \alpha < \psi(x) < \gamma\}$ containing the point x_0. Then for all sufficiently large t, the function $u(\cdot,t,u_0)) - \psi$ has only finitely many zeros in J, all of them simple.

To prove the claim, it is sufficient to show that for some $t > 0$ one has

$$|u(\cdot,t,u_0)) - \psi| > 0 \text{ outside a compact subinterval of } J. \tag{3.7}$$

The conclusion then follows from Lemma 2.2 and Corollary 2.3(ii).

Shifting both ψ and u in x, we assume without loss of generality that $x_0 = 0$. Thus one of the statements (a)–(c) in Lemma 2.1 applies. If either (b) or (c) holds, or if ψ is a constant solution, then the relations $\alpha < u(\cdot,t) < \gamma$ and Lemma 3.3 imply that (3.7) holds for each $t > 0$. If $\psi \in \mathcal{B}$ is a nonconstant periodic solution, then $\psi > \alpha + \epsilon$ for some $\epsilon > 0$. Then Lemma 3.3 and the assumption that $\eta_1, \eta_2 \in [\alpha, 0)$ imply that for all large enough t one has $u(\pm\infty, t, u_0) < \alpha + \epsilon$, which gives (3.7).

Having proved the claim, we return to the entire solution U. We have $\alpha \le U \le \gamma$, since $\alpha \le u(\cdot,\cdot,u_0) \le \gamma$. Remember that we want to prove that $U_0 := U(\cdot,0)$ is identical to a ground state or the constant α.

First of all we show that U_0 cannot be identical to any other steady state. Indeed, assume that $U_0 \equiv \varphi$, where φ is a steady state different from α and any ground state. Then $\varphi \in \mathcal{B}$, hence either $\varphi \equiv \gamma$, or $\varphi \equiv 0$, or φ is a nonconstant periodic solution. The possibility $\varphi \equiv \gamma$ is easily excluded: it would imply that $u(\cdot,t_n,u_0)$ is close to γ on arbitrarily large intervals as $n \to \infty$. But then, by Lemma 3.1, (3.6) would hold contrary to our assumption. Thus either $\varphi \equiv 0$ or φ is a nonconstant periodic solution of (2.1). Set $\tilde{\varphi} \equiv 0$ if $\varphi \not\equiv 0$. If $\varphi \equiv 0$, pick a nonconstant periodic solution $\tilde{\varphi}$ of (2.1) with $\alpha < \tilde{\varphi} < \gamma$. Then $\tilde{\varphi} - \varphi$ has infinitely many simple zeros (see Lemma 2.1(ii)). Consequently, since $u(\cdot + x_n, t_n, u_0) \to U_0 \equiv \varphi$, we have $z(u(\cdot,t_n,u_0) - \tilde{\varphi}) \to \infty$ as $n \to \infty$. On the other hand, as seen in the proof of the Claim, Lemmas 3.3 and 2.2 imply that $z(u(\cdot,t,u_0) - \tilde{\varphi})$ is finite and independent of t for large t, a contradiction.

If $U \equiv \alpha$, there is nothing else to be proved. Assume $U \not\equiv \alpha$. Since we have just shown that $U \not\equiv \gamma$, the comparison principle yields $\alpha < U < \gamma$.

The rest of the proof is by contradiction. Assume that U_0 is not identical to any ground state. For a fixed $x_0 \in \mathbb{R}$, to be chosen below, we let φ be the solution of (2.1) with $\varphi(x_0) = U_0(x_0) \in (\alpha, \gamma)$ and $\varphi'(x_0) = U_0'(x_0)$. We want to apply the Claim to φ. For that we need make a choice of x_0 so that φ is not a ground state. To see that this is possible, recall that all ground states (at level α) are shifts of one fixed ground state ϕ. Therefore, if no x_0 with the given property existed, the following statement would hold: For each $x \in \mathbb{R}$ there is ξ such that

$$U_0(x) = \phi(x + \xi), \quad U_0'(x) = \phi'(x + \xi). \tag{3.8}$$

Clearly, $\xi = \xi(x)$ is uniquely determined ($y \to (\phi(y), \phi'(y))$ is one-to-one) and is of class C^1 by the implicit function theorem (ϕ and ϕ' cannot simultaneously vanish). Elementary considerations using the identities (3.8) now show that $\xi \equiv \xi_0$ for some constant ξ_0, hence $U_0 \equiv \phi(\cdot + \xi_0)$ in contradiction to our assumption on U_0.

Thus we have verified that with a suitable choice of x_0 the solution φ is not a ground state. We can also assume that $\varphi \not\equiv 0$ (otherwise, we make a different choice of x_0, which is possible due to $U_0 \not\equiv 0$). Obviously, $\varphi \not\equiv \alpha$, $\varphi \not\equiv \gamma$, as $\alpha < U < \beta$. Thus φ is a nonconstant solution. As, we proved above, $U_0 - \varphi \not\equiv 0$.

Hence, $U - \varphi$ is a nontrivial entire solution of a linear equation 2.3 and $U(\cdot, 0) - \varphi$ has a multiple zero at $x = x_0$. We derive a contradiction from this conclusion.

First we consider the case when $\varphi \in \mathcal{B}$, which means, by the above choices, that φ is a nonconstant periodic solution. Let $\rho > 0$ be the minimal period of φ. For each n, write x_n in the form $x_n = k_n\rho + \zeta_n$, where $k_n \in \mathbb{Z}$ and $\zeta_n \in [0, \rho)$. We may assume, passing to a subsequence if necessary, that $\zeta_n \to \zeta_0 \in [0, \rho]$. Then $u(\cdot + k_n\rho + \zeta_0, \cdot + t_n, u_0) - u(\cdot + x_n, \cdot + t_n, u_0) \to 0$ in $C^1_{loc}(\mathbb{R}^2)$ and, consequently,

$$u(\cdot + k_n\rho + \zeta_0, \cdot + t_n, u_0) - \varphi \to U - \varphi \text{ in } C^1_{loc}(\mathbb{R}^2). \tag{3.9}$$

Using this and the fact that $U(\cdot, 0) - \varphi$ has a multiple zero at $x = x_0$, it is not difficult to prove (see [9, Lemma 2.6]) that for all large enough n, the function $u(\cdot + k_n\rho + \zeta_0, s + t_n, u_0) - \varphi$ has a multiple zero (near x_0) for some $s \in (-1, 1)$. Therefore, the function

$$u(\cdot, s + t_n, u_0) - \varphi(\cdot - k_n\rho - \zeta_0) = u(\cdot, s + t_n, u_0) - \varphi(\cdot - \zeta_0)$$

has a multiple zero for all large n. Since $s + t_n \to \infty$, this contradicts the Claim with $\psi = \varphi(\cdot - \zeta_0)$.

