Nonexistence of radial time-periodic solutions of reaction-diffusion equations with generic nonlinearities

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Abstract

We consider reaction-diffusion equations $u_t = \Delta u + f(u)$ on the entire space $\mathbb{R}^N$, $N \geq 4$. Assuming that the function f is sufficiently smooth (C^2 is sufficient) and has only nondegenerate zeros, we prove that the equation has no bounded solutions $u(x,t)$ which are radial in x, and periodic and nonconstant in t. We also prove some weaker nonexistence results for $N = 3$. In dimensions $N = 1, 2$, the nonexistence of time-periodic solutions (radial or not) is known by results of Gallay and Slijepčević.

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Contents

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1 Introduction

We consider semilinear parabolic equations of the form

$$u_t = \Delta u + f(u), \qquad x \in \mathbb{R}^N,\ t \in J, \tag{1.1}$$

where Δ in the Laplace operator in the spatial variable $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, f is a C^1 function on $\mathbb{R}$, and $J \subset \mathbb{R}$ is an open interval (usually, $J = (0, \infty)$ or $J = (-\infty, \infty)$).

In the qualitative theory of time-autonomous evolution equations, such as (1.1), one of the most basic questions is whether bounded nonstationary periodic solutions[1] can exist. For (1.1), the question is open in general; we will address it in this paper to some extent, focusing on solutions which are radially symmetric in x (radial solutions, for short).

To put the question in some context, consider first the bounded-domain counterpart of equation (1.1):

$$u_t = \Delta u + f(u), \quad x \in \Omega,\ t \in J. \tag{1.2}$$

Here, Ω is a bounded C^1 domain in $\mathbb{R}^N$, and f and J are as above. Coupling the equation with a common Dirichlet, Neumann, or Robin time-independent boundary condition, the problem of existence of periodic solutions is resolved easily by means of a Lyapunov functional. More specifically, it is well-known that the *energy functional*

$$E_\Omega(w) = \int_\Omega \left(\frac{1}{2} |\nabla w(x)|^2 - F(w(x)) \right) dx, \quad F(u) = \int_0^u f(s)\, ds, \tag{1.3}$$

is decreasing strictly along any nonstationary solution. Thus, trivially, no periodic nonstationary solution can exist.

For the equation on $\mathbb{R}^N$, a similar argument can be applied—replacing Ω by $\mathbb{R}^N$ in (1.3)—but its scope is limited to spatially localized solutions along which the existence and finiteness of the energy integral can be proved. Beyond localized solutions, the energy functional is seemingly of no help in

[1]By a periodic solution, we always mean a solution periodic in t.

resolving the problem of existence of periodic solutions. However, as discovered in [9], very relevant for the problem is the family of functionals (1.3), where one takes $\Omega = B_R$, $R > 0$, $B_R \subset \mathbb{R}^N$ being the ball of radius R centered at the origin. A theorem of [9] says that if $N = 1$ or $N = 2$, then there is no bounded nonstationary solution u of (1.1) satisfying, for some $T > 0$, $R_0 > 0$, the following relations:

$$E_{B_R}(u(\cdot, T)) \geq E_{B_R}(u(\cdot, 0)) \quad (R > R_0). \tag{1.4}$$

This clearly rules out the existence of bounded nonstationary periodic solutions of (1.1). As further shown in [9, 10], these conclusions in dimensions $N = 1, 2$ are valid for a larger class of nonlinearities—allowing the function $f = f(x, u)$ to depend on x—and for a number of different types of equations obeying certain dissipation rules. Of course, insisting on the boundedness of the solution is necessary here. Unbounded periodic solutions exist even for the one-dimensional heat equation $u_t = u_{xx}$; an example is the real part of $e^{\lambda^2 t + \lambda x}$, where λ is any nonzero complex number whose square is on the imaginary axis.

For $N \geq 3$, the energy estimates of [9, 10] are not strong enough to rule out the existence of periodic solutions. In fact, an example of [9] shows that (1.4) can hold, even with $R_0 = 0$, for bounded radial solutions of (1.1), with f replaced by $f(r, u)$, $r = |x|$. Another, example in [9] shows the existence of a bounded radial nonstationary periodic solution of such an equation with $f(r, u) = W(r)u$ linear in u. However, the potential W is unbounded in that example.

Coming back to (1.1), let us now consider a radial solution u of that equation. Viewed as a function of t and the real variable $r = |x| \geq 0$, u is a solution of following problem:

$$u_t = u_{rr} + \frac{N-1}{r} u_r + f(u), \qquad r > 0,\ t \in J, \tag{1.5}$$

$$u_r(0, t) = 0, \qquad t > 0. \tag{1.6}$$

A bounded-domain counterpart of this problem is equation (1.2) where $\Omega \subset \mathbb{R}^N$ is a rotationally symmetric domain, a ball or an annulus, and one of the standard boundary conditions is assumed. The radial solutions are then solutions of (1.5) on an interval satisfying the corresponding boundary conditions (in the case of a ball, the boundary condition at $r = 0$ comes from the symmetry and is the same as in (1.6)). For such solutions, there is an alternative to using the energy functional for showing the nonexistence of periodic solutions, namely, a zero-number argument. For equations on a ball B_R (with a standard boundary condition on ∂B_R), it goes as follows. Assume u is a bounded radial solution defined on an open time-interval J, and consider its time-derivative $v := u_t$, which is a solution of a linear parabolic equation. The zero number of $v(\cdot, t)$, $z(v(\cdot, t))$, is defined as the number of zeros of $v(\cdot, t)$ in the spatial interval $[0, R]$. It is well-known that if $v \not\equiv 0$, then $z(v(\cdot, t))$, as a function of t, is finite and nonincreasing on J, and, moreover, its value drops strictly at any $t_0 \in J$ such that $v(\cdot, t_0)$ has a multiple zero in $[0, R]$ (see [4]; for predecessors of these results concerning equations with one spatial variable see [1, 3, 18]). Now, for a periodic solution u, the monotone periodic function $z(u_t(\cdot, t))$ has to be constant. On the other hand, by the periodicity of $u(0, t)$, there is $t_0 > 0$ such that $u_t(0, t_0) = 0$. Since $r = 0$ is then automatically a multiple zero of $u_t(\cdot, t_0)$—$u_{tr}(0, t) = 0$ holds for all t due the boundary condition (1.6)—$z(u_t(\cdot, t))$ has to drop somewhere and thus cannot stay constant, unless $u_t \equiv 0$. This shows that all bounded periodic solutions are steady states.

The above zero-number argument goes through, with no change, for solutions of (1.5), (1.6), if one *assumes* that $z(u_t(\cdot, t))$ is finite. Thus, with this argument we also have a finiteness issue, not unlike with the energy argument, which limits the argument's applicability in the problem on $\mathbb{R}^N$. This, certainly, is not unexpected; the possibility of an infinite zero number is a common difficulty one faces when using intersection comparison arguments in studies of parabolic equations on unbounded intervals. It has been dealt with in some situations by ad hoc techniques (see [7, 19] for some examples), but it is also known that the behavior of bounded solutions of (1.5), (1.6) may be qualitatively very different than the behavior of solutions on bounded intervals (examples of this can be found in [8, 20, 21] where the power nonlinearities $f(u) = u^p$ are examined). In the present paper, we in a way bypass the difficulty with infinite zero number: we mostly employ intersection comparison arguments on large bounded intervals. We facilitate this approach by preliminary results describing the asymptotics of stationary and (hypothetical) periodic solutions of (1.5), (1.6) as $r \to \infty$.

In dimensions $N \geq 4$, we are able to show, by zero-number arguments, the nonexistence of bounded nonstationary periodic solutions for any C^2-nonlinearity with only nondegenerate zeros. Here is that result more specifically:

Theorem 1.1. *Assume that $N \geq 4$, $f : \mathbb{R} \to \mathbb{R}$ is of class C^2, and $f'(\xi) \neq 0$, whenever $f(\xi) = 0$. Then equation* (1.1) *has no bounded radial periodic solutions other than steady states.*

In the next section, we give a slightly stronger version of this theorem, reducing the regularity requirement (cp. Corollary 2.6).

The above theorem does not include the case $N = 3$, although we do have partial results concerning this dimension; see the next section. Dimensions $N = 1, 2$ are covered by the results of [9, 10], as mentioned above.

Unlike [9, 10], we are only considering radial solutions, which is a limitation of the intersection comparison techniques. Now, the fact that the equation for radial solutions reduces to the one-dimensional (singular) problem (1.5), (1.6) suggests a thought that it may be possible to adapt to this problem the energy arguments used in [9, 10] for one-dimensional equations (1.1). However, the examples of radial periodic solutions and radial solutions satisfying (1.4), as given in [9] and mentioned briefly above, speak against such a possibility.

For $N \geq 4$, our theorem shows that generically with respect f (in suitable topological spaces of C^2 functions), equation (1.1) has no bounded nonstationary radial periodic solutions. We emphasize, however, that the genericity is expressed by the explicit nondegeneracy condition: $f'(\xi) \neq 0$, whenever $f(\xi) = 0$.

Let us elaborate a little on the role of the nondegeneracy condition in our results. One of the theorems given in the next section, which is valid in any dimension, says that if u is a bounded periodic solution of (1.5), (1.6), then the limit $\zeta = \lim_{r \to \infty} u(r, t)$ exists and is a zero of f (independent of t). The proof of this theorem does use the nondegeneracy condition but in a very unessential way (much weaker conditions are sufficient). Once the existence of the limit $\zeta \in f^{-1}\{0\}$ is known for a given periodic solution u, to prove that u is a steady state we only need the nondegeneracy at that particular zero of f: $f'(\zeta) \neq 0$; no other nondegeneracy conditions are needed. The proof is simple in the case $f'(\zeta) < 0$; it works in any dimension (including $N = 3$) and under the weaker condition requiring merely that $f'(u) \leq 0$ on a neighborhood of ζ. The proof in the case $f'(\zeta) > 0$ is rather involved and it is in that case that we need the condition $N \geq 4$. For $N = 3$ our proof applies under more restrictive conditions; for example, it works if f is of class C^3 and the function $u \mapsto f(u - \zeta)$ is odd.

To conclude our introductory discussion, we make a remark about the connection between equation (1.5) and the one-dimensional equation

$$u_t = u_{rr} + f(u), \quad r \in \mathbb{R}. \tag{1.7}$$

The latter can viewed, at least formally, as the "limit equation" of (1.5) as $r \to \infty$, so one naturally hopes to gain some information about solutions of (1.5) from (1.7). We use this idea at several places. For example, in our proof of the existence of the limit $\zeta = \lim_{r\to\infty} u(r,t)$ for a periodic solution of (1.7), we employ equation (1.7) and use the fact that it has no bounded nonstationary periodic solution [9, 10]. Of course, there is nothing new about the observation that equations (1.5) and (1.7) are related; this has been used in a number of other results; see [14, 24] for some early ones. We also mention the recent papers [6, 22], where an interesting aspect of the relation between (1.5) and (1.7) is shown: the asymptotic shape (as $t \to \infty$) of propagating solutions of (1.5) is described in term of steady states of (1.5) and propagating terraces of (1.7).