Next we consider the case when $\varphi \notin \mathcal{B}$, hence one of the statement (b), (c) in Lemma 2.1(i) applies. Let I be the connected component of the set $\{x : \alpha < \varphi(x) < \beta\}$ containing x_0.

We distinguish two cases:

(i) x_n can be replaced by a subsequence such that $x_n \to y_0 \in \mathbb{R}$,

(ii) $|x_n| \to \infty$.

In the case (i), we use a similar argument as for the periodic φ. We have

$$u(\cdot, \cdot + t_n, u_0) = u(\cdot - x_n + x_n, \cdot + t_n, u_0) \to U(\cdot - y_0, t)$$

in $C^1_{loc}(\mathbb{R}^2)$. Therefore, similarly as above, [9, Lemma 2.6] implies that if n is large enough, then $u(\cdot, s + t_n, u_0) - \varphi(\cdot - y_0)$ has a multiple zero in $I + y_0$ for some $s \in (-1, 1)$. This contradicts the Claim with $\psi = \varphi(\cdot - y_0)$.

Finally, we derive a contradiction in the case (ii). Passing to a subsequence, we may assume that $x_n \to \infty$ or $x_n \to -\infty$. We just consider the former, the latter can be treated in the same way. Once again, we use the property that for all large n

$$u(\cdot + x_n, s + t_n, u_0) - \varphi \text{ has a multiple zero in } I \text{ for some } s \in (-1, 1). \tag{3.10}$$

Remember that one of the statements (b), (c) of Lemma 2.1(i) applies to the solution φ. Fixing any $t_0 > 0$, we have

$$u(x, t_0, u_0) - y(t_0, \eta_2),\ u_x(x, t_0, u_0) \to 0 \text{ as } x \to \infty$$

(see (3.5) and Lemma 3.3). Therefore, it is easy to show that if n is sufficiently large, then one of the following statements holds:

(zb) $u(\cdot, t_0, u_0) - \varphi(\cdot - x_n)$ has exactly two zeros in $I + x_n$, both of them simple (this holds if (b) applies to φ).

(zc) $u(\cdot, t_0, u_0) - \varphi(\cdot - x_n)$ has a unique zero in $I + x_n$ and the zero is simple (this holds if (c) applies to φ).

Choose any n so large that one of these statements applies and such that, in addition, $t_n - 1 > t_0$ and (3.10) holds. By the monotonicity of the zero number, we have $z_{I+x_n}(u(\cdot,t,u_0) - \varphi(\cdot - x_n)) \le 1$ or $z_{I+x_n}(u(\cdot,t,u_0) - \varphi(\cdot - x_n)) \le 2$, for all $t > t_0$, according to whether (zb) or (zc) holds.

By Lemma 2.2 and (3.10), $z_{I+x_n}(u(\cdot,t,u_0) - \varphi(\cdot - x_n))$ drops at $t = s + t_n > t_0$. This is clearly impossible if (zc) holds (in this case, the zero number is at least 1 at all times since $\alpha < u(\cdot,t_0,u_0) < \gamma$). Thus (zb) holds and in this case we necessarily have $u(\cdot, s+t_n, u_0) \ge \varphi(\cdot - x_n)$. Then, by comparison,

$$u(\cdot,t,u_0) > u(\cdot, t-(s+t_n), \psi^*) \quad (t > s+t_n), \tag{3.11}$$

where ψ^* is defined by

$$\psi^*(x) = \begin{cases} \varphi(x - x_n), & \text{if } x \in I + x_n, \\ \alpha, & \text{if } x \in \mathbb{R} \setminus (I + x_n). \end{cases}$$

It is a well-known fact that, since α and ψ are both solutions of (2.1), the function ψ^* is a strict generalized subsolution of (1.1). This is to say that $u(\cdot,t,\psi^*) > \psi^*$ for all $t > 0$ and $u(\cdot,t,\psi^*)$ is increasing in t. Thus, as $t \to \infty$, $u(t,\cdot,\psi^*)$ converges to a steady state, say φ^*. This steady state is in $\mathcal{B}$ and above ψ^*, thus the only possibility is $\varphi^* \equiv \gamma$ (see Lemma 2.1(iii)). Thus (3.11) implies that (3.6) holds, contrary to our starting assumption. Thus we have derived a contradiction in this last case as well. This concludes the proof Lemma 3.4. □

In the next lemma, we consider a 3-step function g defined by

$$g(x) = \begin{cases} \beta & (x \in (-\infty, -q), \\ \vartheta & (x \in [-q, 0]), \\ \alpha & (x \in (0, \infty]), \end{cases} \tag{3.12}$$

with suitable constants $\beta \in (\alpha, 0)$, $q > 0$, and $\vartheta \in (0, \gamma)$ (see Figure 2).

Lemma 3.5. *Let $q := \ell(\gamma/2)$ (the constant ℓ in Lemma 3.1 corresponding to $\theta = \gamma/2$). Given arbitrary $\beta \in (\alpha, 0)$, there exists $\vartheta \in (0, \gamma)$ with the following property. If g is as in* (3.12)*, then $\lim_{t\to\infty} u(\cdot,t,g) = \phi$, where ϕ is a ground state of* (1.1) *and the convergence is in $L^\infty(\mathbb{R})$.*

Remark 3.6. We will use the solution $u(\cdot,t,g)$ in intersection comparison arguments below. For that purpose it will be useful to have noticed that $u(\cdot,t,g)$ has the following features (regardless of the specific value of $\vartheta \in (0,\gamma)$).

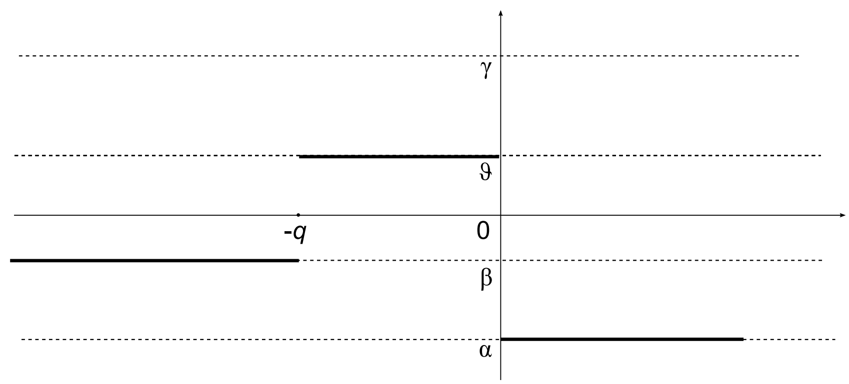

Figure 2: The graph of the function g

(i) For all sufficiently small $t > 0$, the function $u(\cdot, t, g)$ has a unique critical point, which is its global maximizer.