The rest of the paper is organized as follows. Our main results are given in Section 2. In Section 3, after stating some zero-number results, we prove that for a given bounded periodic solution u of (1.5), (1.6) the limit $\zeta = \lim_{r\to\infty} u(r,t)$ exists. In Sections 4 and 5, we then separately consider the cases $f'(\zeta) < 0$ and $f'(\zeta) > 0$, and prove that u is a steady state.

2 Main results

In the rest of the paper, we assume that $N \geq 3$ and make the following our standing regularity assumption on the nonlinearity f:

(R) $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function with bounded derivative

Of course, the boundedness of f', or the global Lipschitz continuity of f, is just a convenience assumption which has no effect on the validity of our results concerning individual bounded solutions. We can always modify f outside the range of the solutions in question to achieve the global Lipschitz continuity.

In most results, we also need some nondegeneracy condition. The following global nondegeneracy condition is assumed in some results, as specified.

(ND) One has $f'(\zeta) \neq 0$ whenever $f(\zeta) = 0$.

Proposition 2.1. *Assuming* (ND)*, let* $u(r,t)$ *be a radial bounded periodic solution of* (1.1)*. Then there is* $\zeta \in f^{-1}\{0\}$ *such that*

$$\lim_{r\to\infty} u(r,t) = \zeta, \quad \textit{uniformly in } t \in \mathbb{R}. \tag{2.1}$$

The proof of this proposition is given in Section 3.

Remark 2.2. (i) For steady states $u = u(r)$, the existence of the limit in (2.1) is known from earlier papers. It was proved in [13] (see also [15, 17]) that if ψ is any bounded radial steady state of (1.1), or more generally, any solution of equation (3.3) below which is bounded on $[1,\infty)$, then

$$(\psi(r), \psi'(r)) \to (\zeta, 0), \text{ where } \zeta \in f^{-1}\{0\}. \tag{2.2}$$

In fact, as shown in [13, 15], this conclusion is valid under much weaker nondegeneracy conditions than (ND). For example, for $N \geq 3$ it is sufficient that for any $\zeta \in f^{-1}\{0\}$ there be $\epsilon > 0$ with the following property: if I is any of the intervals $(\zeta - \epsilon, \zeta)$, $(\zeta, \zeta + \epsilon)$, then the function f' does not change sign in I (that is, it does not assume both positive and negative values in I). This condition is clearly satisfied if f is analytic in a neighborhood of ζ or, more generally, if $f'(u) = c(u-\zeta)^k + o(|u-\zeta|^q)$ as $u \to \zeta$, where k is a nonnegative integer and q, c are constants satisfying $q > k$, $c \neq 0$. Examples of equations (1.1) with nonconvergent bounded radial steady states were given in [16] for $N = 2$. We are not aware of such examples for $N \geq 3$.

(ii) In the proof of Proposition 2.1, we will actually show, without assuming (ND), that the convergence for the steady states implies the convergence for periodic solutions: if the statement of Proposition 2.1 holds with the extra assumption that $u = u(r)$ is a steady state, then the statement also holds without this extra assumption.

We now focus our attention on periodic solutions satisfying (2.1) for some $\zeta \in f^{-1}\{0\}$. We will assume that ζ is a nondegenerate zero of f: $f'(\zeta) \neq 0$, and consider separately the cases $f'(\zeta) < 0$ and $f'(\zeta) > 0$. We remark that the nondegeneracy of other zeros of f is not needed in the corresponding theorems.

In the case $f'(\zeta) < 0$, we have the following general theorem.

Theorem 2.3. *Let* $u(r,t)$ *be a radial bounded periodic solution of* (1.1)*. Assume that* (2.1) *holds with* $f(\zeta) = 0 > f'(\zeta)$*. Then* u *is a steady state.*

Note that this theorem needs no additional assumptions; just the standing regularity assumption (R) and the condition $f'(\zeta) < 0$ are assumed. The condition $f'(\zeta) < 0$ can be replaced by the weaker condition that $f'(u) \leq 0$ on a neighborhood of ζ. We will prove the conclusion of the theorem under this weaker assumption in Section 4.

In our theorem for $f'(\zeta) > 0$, we have additional hypotheses:

Theorem 2.4. *Let $u(r,t)$ be a radial bounded periodic solution of* (1.1). *Assume that* (2.1) *holds with $f(\zeta) = 0 < f'(\zeta)$. If $N \geq 4$, assume further that f is of class $C^{1,\sigma}$ for some $\sigma \in (2/(N-1), 1]$; if $N = 3$, assume further that f is of class $C^{2,\sigma}$ for some $\sigma \in (0,1]$ and $f''(\zeta) = 0$. Then u is a steady state.*

The proof of this theorem is given in Section 5.

Remark 2.5. The extra assumptions in this theorem are used in the proof of a certain property of radial steady states of (1.1) (see Lemma 5.2 below). Most likely, they are just technical assumptions. Obviously, there is a significant difference between the assumptions for $N \geq 4$, which just add a minor extra regularity requirement to the standing hypothesis (R), and the assumptions for $N = 3$, which include the condition $f''(\zeta) = 0$. While the assumed nondegeneracy condition $f'(\zeta) \neq 0$ is generically satisfied, the "degeneracy condition" $f''(\zeta) = 0$ is quite restrictive. It holds, however, if $v \mapsto f(\zeta + v)$ is an odd function.

Incidentally, the extra assumptions in Theorem 2.4 are the same as the technical assumptions used in [12] for the proof a result concerning radial steady states, which we recall in Lemma 5.1(i) below (these technical assumptions were later removed in [11]).

The following result is a direct consequence of Proposition 2.1 and Theorems 2.3, 2.4. It is slightly stronger than Theorem 1.1 given in the introduction.

Corollary 2.6. *Assume that $N \geq 4$, f is of class $C^{1,\sigma}$ with $2/(N-1) < \sigma \leq 1$, and condition* (ND) *holds. Then equation* (1.1) *has no bounded radial periodic solutions other than steady states.*

3 Steady states and intersection comparison principles

In this section, we first recall several zero number properties of radial solutions of linear parabolic equations and then use them to prove some useful conclusions concerning radial solutions of (1.1). With a slight abuse of notation, we often view radial functions $\psi(x)$ on $\mathbb{R}^N$ as functions of the real variable $r = |x|$.

Let J be an open interval. For any two solutions u, $\tilde{u}$ of (1.1), the function $v := u - \tilde{u}$ solves the linear equation

$$v_t = \Delta v + c(x,t)v, \qquad |x| < r_1,\ t \in J, \tag{3.1}$$

where $r_1 = \infty$ and c is a continuous bounded function given by

$$c(x,t) = \int_0^1 f'(\tilde{u}(x,t) + s(u(x,t) - \tilde{u}(x,t))\, ds.$$

If u, $\tilde{u}$ are radial solutions, then $c = c(r,t)$ is radial as well and the equation for $v = v(r,t)$ takes the form

$$v_t = v_{rr} + \frac{N-1}{r} v_r + c(r,t)v, \qquad r_0 < r < r_1,\ t \in J, \tag{3.2}$$

with $r_0 = 0$, $r_1 = \infty$, and we also have $v_r(0,t) = 0$ for $t \in J$. The zeros of the function $r \mapsto v(r,t)$, which we examine in this section, give the points of intersection of the graphs of the radial solutions $u(\cdot,t)$, $\tilde{u}(\cdot,t)$ (this is the reason for the "intersection comparison principles" in the title of this section). In most cases, we will assume that $\tilde{u}$ is a radial steady state of (1.1), but we also consider differences $v := u - \psi$, where ψ is just a solution of the equation

$$\psi_{rr} + \frac{N-1}{r}\psi_r + f(\psi) = 0, \qquad r > 0. \tag{3.3}$$

Such a solution is not necessarily a steady state of equation (1.1); it may well be unbounded as $r \searrow 0$ (one can then refer to ψ as a singular steady state, but we do adopt this terminology here). This is the reason why we sometimes consider equations (3.2) with $r_0 > 0$. At occasions, it will also be convenient to take $r_1 < \infty$.

Note that (3.2) is a regular parabolic equation if $r_0 > 0$. Likewise, (3.3) is a regular ordinary differential equation on $(0, \infty)$, and our standing assumption (R) implies that for any $(a, b) \in \mathbb{R}^2$ and $r_0 > 0$ the solution of (3.3) satisfying the initial conditions

$$\psi(r_0) = a, \quad \psi_r(r_0) = b, \tag{3.4}$$

is defined globally on $(0, \infty)$. It is also well known that for $r_0 = 0$, $b = 0$ the initial value problem (3.3), (3.4) is well posed: it has a unique solution $\psi(\cdot, a) \in C^1[0, \infty) \cap C^2(0, \infty)$, and for any $R > 0$ the $C^1[0, R]$-valued map $a \mapsto \psi(\cdot, a)$ depends continuously on the initial value a. This can be proved in a standard way by a fixed-point argument (considering first a short r-interval) applied to the integral equation

$$u(r) = a - \int_0^r s^{-(N-1)} \int_0^s \theta^{N-1} f(u(\theta))\, d\theta ds. \tag{3.5}$$

The following lemma regarding the behavior of solutions as $r \to 0$ will be useful below.

Lemma 3.1. *Let ψ be a solution of* (3.3)*. Then either $\psi(r)$ is unbounded as $r \to 0+$ or else $\psi = \psi(\cdot, a)$ for some $a \in \mathbb{R}$ (that is, ψ is the solution of* (3.3), (3.4) *with $r_0 = 0$ and $b = 0$).*

Proof. From (3.3), we have

$$r^{N-1}\psi_r(r) = \rho^{N-1}\psi_r(\rho) - \int_\rho^r \theta^{N-1} f(\psi(\theta))\, d\theta \quad (r > \rho > 0). \tag{3.6}$$

Assume that $\psi(r)$ stays bounded as $r \to 0+$. Then, necessarily, $\psi_r(r)$ stays bounded along a sequence $r_n \searrow 0$. Putting $\rho = r_n$ in (3.6) and taking the limit as $n \to \infty$, we obtain

$$r^{N-1}\psi_r(r) = -\int_0^r \theta^{N-1} f(\psi(\theta))\, d\theta. \tag{3.7}$$

This implies that $\psi_r(r) \to 0$ as $r \to 0$. Consequently, the limit $a := \lim_{r\to 0} \psi(r)$ exists as well. Defining $\psi(0) = a$, we see from (3.7) that ψ satisfies the integral equation (3.5) which is equivalent to problem (3.3), (3.4) with $r_0 = 0$ and $b = 0$. Thus $\psi = \psi(\cdot, a)$. $\square$

We now recall some zero-number results. When referring to equation (3.1), we always assume that c is a continuous bounded radial function on $\{x \in \mathbb{R}^N : |x| < r_1\} \times J$. Similarly, in equation (3.2), c is always assumed to be a continuous bounded function on $(r_0, r_1) \times J$.