(ii) If $u_0 \in \mathcal{B}$, $u_0 \equiv \tilde{\eta} \in (\alpha, \beta)$ near $\pm\infty$, and $\tilde{g} := g(\cdot + x_0)$ with a sufficiently large x_0, then

$$z(u(\cdot, t_0, \tilde{g}) - u(\cdot, t_0, u_0)) = 1 \tag{3.13}$$

for all sufficiently small $t_0 > 0$.

These properties are verified easily by considering smooth approximations of the function g. To prove (i), take smooth functions g_n with a unique critical point such that $g_n - g \to 0$ in $L^1(\mathbb{R})$. Each function $u_x(\cdot, t, g_n)$ has a unique zero by Lemma 2.2. Approximating $u_x(\cdot, t, g)$ by $u_x(\cdot, t, g_n)$ (see Lemma 2.4) and applying Lemma 2.2 to $u_x(\cdot, t, g)$, we see that $u_x(\cdot, t, g)$ has at most one zero for any $t > 0$ and that zero, when it exists, must be simple. Statement (ii) is proved by a similar approximation argument, using g_n with a sharp transition from positive values to values less than $\tilde{\eta}$.

Proof of Lemma 3.5. We find $\vartheta \in (0, \gamma)$ as a threshold value. To simplify the notation, let $u^\vartheta = u(\cdot, \cdot, g)$, where g is as in (3.12). Consider the following two statements regarding the solution u^ϑ:

(L0) There is $T \geq 0$ such that $u^\vartheta(x, T) < 0 \quad (x \in \mathbb{R})$.

(G0) There is $T \geq 0$ such that $u^\vartheta(\cdot, T) > \gamma/2$ on a closed interval of length q.

Set

$$K_0 := \{\vartheta \in [0,\gamma] : \text{(L0) holds}\},$$
$$K_1 := \{\vartheta \in [0,\gamma] : \text{(G0) holds}\}.$$

Lemma 3.3, and the continuous dependence of solutions on their initial data (see Lemma 2.4) imply that the sets K_0 and K_1 are both open in $[0,\gamma]$. By the definition of g, we have $0 \in K_0$ and $(\gamma/2,\gamma] \subset K_1$. By Lemma 3.1, $K_0 \cap K_1 = \emptyset$. Thus there is $\vartheta \in (0,\gamma)$, for which neither (L0) nor (G0) holds. Fix any such ϑ. We show that it has the property stated in Lemma 3.5.

First we note that, since (L0) does not hold, for each $t > 0$ the function $u^\vartheta(\cdot,t)$ assumes positive values. At the same time, it is negative near $\pm\infty$, by Lemma 3.3. Hence $u^\vartheta(\cdot,t)$ has a global maximizer $\xi(t)$. Actually, $\xi(t)$ is the unique critical point of $u^\vartheta(\cdot,t)$. This is obvious for small t (cp. Remark 3.6) and for any t it follows from the nonincrease of the zero number $z(u_x^\vartheta(\cdot,t))$. Thus for each $t > 0$

$$u_x^\vartheta(\cdot,t) > 0 \quad (x < \xi(t)), \qquad u_x^\vartheta(\cdot,t) < 0 \quad (x > \xi(t)). \tag{3.14}$$

The next information on $u^\vartheta(\cdot,t)$ is obtained from Lemma 3.4:

$$\lim_{t\to\infty} u^\vartheta(\cdot + \xi(t),t) = \phi_0 \text{ in } L^\infty_{loc}(\mathbb{R}), \tag{3.15}$$

where ϕ_0 is the even ground state. Indeed, since (G0) does not hold, Lemma 3.4 guarantees that any sequence $t_n \to \infty$ has a subsequence such that $u^\vartheta(\cdot + \xi(t_n),t_n)$ converges in $L^\infty_{loc}(\mathbb{R})$ to a steady state φ, either a ground state or the constant α. Since $u^\vartheta(\cdot+\xi(t),t)$ assumes a positive value at $x = 0$, which is its global maximizer, necessarily $\varphi = \phi_0$. And since this is true for any sequence $t_n \to \infty$, (3.15) holds.

The convergence in (3.15) is in fact uniform, that is, $L^\infty_{loc}(\mathbb{R})$ can be replaced by $L^\infty(\mathbb{R})$:

$$\lim_{t\to\infty} u^\vartheta(\cdot + \xi(t),t) = \phi_0 \text{ in } L^\infty(\mathbb{R}). \tag{3.16}$$

This follows easily from the monotonicity of $u^\vartheta(\cdot + \xi(t),t)$ and ϕ_0 in the intervals $(-\infty,0)$, $(0,\infty)$ and the relations

$$u^\vartheta(\cdot,t) \geq \alpha = \phi_0(\pm\infty) \quad (t \geq 0).$$

Our final step is to prove that $u^\vartheta(\cdot,t)$ converges to a ground state, a shift of ϕ_0, in $L^\infty(\mathbb{R})$. In particular, this will prove that $u^\vartheta(\cdot,t)$ cannot drift to infinity along the manifold $\{\phi_0(\cdot+\zeta):\zeta\in\mathbb{R}\}$. (It is worthwhile to mention that such drifts do occur for equations on a half-line with Dirichlet or Robin boundary conditions, see [5, 13]). Our proof uses the normal hyperbolicity of this manifold of steady states in a similar way as in some proofs of convergence of localized solutions [4, 15, 17]. A minor additional difficulty is that here the solution is not a priori known to converge to a fixed steady state along a sequence of times. This is dealt with by space-shifting the solution suitably.

We introduce the time-one map for equation (1.1). Take $X := L^\infty(\mathbb{R})$ and for $u_0\in X$ set $\Pi(u_0) := u(\cdot,1,u_0)$. Without loss of generality, modifying the nonlinearity f outside the interval $[\alpha,\gamma]$ if necessary, we may assume that all solutions are global, hence Π is well defined on X. Also, it is a C^1 map on X (see [21]). Obviously, $\Pi^n(u_0) = u(\cdot,n,u_0)$, $n = 0,1,\dots$.

To prove the desired convergence result, it is sufficient to show that the sequence

$$\Pi^n(g) = u^\vartheta(\cdot,n), \quad n = 0,1,\dots,$$

converges to a ground state ϕ; since ϕ is a steady state, the continuity with respect to initial data then implies that $u^\vartheta(\cdot,t)$ also converges to ϕ. In fact, it is sufficient to prove that $\{\Pi^n(g)\}$ is convergent in $L^\infty(\mathbb{R})$. By (3.15), its limit is then necessarily a shift of ϕ_0 (and $\xi(t)$ is also convergent).

We prove the convergence by contradiction. Assume that the sequence $\{\Pi^n(g)\}$ in not Cauchy in X: there are n_k, m_k, $k=1,2,\dots$, such that

$$\liminf \|\Pi^{n_k}(g) - \Pi^{n_k+m_k}(g)\|_X > 0, \quad n_k\to\infty. \tag{3.17}$$

Denote

$$x_k := \xi(n_k), \quad u_k := u^\vartheta(\cdot + x_k, n_k).$$

By (3.16), $u_k\to\phi_0$ in X. Also, by (3.17) and the translation invariance of (1.1),

$$\liminf \|u_k - \Pi^{m_k}(u_k)\|_X > 0.$$

Therefore, given any small $\epsilon>0$, for all sufficiently large k there is p_k with the following property:

$$\|\phi_0 - \Pi^j(u_k)\|_X < \epsilon \quad (j = 0,1,\dots,p_k); \quad \|\phi_0 - \Pi^{p_k+1}(u_k)\|_X \ge \epsilon. \tag{3.18}$$

This is the setup of Lemma 1 of [3], which we intend to apply. We recall here a special case of that lemma using the present notation:

Lemma BP. *Let* Π *be a* C^1 *map on a Banach space* X *and* ϕ_0 *a fixed point of* Π*. Assume that the following hypotheses are satisfied with a sufficiently small* $\epsilon > 0$*:*

(a) $\sigma(\Pi'(\phi_0)) = \sigma_s \cup \sigma_c \cup \sigma_u$*, where* σ_s*,* σ_c *and* σ_u *are closed subsets of* $\{\lambda : |\lambda| < 1\}$*,* $\{\lambda : |\lambda| = 1\}$ *and* $\{\lambda : |\lambda| > 1\}$*, respectively. Moreover, the range of the spectral projection of* $\Pi'(\phi_0)$ *associated with the spectral set* $\sigma_u \cup \sigma_c$ *is finite-dimensional.*

(b) ϕ_0 *is stable for the restriction* $\Pi|_{W^c_{loc}}$*, where* W^c_{loc} *is a local center manifold of* ϕ_0*.*

(c) *There are sequences* u_k *and* p_k *such that* $u_k \to \phi_0$ *and* (3.18) *holds.*

Then there is a subsequence of $\{\Pi^{p_k+1}(u_k)\}$ *which converges to an element of* $W^u \setminus \{\phi\}$ *where* W^u *is the (strong) unstable manifold of* ϕ_0 *relative to the map* Π*.*

Hypotheses (a), (b) are verified in the present situation in a standard way using spectral properties of Schrödinger operators and the spectral mapping theorem (see for example [4, 15, 17]). For the one-dimensional problem considered here, the local center manifold W^c_{loc} is one-dimensional and consists of shifts of ϕ_0 (so the stability requirement holds trivially). It is not necessary to recall the precise meaning of the unstable manifold W^u here; the only property we will use is that $W^u \setminus \{\phi\}$ does not contain any fixed point of Π.

Thus, Lemma BP implies that passing to a subsequence one has

$$\Pi^{p_k+1}(u_k)\} \to w \text{ in } X, \tag{3.19}$$

where w is not a a steady state of (1.1). Recalling that $u_k = u^\vartheta(\cdot + x_k, n_k)$ and using the translation invariance of equation (1.1), we obtain

$$\Pi^{p_k+1}(u_k) = u^\vartheta(\cdot + x_k, n_k + p_k + 1).$$

Since the convergence in X is the uniform convergence on $\mathbb{R}$, (3.19) implies

$$u^\vartheta(\cdot + \xi(n_k + p_k + 1), n_k + p_k + 1) - w(\cdot - x_k + \xi(n_k + p_k + 1)) \to 0.$$

Using this and (3.16), we obtain

$$\phi_0 - w(\cdot - x_k + \xi(n_k + p_k + 1)) \to 0$$

in X, which is clearly impossible as w is not a fixed point of Π (in particular, it is not a shift of ϕ_0).

This contradiction shows that the sequence $\{\Pi^n(g)\}$ in Cauchy in X, completing the proof of Lemma 3.5. □

Proof of Proposition 3.2. To simplify the notation, we set $u^\mu = u(\cdot,\cdot,\psi_\mu)$. Similarly as in the proof of Lemma 3.5, we consider the following two statements:

(L0) There is $T \geq 0$ such that $u^\mu(x,T) < 0 \quad (x \in \mathbb{R})$.

(G0) There is T such that $u^\mu(\cdot,T) > \theta$ on a closed interval of length ℓ.

If $\mu \in [0,1]$ is such that (L0) holds, then, by Lemma 3.3 one has $\zeta := \sup_{x\in\mathbb{R}} u^\mu(x,\tau) < 0$. Therefore, by comparison with the ODE solution $y(t,\zeta)$, we obtain $\lim_{t\to\infty} u^\mu(\cdot,t) = \alpha$ in $L^\infty(\mathbb{R})$. If, on the other hand, (G0) holds, then $\lim_{t\to\infty} u^\mu(\cdot,t) = \gamma$ in $L^\infty_{loc}(\mathbb{R})$ by Lemma 3.1.

Set

$$M_0 := \{\mu \in [0,1] : \text{(L0) holds}\},$$
$$M_1 := \{\mu \in [0,1] : \text{(G0) holds}\}.$$

Assumption (a2), Lemma 3.3, and the continuous dependence (in $L^\infty(\mathbb{R})$) of solutions on their initial data imply that the sets M_0 and M_1 are both open in $[0,1]$. Also, by (a2) and the comparison principle, these sets are intervals. Now, assumption (a1) implies that $0 \in M_0$ and $1 \in M_1$. Obviously, $M_0 \cap M_1 = \emptyset$. Thus the set of $\mu \in [0,1]$ for which none of (L0), (G0) holds is a nonempty closed set J in $(0,1)$ containing in particular $\mu^* := \sup M_0$. To complete the proof of Proposition 3.2, we prove that for each $\mu \in J$ one has

$$\lim_{t\to\infty} u^\mu(\cdot,t) = \phi_0 \text{ in } L^\infty(\mathbb{R}). \tag{3.20}$$

First we show that (3.20) holds with $L^\infty(\mathbb{R})$ replaced by $L^\infty_{loc}(\mathbb{R})$:

$$\lim_{t\to\infty} u^\mu(\cdot,t) = \phi_0 \text{ in } L^\infty_{loc}(\mathbb{R}). \tag{3.21}$$

We start by showing that $u^\mu(\cdot,t)$ has a limit in $L^\infty_{loc}(\mathbb{R})$ or, in other words, that $\omega(u^\mu)$ consists of a single function. Clearly, $\omega(u^\mu)$ consists of even functions. Therefore, by a straightforward application of Lemma 3.4 one shows that $\omega(u^\mu)$ can only contain the constant steady state α or the ground state

ϕ_0 (note that (3.6) is excluded, as (G0) does not hold). The connectedness of $\omega(u^\mu)$ implies that either $\omega(u^\mu) = \{\alpha\}$ or $\omega(u^\mu) = \{\phi_0\}$.