If $I \subset [0, \infty)$ is an interval and $g : I \to \mathbb{R}$ is a continuous function, we denote by $z_I(g)$ the number of zeros of g in I. If $I = [0, \infty)$, we often omit the subscript I: $z(g) = z_{[0,\infty)}g$.

Lemma 3.2. *Let $0 \leq r_0 < r_1 \leq \infty$; and $I := [r_0, r_1]$ if $r_1 < \infty$, $I = [r_0, \infty)$, if $r_1 = \infty$. Assume that either $r_0 = 0$ and $v(r, t)$ is a nontrivial bounded radial solution of (3.1), or $r_0 > 0$ and v is a nontrivial bounded solution of (3.2) such that $v \in C(I \times J)$ and $v(r_0, t) \neq 0$ for all $t \in J$. Finally, if $r_1 < \infty$ assume also that $v(r_1, t) \neq 0$ for all $t \in J$. Then the following statements are valid:*

(i) For each $t \in J$, the zeros of $v(\cdot, t)$ in I are isolated. In particular, if $r_1 < \infty$, then $z_I(v(\cdot, t)) < \infty$ for all $t \in J$.

(ii) The function $t \mapsto z_I(v(\cdot, t))$ is monotone nonincreasing.

(iii) If for some $t_0 \in J$ the function $v(\cdot, t_0)$ has a multiple zero ρ_0 in I (that is, $v(\rho_0, t_0) = v_\rho(\rho_0, t_0) = 0$) and $z_I(v(\cdot, t_0)) < \infty$, then for any $t_1, t_2 \in J$ with $t_1 < t_0 < t_2$, one has $z_I(v(\cdot, t_1)) > z_I(v(\cdot, t_2))$.

Proof. For $r_0 = 0$ the lemma is proved in [4]; for $r_0 > 0$, proofs can be found in [1, 3]. □

Remark 3.3. Under the assumptions of Lemma 3.2, if $r_1 < \infty$, then the times $t_0 \in J$ such that the function $v(\cdot, t_0)$ has a multiple zero are isolated. Indeed, if $\bar{t} \in J$ were an accumulation point of such times t_0, then, picking $t_1, t_2 \in J$ with $t_1 < \bar{t} < t_2$, the finite zero number $z_I(v(\cdot, t))$ would have to drop infinitely many times in $[t_1, t_2]$ which is impossible by the monotonicity.

We also need the following robustness property.

Lemma 3.4. *Let $w_n(r, t)$ be a sequence of functions converging to $v(r, t)$ in $C^1(\bar{I} \times [s, T])$, where $s < T$ are numbers in J, and either $I = (r_0, r_1)$ for some $0 < r_0 < r_1 < \infty$ or $I = [0, r_1)$ for some $r_1 \in (0, \infty)$. If $I = (r_0, r_1)$, assume that v is a solution of (3.2); and if $I = [0, r_1)$, assume that v is a radial solution of (3.1) and $\partial_r w_n(0, t) = 0$ for all $t \in [s, T]$ and $n = 1, 2, \ldots$.*

Finally, assume that $v \not\equiv 0$ and $v(\cdot, t_0)$ has a multiple zero $\rho_0 \in I$ for some $t_0 \in (s, T)$. Then there exist sequences $\{r_n\}$ in I and $\{t_n\}$ in J such that $r_n \to \rho_0$, $t_n \to t_0$, and for all sufficiently large n the function $w_n(\cdot, t_n)$ has a multiple zero at r_n: $w_n(r_n, t_n) = \partial_{x_n} w_n(r_n, t_n) = 0$.

Proof. In the case $I = (r_0, r_1)$ with $0 < r_0 < r_1 < \infty$, the lemma is a reformulation of [5, Lemma 2.6]; we just remark that although [5, Lemma 2.6] is stated for equations of the form $v_t = v_{rr} + c(r,t)v$, without the drift term, its proof applies equally well to equations with the drift term $(N-1)v_r/r$ which is nonsingular on $[r_0, r_1]$. If $I = [0, r_1)$, and $\rho_0 > 0$, we can choose a small $r_0 > 0$ (in particular, $r_0 < \rho_0$) and obtain the desired conclusion by applying [5, Lemma 2.6] with $I = [0, r_1)$ replaced by $I = (r_0, r_1)$.