Thus, to prove (3.21), we just need to rule out the possibility $\omega(u^\mu) = \{\alpha\}$. Assume it holds. Then, since (L0) does not hold (remember that $\mu \in J$) and $u^\mu(\cdot, t)$ is even, for each $t > 0$ there is $\zeta(t) > 0$ such that $u^\mu(\pm\zeta(t), t) > 0$. From the assumed convergence $u^\mu(\cdot, t) \to \alpha$ in $L^\infty_{loc}(\mathbb{R})$ it follows that $\zeta(t) \to \infty$ as $t \to \infty$. We now employ the solution $u(\cdot, t, g)$ from Lemma 3.5, where we choose $\beta \in (\eta, 0)$. Take a shift $\tilde{g} = g(\cdot + x_0)$ so that $u(\cdot, t_0, \tilde{g}) - u(\cdot, t_0, u_0)$ has a unique zero for some $t_0 > 0$ (cp. Remark 3.6). Then, by the monotonicity of the zero number, $z(u(\cdot, t, \tilde{g}) - u^\mu(\cdot, t)) \leq 1$ for all $t > t_0$. By the translation invariance of (1.1), we have $u(\cdot, t, \tilde{g}) = u(\cdot + x_0, t, g)$, hence $u(\cdot, t, \tilde{g})$ converges in $L^\infty(\mathbb{R})$ to the ground state $\tilde{\phi} = \phi(\cdot + x_0)$, ϕ being the limit ground state of $u(\cdot, t, g)$ (cp. Lemma 3.5). Let x_1 be the maximizer of $\tilde{\phi}$. Since $\zeta(t) \to \infty$ and $u^\mu(x_1, t) \to \alpha$, for large enough t one has $-\zeta(t) < x_1 < \zeta(t)$ and

$$u^\mu(\pm\zeta(t), t) > 0 > \tilde{\phi}(\pm\zeta(t)), \quad \tilde{\phi}(x_1) > 0 > u^\mu(x_1, t). \tag{3.22}$$

These relations and the fact that $u(\cdot, t, \tilde{g}) \to \tilde{\phi}$ in $L^\infty(\mathbb{R})$ imply that $u(\cdot, t, \tilde{g}) - u^\mu(\cdot, t)$ has at least 2 zeros for large t, which is a contradiction.

Thus we have proved that $\omega(u^\mu) = \{\phi_0\}$, that is, (3.21) holds.

The last step is to prove that the limit is in fact uniform on $\mathbb{R}$: (3.20) holds. Assume it is not. In view of (3.21) and the relations $u^\mu \geq \alpha = \phi_0(\pm\infty)$, there must then exist a constant $\epsilon_0 > 0$ and sequences $t_n \to \infty$, $\lambda_n \to \infty$ such that

$$u^\mu(\pm\lambda_n, t_n) > \phi(\pm\lambda_n) + \epsilon_0 > \alpha + \epsilon_0 \quad (n = 1, 2, \dots)). \tag{3.23}$$

We employ the solution $u(\cdot, t, g)$ from Lemma 3.5 one more time. Fix $\beta \in (\eta, 0)$ and let g be as in Lemma 3.5. Let ϕ be the limit ground state of the solution $u(\cdot, t, g)$. Next take a shift $\tilde{g} = g(\cdot + x_0)$ with a large x_0 such that, first,

$$z(u(\cdot, t, \tilde{g}) - u^\mu(\cdot, t)) \leq 1 \tag{3.24}$$

for all small t (cp. Remark 3.6) and, second, the maximizer x_1 of the ground state $\tilde{\phi} = \phi(\cdot + x_0))$ is negative. The latter implies that

$$\begin{aligned} &\phi_0 < \tilde{\phi} \text{ on } (-\infty, x_1], \\ &\phi_0 > \tilde{\phi} \text{ on } [0, \infty). \end{aligned} \tag{3.25}$$

Now, relations (3.23) and the fact that $\tilde{\phi} \approx \alpha$ near $-\infty$ imply that for large n one has

$$u^\mu(-\lambda_n, t_n) > \alpha + \epsilon_0 > \tilde{\phi}(-\lambda_n).$$

Using this, (3.25), and the facts that $u(\cdot, t, \tilde{g}) \to \tilde{\phi}$ in $L^\infty(\mathbb{R})$, $u^\mu(\cdot, t) \to \phi_0$ in $L^\infty_{loc}(\mathbb{R})$, we obtain the following relations for all sufficiently large n:

$$u^\mu(-\lambda_n, t_n) > u(-\lambda_n, t_n, \tilde{g}),\ u^\mu(x_1, t_n) < u(x_1, t_n, \tilde{g}),\ u^\mu(0, t_n) > u(0, t_n, \tilde{g}).$$

Consequently, for all large enough n the function $u(\cdot, t_n, \tilde{g}) - u^\mu(\cdot, t_n)$ has at least two zeros, one in $(-\infty, x_1)$ and another one in $(x_1, 0)$. This is in contradiction with (3.24) and the monotonicity of the zero number.

This contradiction proves (3.20) and completes the proof. □

We refer to μ^* as the threshold value (relative to the family ψ_μ, $\mu \in [0, 1]$), to the solution in (t2) as the *threshold solution*, and to the solutions in (t1) as *subthreshold solutions*.

Remark 3.7. Although this is not needed below, we mention that the threshold value μ^* in Proposition 3.2 is sharp: for all $\mu \in (\mu^*, 1]$ one has $\lim_{t\to\infty} u^\mu(\cdot, t) = \gamma$ in $L^\infty_{loc}(\mathbb{R})$. Indeed, we have proved in the proof of Proposition 3.2 that (3.20) is the only alternative to (L0) and (G0). Now one can prove in a number of ways that (3.20) cannot hold for $\mu > \mu^*$. One possibility is to use an argument based on the zero number as in [9, 27] (the case $\eta = \alpha$ is covered by [9] and the case $\eta \in (\alpha, 0)$ is even simpler to handle by similar arguments). Alternatively, one could use the linear instability of ϕ_0 and invariant manifold techniques, or arguments based on exponential separation similarly as in [24].

4 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on a recursive construction. The idea is, similarly as in [26], to define $u_0 \in \mathcal{B}$ successively on a sequence of larger and larger intervals. The definition on each of the intervals guarantees a certain subthreshold behavior of the solution $u(\cdot, t, u_0)$, regardless of the values of u_0 outside that interval. This allows us to prescribe the behavior of the solution along two sequences of times approaching infinity. This way we show that the constant steady state α as well as a ground state are contained in $\omega(u_0)$.

Zero number arguments are then used to prove that $\omega(u_0)$ does not contain any other nonconstant steady state.

Our recursive construction is detailed in the following lemma.

Lemma 4.1. *Let $\{\eta_k\}_k$, $\{\theta_k\}_k$ be monotone sequences in $(\alpha, 0)$, $(0, \gamma)$, respectively, such that $\eta_k \nearrow 0$ and $\theta_k \searrow 0$. Let ϕ_0 be the ground state of* (1.7) *with maximum at $x = 0$. There exist $t_0 \in (0, 1)$ and a sequence $(u_k, R_k, \rho_k, t_k, \delta_k)$, $k = 1, 2, \dots,$ in $\mathcal{B} \times (0, \infty)^4$ such that the statements* (i)-(iv) *below are valid for all $k = 1, 2, \dots,$ and statements* (v), (vi) *are valid for all $k = 2, 3 \dots$.*

(i) $t_{2k} > t_{2k-1} > 1$, $\rho_k > R_k > 2$,

(ii) u_k *is piecewise linear and even, and* $\operatorname{spt}(u_k - \eta_k) \subset (-R_k, R_k)$.

(iii) *The solution $u(\cdot, \cdot, u_k)$ has the following properties:*

$$\lim_{t\to\infty} \|u(\cdot, t, u_k) - \alpha\|_{L^\infty(\mathbb{R})} = 0, \tag{4.1}$$

$$u(x, t, u_k) < 0 \quad (x \in \mathbb{R} \setminus (-R_k, R_k),\ t \in [0, t_{2k}]), \tag{4.2}$$

$$\|u(\cdot, t_{2k-1}, u_k) - \phi_0\|_{L^\infty(\mathbb{R})},\ \|u(\cdot, t_{2k}, u_k) - \alpha\|_{L^\infty(\mathbb{R})} < \frac{1}{k}, \tag{4.3}$$

$$u(\cdot, t_{2k-2}, u_k) \text{ has exactly four zeros, all of them simple,} \tag{4.4}$$

$$u(x, t_{2k}, u_k) < 0 \quad (x \in \mathbb{R}). \tag{4.5}$$

(iv) *For each $u_0 \in \mathcal{B}$ with $\|u_k - u_0\|_{L^\infty(-\rho_k, \rho_k)} < \delta_k$, the following relations hold:*

$$u(x, t, u_0) < 0 \quad (x = \pm R_k,\ t \in [0, t_{2k}]), \tag{4.6}$$

$$\|u(\cdot, t_{2k-1}, u_0) - \phi_0\|_{L^\infty(-R_k, R_k)},\ \|u(\cdot, t_{2k}, u_0) - \alpha\|_{L^\infty(-R_k, R_k)} < \frac{2}{k}, \tag{4.7}$$

$$u(\cdot, t_{2k-2}, u_0) \text{ has exactly four zeros in } (-R_k, R_k), \text{ all of them simple,} \tag{4.8}$$

$$z_{(-R_k, R_k)}(u(\cdot, t, u_0)) \le 4 \quad (t \in [t_{2k-2}, t_{2k}]), \tag{4.9}$$

$$u(x, t_{2k}, u_0) < 0 \quad (x \in [-R_k, R_k]). \tag{4.10}$$

(v) $t_{2k-1} > t_{2k-2} + 1$, $R_k > R_{k-1} + 1$, $\delta_{k+1} < \delta_k/2$.

(vi) $u_k \equiv u_{k-1}$ *on* $[-\rho_{k-1}, \rho_{k-1}]$ *and* $\eta_{k-1} \le u_k \le \theta_k$ *on* $\mathbb{R} \setminus [-\rho_{k-1}, \rho_{k-1}]$.

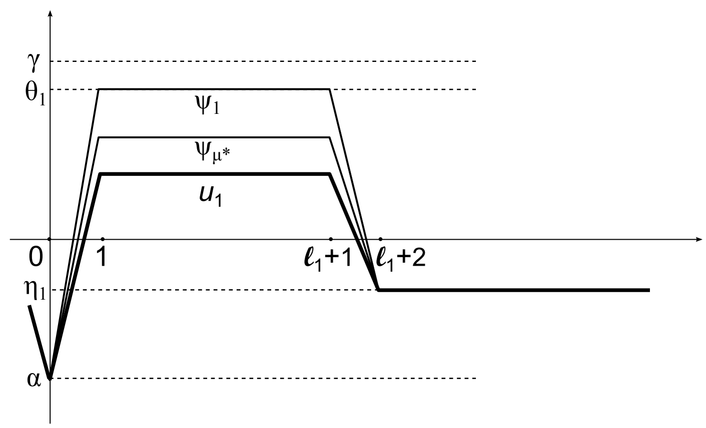

Figure 3: The graphs of ψ_1, ψ_{μ^*}, and u_1. One has $u_1 = \psi_\mu$, for some $\mu < \mu^*$, $\mu \approx \mu^*$.

Proof. We shall repeatedly use Lemma 3.1 and Proposition 3.2 with $\theta = \theta_k$, $\eta = \eta_k$. The symbol $\ell(\theta)$ has the same meaning as in Lemma 3.1.