Assume now that $r_0 = 0 = \rho_0$. We use very similar arguments as in [5]. First of all we note that, since the zeros of $v(\cdot, t_0)$ are isolated, for any sufficiently small $r_1 > 0$, $\rho_0 = 0$ is the only zero of $v(\cdot, t_0)$ in $[0, r_1]$. Fix any such r_1. If $\epsilon > 0$ is sufficiently small, then, by continuity, $|v(r_1, t)| > 0$ for all $t \in [t_0 - \epsilon, t_0 + \epsilon]$ and, by Remark 3.3, the functions $v(\cdot, t_0 \pm \epsilon)$ have only simple zeros in $[0, r_1)$. Applying Lemma 3.2 to the solution v on the rectangle $[0, r_1] \times [t_0 - \epsilon, t_0 + \epsilon]$, we obtain that

$$z_{[0,r_1]}(v(\cdot, t_0 - \epsilon)) > z_{[0,r_1]}(v(\cdot, t_0 + \epsilon)).$$

Now, from the convergence of $w_n(r,t)$ to $v(r,t)$ in $C^1(\bar{I} \times [s,T])$, we infer that for all sufficiently large n the function w_n has the following two properties:

$$|w_n(r_1, t)| > 0 \quad (t \in [t_0 - \epsilon, t_0 + \epsilon]), \tag{3.8}$$

$$z_{[0,r_1]}(w_n(\cdot, t_0 - \epsilon)) > z_{[0,r_1]}(w_n(\cdot, t_0 + \epsilon)). \tag{3.9}$$

If $w_n(\cdot, t)$ has only simple zeros in $[0, r_1)$ (in particular, $w_n(0,t) \neq 0$) for all $t \in [t_0 - \epsilon, t_0 + \epsilon]$, then a simple continuation argument using (3.8) and the implicit function theorem shows that $z_{[0,r_1]}(w_n(\cdot, t))$ is independent of $t \in [t_0 - \epsilon, t_0 + \epsilon]$. This is impossible due to (3.9). Thus, there is $t_n \in [t_0 - \epsilon, t_0 + \epsilon]$ such that $w_n(\cdot, t)$ has a multiple zero r_n in $[r_0, r_1)$. Since the positive constants r_1 and ϵ can be chosen arbitrarily small, the conclusion of the lemma is proved. $\square$

Using the above results, we now prove the following useful proposition. It gives conditions which guarantee that a given bounded radial solution of (1.1) cannot be periodic. Note that no nondegeneracy condition is assumed in this proposition.

Proposition 3.5. *Let $u(r,t)$ be a radial bounded solution of* (1.1) *with $J = (0,\infty)$. Assume that ψ is a solution of* (3.3) *such that for some $r_1 \in (0,\infty)$ the following conditions are satisfied:*

(ci) *$u(r_1,t) - \psi(r_1) \neq 0$ for all $t > 0$.*

(cii) *There is $t_0 \in \mathbb{R}$ such the function $u(\cdot,t_0) - \psi$ has a multiple zero $\rho \in [0,r_1)$ ($\rho = 0$ is allowed only if ψ is a radial steady state of* (1.1)*, cp. Lemma 3.1).*

Then the solution u is not periodic in t.

Proof. We use Lemma 3.2 with $v := u - \psi$, $J := (0,\infty)$, $I := [r_0, r_1]$, where r_0 is defined as follows. By Lemma 3.1, either ψ is a radial steady state of (1.1) or $\psi(r)$ is unbounded as $r \searrow 0$. Set $r_0 := 0$ in the former case. In the latter case, pick any $r_0 \in (0,\rho)$ such that $u(r_0,t) - \psi(r_0) \neq 0$ for all $t > 0$ (r_0 exists as u is bounded).

This choice of r_0 and condition (ci) imply that the hypotheses of Lemma 3.2 are satisfied. Thus, the map $t \mapsto z_I(u(\cdot,t)-\psi)$ in monotone nonincreasing and, since $r_1 < \infty$, there is a constant m such $z_I(v(\cdot,t)) \leq m$ for all $t \geq 1$. If the solution u were periodic in t, then condition (cii) would imply that there is an unbounded sequence of times $t \geq 1$ such that $u(\cdot,t) - \psi$ has a multiple zero. By Lemma 3.2(iii), $z_I(u(\cdot,t) - \psi)$ would then drop infinitely many times as $t \geq 1$ increases, which is absurd. Thus, u cannot be a periodic solution. □

We can now prove Proposition 2.1.

Proof of Proposition 2.1. We prove that the statement in Proposition 2.1 holds, provided it holds for any bounded radial steady state $u = u(r)$ of (1.1) (cp. Remark 2.2(ii)). This will prove Proposition 2.1, for, as noted in Remark 2.2(i), the convergence for steady states is known to hold under the nondegeneracy condition (ND).

Let $u(r,t)$ be a radial bounded periodic solution of (1.1). We first prove that $u_t(r,t) \to 0$ as $r \to \infty$, uniformly in t. If not, there exist sequences $\{r_n\}$, $\{t_n\}$ with $r_n \to \infty$ and t_n in a bounded period interval of $u(r,\cdot)$ such that for all n one has $|u_t(r_n,t_n)| > \delta$ with $\delta > 0$. Consider the functions $u_n(r,t) := u(r_n + r, t_n + t)$, $n = 1, 2, \ldots$. Note that u_n is a bounded (uniformly in n) solution of the equation

$$\tilde{u}_t = \tilde{u}_{rr} + \frac{N-1}{r_n + r}\tilde{u}_r + f(\tilde{u}), \qquad r > -r_n,\ t \in \mathbb{R},$$

and, of course, it is periodic with the same period as u. Using the boundedness and standard parabolic estimates, and passing to a subsequence if necessary, we obtain that $u_n \to \bar{u}$, $\partial_t u_n \to \partial_t \bar{u}$, locally uniformly[2] on $\mathbb{R}^2$, where $\bar{u}$ is a bounded periodic solution of the one-dimensional equation

$$\bar{u}_t = \bar{u}_{rr} + f(\bar{u}), \qquad r \in \mathbb{R},\ t \in \mathbb{R}.$$

By [9, 10], $\bar{u}$ is a necessarily a steady state and $\bar{u}_t = 0$. This yields a contradiction to $|u_t(r_n, t_n)| > \delta$, $n = 1, 2, \dots$. Thus, $u_t(r, t) \to 0$ as $r \to \infty$, uniformly in t, is true, as claimed.