STEP 1. Let $k = 1$. Set $\ell_1 := \ell(\theta_1)$. For each $\mu \in [0, 1]$ define an even piecewise linear function ψ_μ as follows (see Figure 3):

$$\psi_\mu(x) = \begin{cases} \alpha + (\mu\theta_1 - \alpha)x & (x \in [0,1]), \\ \mu\theta_1 & (x \in [1, \ell_1 + 1]), \\ (\mu\theta_1 - \eta_1)(\ell_1 + 2 - x) + \eta_1 & (x \in [\ell_1 + 1, \ell_1 + 2]), \\ \eta_1 & (x \geq \ell_1 + 2), \\ \psi_\mu(-x) & (x < 0). \end{cases}$$

It is straightforward to verify that the family ψ_μ, $\mu \in [0, 1]$, satisfies the assumptions of Proposition 3.2 with $\theta = \theta_1$, $\eta = \eta_1$; we just remark that condition (3.1) is satisfied due to the comparison principle, as $\psi_0 \leq 0$ and $\psi_0 \equiv \eta_1 < 0$ near $\pm\infty$. Let $\mu^* \in (0, 1)$ be the threshold value, as in Proposition 3.2. Since the ψ_μ are all even in x, the limit ground state of the threshold solution $u(\cdot, \cdot, \psi_{\mu^*})$ is ϕ_0:

$$\lim_{t\to\infty} u(\cdot, t, \psi_{\mu^*}) = \phi_0 \text{ in } L^\infty(\mathbb{R}).$$

Now, by the continuity of solutions with respect to initial data (in $L^\infty(\mathbb{R})$), if $\mu < \mu^*$ is close to μ^*, then the subthreshold solution $u(\cdot, \cdot, \psi_\mu)$ gets close to ϕ_0 at a large time t_1, and then approaches α in $L^\infty(\mathbb{R})$, as $t \to \infty$. Thus we can choose $\mu \in (0, \mu^*)$ and times $t_2 > t_1 > 1$ such that for $u_1 := \psi_\mu$ relations (4.1), (4.3), and (4.5) are valid. Next, Lemma 3.3 and the assumption $\eta_1 < 0$ imply that relation (4.2) holds for each sufficiently large R_1. We pick such R_1 satisfying also $R_1 > \ell_1 + 2$ so that $\mathrm{spt}(u_1 - \eta_1) \subset (-R_1, R_1)$ and statement (ii) holds. Finally, since $\mu\theta_1 > 0$, the definition of u_1 implies that $z(u(\cdot, t, u_1)) \geq 4 = z(u(\cdot, 0, u_1))$ for all small $t \geq 0$. By the monotonicity of the zero number, the equality must hold there. Using Corollary 2.3, we pick $t_0 \in (0, 1)$ such that (4.4) holds.

Relations (4.2)-(4.5) and Corollary 2.6 clearly imply the existence of constants $\rho_1 > R_1$ and $\delta_1 > 0$ such that statement (iv) holds with (4.9) excluded. Relation (4.9) follows by an application of Lemma 2.2(ii) with $I = (-R_1, R_1)$, which is legitimate by (4.6).

STEP 2 (the induction argument). Suppose that for some $n \geq 1$,

$$(u_k, R_k, \rho_k, t_k, \delta_k) \in \mathcal{B} \times (0, \infty)^4, \quad k = 1, \ldots, n,$$

have been defined such that statements (i)-(iv) hold for all $k = 1, \ldots, n$, and, in case $n \geq 2$, statements (v), (vi) hold for all $k = 2, \ldots, n$. We need to define $(u_{n+1}, R_{n+1}, \rho_{n+1}, t_{n+1}, \delta_{n+1})$ in such a way that statements (i)-(vi) are valid for $k = n + 1$.

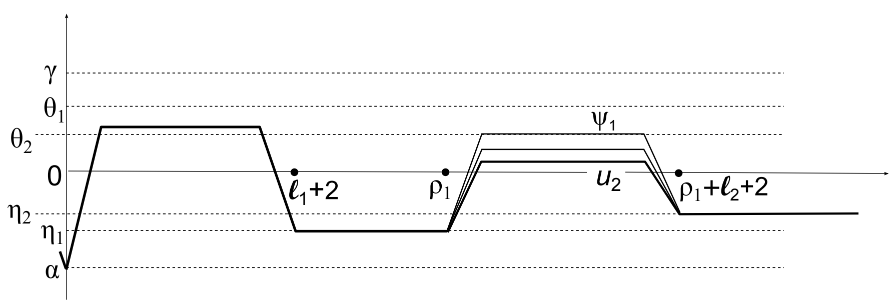

Figure 4: The graphs of ψ_1 and u_2, with the graph of ψ_{μ^*} in between. They all coincide with the graph of u_1 on $[0, \rho_1]$.

Set $\ell_{n+1} = \ell(\theta_{n+1})$. For each $\mu \in [0, 1]$ define an even piecewise linear function ψ_μ as follows (see Figure 4):

$$\psi_\mu(x) = \begin{cases} u_n(x) & (x \in [0, \rho_n]), \\ (\mu\theta_{n+1} - \eta_n)(x - \rho_n) + \eta_n & (x \in [\rho_n, \rho_n + 1]), \\ \mu\theta_{n+1} & (x \in [\rho_n + 1, \rho_n + \ell_{n+1} + 1]), \\ (\mu\theta_{n+1} - \eta_{n+1})(\rho_n + \ell_{n+1} + 2 - x) + \eta_{n+1} & \\ & (x \in [\rho_n + \ell_{n+1} + 1, \rho_n + \ell_{n+1} + 2]), \\ \eta_{n+1} & (x \geq \rho_n + \ell_{n+1} + 2), \\ \psi_\mu(-x) & (x < 0). \end{cases}$$

Since $\mathrm{spt}(u_n - \eta_n) \subset (-R_n, R_n) \subset (-\rho_n, \rho_n)$, this definition implies that $\psi_\mu \in \mathcal{B}$.

Since $\psi_\mu \equiv u_n$ on $[-\rho_n, \rho_n]$, statement (iv) with $k = n$ applies to $u_0 = \psi_\mu$ for each $\mu \in [0, 1]$. In particular, (4.6) and (4.10) give

$$u(R_n, t, \psi_\mu) < 0 \quad (t \in [0, t_{2n}],\ \mu \in [0, 1]), \tag{4.11}$$
$$u(x, t_{2n}, \psi_\mu) < 0 \quad (x \in [-R_n, R_n],\ \mu \in [0, 1]). \tag{4.12}$$

We first use these relations to verify that the family ψ_μ, $\mu \in [0, 1]$, satisfies the hypotheses of Proposition 3.2 with $\theta = \theta_{n+1}$, $\eta = \eta_{n+1}$. We just need to prove that condition (3.1) holds, all the other hypotheses are obviously satisfied. Since $\mathrm{spt}(u_n - \eta_n) \subset (-R_n, R_n)$, the definition of ψ_0 gives $\psi_0 \leq 0$ in (R_n, ∞). Combining this with (4.11) and using the comparison principle, we obtain

$$u(x, t_{2n}, \psi_0) < 0 \quad (x \geq R_n).$$

This, (4.12), and the evenness of $u(\cdot, t, \psi_0)$ show that (3.1) holds with $s_0 = t_{2n}$.

Thus Proposition 3.2 applies. Let $\mu^* \in (0, 1)$ be the threshold value as in that proposition. Again, by the evenness of ψ_μ, the limit ground state of the threshold solution $u(\cdot, \cdot, \psi_{\mu^*})$ must be ϕ_0.

We claim that for each $\mu < \mu^*$ sufficiently close to μ^*,

$$u(\cdot, t_{2n}, \psi_\mu) \text{ has exactly four zeros, all of them simple.} \tag{4.13}$$

To prove this claim, we first show that for each $t \in [0, t_{2n}]$

$$u(\cdot, t, \psi_{\mu^*}) > 0 \text{ somewhere in } (R_n, \infty). \tag{4.14}$$

We proceed by contradiction. Assume that, to the contrary,

$$u(\cdot,\tau,\psi_{\mu^*}) \le 0 \text{ on } (R_n,\infty) \text{ at some } \tau \in [0,t_{2n}]. \tag{4.15}$$

Then (4.11) and the comparison principle give $u(x,t,\psi_{\mu^*}) \le 0$ for all $(x,t) \in (R_n,\infty)\times[\tau,t_{2n}]$. Combining this with (4.12) and the evenness in x, we obtain $u(\cdot,t_{2n},\psi_{\mu^*}) \le 0$. But then the comparison principle shows that $u(\cdot,\cdot,\psi_{\mu^*})$ cannot converge to ϕ_0, a contradiction.

Hence we have proved that (4.14) holds for each $t \in [0,t_{2n}]$. In view of (4.11) and Lemma 3.3 (which we use to show that $u < 0$ near $x = \infty$), relation (4.14) means that

$$z_{(R_n,\infty)}(u(\cdot,t,\psi_{\mu^*})) \ge 2 \quad (t \in [0,t_{2n}]). \tag{4.16}$$

Now, by (4.11) and Lemma 2.2,

$$z_{(R_n,\infty)}(u(\cdot,t,\psi_\mu)) \le z_{(R_n,\infty)}(\psi_\mu) \le 2 \quad (t \in [0,t_{2n}],\ \mu \in [0,1]), \tag{4.17}$$

where the last inequality is by the definition of ψ_μ. In particular, the equality holds in (4.16). Therefore, by Corollary 2.3, $u(\cdot,t_{2n},\psi_{\mu^*})$ has exactly two zeros in (R_n,∞) both of them simple. The same is then true for all $\mu \approx \mu^*$, by (4.17) and the continuity with respect to the initial data. This, the evenness in x, and (4.12) imply (4.13).

As in STEP 1, we use the continuity with respect to initial data in $L^\infty(\mathbb{R})$, to find $\mu \in (0,\mu^*)$ so close to μ^* that (4.13) holds and that the subthreshold solution $u(\cdot,\cdot,\psi_\mu)$ is close to ϕ_0 at a large time $t_{2n+1} > t_{2n}+1$ and close to α at a later time t_{2n+2}. Relations (4.3), (4.5) are then valid with $u_{n+1} := \psi_\mu$ and $k = n+1$. Of course, (4.1) is valid for the subthreshold solution $u(\cdot,\cdot,\psi_\mu)$.

Having defined u_{n+1} and $t_{2n+2} > t_{2n+1}$, we use Lemma 3.3 to pick R_{n+1} so large that (4.2) holds with $k = n+1$ and, in addition, $R_{n+1} > \rho_n + \ell_{n+1} + 2$ (the latter gives $\operatorname{spt} u_{n+1} \subset (-R_{n+1}, R_{n+1})$). Hence statements (i)-(iii) hold with $k = n+1$. We next use Corollary 2.6 to find $\rho_{n+1} > R_{n+1}$ and $\delta_{n+1} \in (0,\delta_n/2)$ such that statement (iv) holds (to verify (4.9) one uses Lemma 2.2 and (4.6)).

Relations (v) and (vi) hold by construction and the monotonicity of the sequences $\{\eta_n\}$, $\{\theta_n\}$.

This completes the induction argument and thereby the proof of Lemma 4.1. □

Completion of the proof of Theorem 1.1. With ϕ_0, η_k, θ_k, and $(u_k, R_k, \rho_k, t_k, \delta_k)$ as in Lemma 4.1, take any $u_0 \in \mathcal{B}$ with

$$\|u_k - u_0\|_{L^\infty(-\rho_k,\rho_k)} < \delta_k \quad (k = 1,2,\dots). \tag{4.18}$$

For example, set

$$u_0(x) \equiv u_k(x) \quad (|x| \le \rho_k,\ k = 1, 2, \dots), \tag{4.19}$$

which is a correctly defined function, in view of (vi), and it is in $\mathcal{B}$, as $u_k \in \mathcal{B}$.

Since $\eta_k, \theta_k, \delta_k \to 0$, and $\rho_k \to \infty$ (see statements (i) and (v)), relations (4.18) and (vi) imply that $u_0 \in C_0(\mathbb{R})$.

By (4.18), statement (iv) applies to u_0 for each k. Since $R_k, t_k \to \infty$, from (4.7) we obtain that the equilibria α and ϕ_0 are contained in $\omega(u_0)$. We next show that if φ is a nonconstant equilibrium of (1.1), then $\varphi \notin \omega(u_0)$.

First of all, no ground state other than ϕ_0 can be contained in $\omega(u_0)$ by the evenness of u_0. If $\varphi \in \mathcal{B}$ is a nonconstant equilibrium of (1.1) which is not a ground state, then it is periodic and φ has infinitely many zeros, all of them simple (cp. Lemma 2.1). Assume $\varphi \in \omega(u)$. Then there is a sequence $s_k \to \infty$ such that $u(\cdot, s_k) \to \varphi$ in $L^\infty_{loc}(\mathbb{R})$. Consequently, if $b > 0$ is sufficiently large, then there is k_0 such that

$$z_{(-b,b)}(u(\cdot, s_k, u_0)) \ge 5 \quad (k = k_0, k_0 + 1, \dots).$$

On the other hand, relations (4.9), (i), and (v) imply that for each $b > 0$ there is $\tau_b > 0$ such

$$z_{(-b,b)}(u(\cdot, t, u_0)) \le 4 \quad (t \ge \tau_b).$$

This contradiction shows that no nonconstant periodic equilibrium can be contained in $\omega(u_0)$. Thus, since $\alpha, \phi_0 \in \omega(u_0)$, the connectedness of $\omega(u_0)$ implies that it contains some functions which are not equilibria of (1.1). The proof is now complete. □

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