Pick now any $\tau \in \mathbb{R}$, and set $a := u(0, \tau)$, $b := 0 = u_r(0, \tau)$. Let ψ be the solution of (3.3), (3.4) with $r_0 = 0$ (so $\psi = \psi(\cdot, a)$ is a radial steady state of (1.1)). First we verify that ψ has to be bounded. Assume it is not. Then we can find $r_1 > 0$ such that $|\psi(r_1)| > \sup_{(x,t)\in\mathbb{R}^2} |u(x, t)|$. In particular, $u(r_1, t) - \psi(r_1) \neq 0$ for all t. Since $r = 0$ is a multiple zero of the function $u(\cdot, \tau) - \psi$, Proposition 3.5 tells us that u is not periodic, which is a contradiction.

Having proved that ψ is bounded, we obtain, due to our starting assumption, that the limit $\zeta := \psi(\infty) \in \mathbb{R}$ exists and $f(\zeta) = 0$. We show that $u(r, t) \to \zeta$ as $r \to \infty$, uniformly in $t \in \mathbb{R}$. Assume this is not true. Then, since u is periodic and $u_t(r, t) \to 0$ as $r \to \infty$, uniformly in t, there exists $r_1 \in (0, \infty)$ such that $u(r_1, t) \neq \psi(r_1)$ for all $t \in \mathbb{R}$. Thus, as above, Proposition 3.5 says that u is not periodic, which is a contradiction. The proof is complete. □

Remark 3.6. Notice that the arguments in the last two paragraphs of the above proof also show that if u and ζ are as in Proposition 2.1, and ψ is a solution of (3.3) such that for some τ the function $u(\cdot, \tau) - \psi$ has a multiple zero, then necessarily $\psi(\infty) = \zeta$.

4 Stable limit ζ: proof of Theorem 2.3

In this section, we assume that $u(r, t)$ is a radial bounded periodic solution of (1.1) and that (2.1) holds with the limit $\zeta \in f^{-1}\{0\}$ satisfying the following

[2] This is a slight abuse of language, which we also use at other places below. What we mean here is that for any bounded set $K \subset \mathbb{R}^2$ the functions u_n, $\partial_t u_n$ are *defined on* K for all large enough n and $u_n \to \bar{u}$, $\partial_t u_n \to \partial_t \bar{u}$ uniformly on K.

condition: there is $\delta > 0$ such that

$$f'(u) \le 0 \quad (u \in [\zeta - \delta, \zeta + \delta]). \tag{4.1}$$

This condition holds in particular if $f'(\zeta) < 0$, as assumed in Theorem 2.3. We prove that u is a steady state.

With $\delta > 0$ as in (4.1), pick ρ_0 such that

$$|u(r,t) - \zeta| < \delta \quad (r \ge \rho_0,\ t \in \mathbb{R}). \tag{4.2}$$

Let ψ be the solution of the following initial-value problem:

$$\psi_{rr} + \frac{N-1}{r}\psi_r + f(\psi) = 0, \qquad r > 0, \tag{4.3}$$

$$\psi(\rho_0) = u(\rho_0, 0),\ \psi'(\rho_0) = u_r(\rho_0, 0). \tag{4.4}$$

Note that $u(\cdot, 0) - \psi$ has a multiple zero at $r = \rho_0$. We claim that $u \equiv \psi_0$.

We prove our claim by contradiction. Assume $u \not\equiv \psi_0$. If there is $r_1 \ge \rho_0$ such that $u(r_1, t) - \psi(r_1) \ne 0$ for all t, then, according to Proposition 3.5, u cannot be periodic. So no such r_1 can exist. This and (4.2) imply that $|\psi| < \delta$ for all $r \ge \rho_0$. Pick $\epsilon > 0$ so that $|\psi| < \delta$ holds also in $[\rho_0 - \epsilon, \rho_0]$. Let now $\tilde{\psi}$ be the solution of (4.3) with the following initial conditions

$$\tilde{\psi}(\rho_0) = \psi(\rho_0),\ \tilde{\psi}'(\rho_0) = \gamma, \tag{4.5}$$

where $\gamma > \psi'(\rho_0)$ is chosen as follows. Appealing to the continuous dependence of solutions of (4.3) on the initial data, we choose $\gamma > \psi'(\rho_0)$ sufficiently close to $\psi'(\rho_0)$ so that the function $\tilde{\psi}$ has the following two properties:

(i) $|\tilde{\psi}| < \delta$ on $[\rho_0 - \epsilon, \rho_0 + \epsilon]$;

(ii) there is $\tilde{t}_0$ such that the function $u(\cdot, \tilde{t}_0) - \tilde{\psi}$ has a multiple zero $\tilde{\rho}_0 \in (\rho_0 - \epsilon, \rho_0 + \epsilon)$ (here we also use Lemma 3.4).

Just like for ψ, applying Proposition 3.5, we obtain that there can be no $r_1 \ge \rho_0 + \epsilon$ such that $u(r_1, t) - \tilde{\psi}(r_1) \ne 0$ for all t. Therefore, $|\tilde{\psi}| < \delta$ on $[\rho_0 + \epsilon, \infty)$. Thus, ψ and $\tilde{\psi}$ are distinct solutions of (4.3) satisfying $\psi(\rho_0) = \tilde{\psi}(\rho_0)$ and $|\psi|, |\tilde{\psi}| < \delta$ on $[\rho_0, \infty)$. Moreover, by Remark 3.6, $\psi(\infty) = \tilde{\psi}(\infty) = \zeta$. The difference $w := \psi - \tilde{\psi}$ is a solution of the linear equation

$$w_{rr} + \frac{N-1}{r}w_r + c(r)w = 0, \qquad r > \rho_0,$$

where

$$c(r) = \int_0^1 f'(\tilde\psi(r) + s(\psi(r) - \tilde\psi(r))\, ds.$$

We have $w(\rho_0) = 0 = w(\infty)$ and, due to (4.1), $c(r) \leq 0$ in $[\rho_0, \infty)$. The relation $w \not\equiv 0$ now gives a contradiction to the maximum principle.

We have shown that the assumption $u \not\equiv \psi$ leads to a contradiction, thus proving that $u \equiv \psi$ must hold, as claimed.

5 Unstable limit ζ: proof of Theorem 2.4

Here we assume that $u(r,t)$ is a radial bounded periodic solution of (1.1) and that (2.1) holds with $\zeta \in f^{-1}\{0\}$, $f'(\zeta) > 0$. We prove that under the additional hypotheses given in Theorem 2.4, u is a steady state. We remark that there is only one place where the extra hypotheses of Theorem 2.4 are needed, namely, Lemma 5.2; all the other lemmas and arguments in this section are valid without them.

At no cost to generality, we may assume that $\zeta = 0$ (replace f by $f(\cdot + \zeta)$ and u by $u - \zeta$). We use the following notation

$$\zeta = 0, \quad \beta := \sqrt{f'(0)}. \tag{5.1}$$

We need additional results concerning solutions ψ of (3.3). As above, we denote by $\psi(\cdot, a)$ the solution of (3.3), (3.4) with $r_0 = 0$ and $b = 0$; $|(a,b)|$ denotes the Euclidean norm of a vector $(a,b) \in \mathbb{R}$.

Lemma 5.1. *(i) If ψ is a solution of* (3.3)*, then the function $r^{(N-1)/2}\psi(r)$ is bounded as $r \to \infty$ and there are constants A, B such that*

$$\psi(r) = r^{-(N-1)/2}(A\cos\beta r + B\sin\beta r + o(1)) \quad \text{as } r \to \infty. \tag{5.2}$$

(ii) There are positive constants d, c_1 such that if $r_0 \geq 1$ and ψ is a solution of (3.3) *with $|(\psi(r_0), \psi'(r_0))| \leq d$, then*

$$|(\psi(r), \psi'(r))| \geq c_1(1+r-s)^{-(N-1)}|(\psi(s), \psi'(s))| \quad (r > s \geq r_0). \tag{5.3}$$

(iii) There are positive constants δ and ℓ, such that if $r_0 \geq 1/\delta$ and ψ is a solution of (3.3) *with $|(\psi(r_0), \psi'(r_0))| \leq \delta$, then ψ has a critical point in every subinterval of (r_0, ∞) of length ℓ.*

Proof of Lemma 5.1. For the proof of statement (i) we refer the reader to [11].

To prove statement (ii), we use the standard energy functional in a similar way as in [2, Proposition 2.6]. Assuming ψ is a solution of (3.3), let

$$E(r) := \frac{\psi_r^2(r)}{2} + F(\psi(r)), \quad F(u) := \int_0^u f(s)\,ds. \tag{5.4}$$

Then, by (3.3),

$$E'(r) = -\frac{N-1}{r}\psi_r^2(r) = -\frac{N-1}{r}(2E(r) - 2F(\psi(r))). \tag{5.5}$$

Rewriting this as

$$E'(r) + \frac{2(N-1)}{r}E(r) = \frac{2(N-1)}{r}F(\psi(r)),$$

and multiplying by the integrating factor $r^{2(N-1)}$, we obtain

$$\left(r^{2(N-1)}E(r)\right)' = 2(N-1)r^{2(N-1)-1}F(\psi(r)).$$

Since $F(0) = F'(0) = f(0) = 0$, and $F''(0) = f'(0) > 0$, there is a constant $\bar{d} > 0$ such that $F(u) \geq 0$ for $|u| \leq \bar{d}$. It follows that if $r_0 \geq 1$ and $|\psi(r)| \leq \bar{d}$ for all $r \geq r_0$, then $F(\psi(r)) \geq 0$ and the function $r^{2(N-1)}E(r)$ is nondecreasing in $[r_0, \infty)$. In this case, for all $r > s \geq r_0$ one has

$$E(s) \leq \left(\frac{r}{s}\right)^{2(N-1)} E(r) = \left(1 + \frac{r-s}{s}\right)^{2(N-1)} E(r) \leq (1 + r - s)^{2(N-1)}E(r). \tag{5.6}$$

From the relations $F(0) = F'(0) = 0 < F''(0)$ it also follows that, making $\bar{d} > 0$ smaller if necessary, there is a constant $c \geq 1$ such that

$$c^{-1}|(\psi(r), \psi'(r))|^2 \leq E(r) \leq c|(\psi(r), \psi'(r))|^2 \tag{5.7}$$

whenever $|(\psi(r), \psi'(r))| \leq \bar{d}$. If the latter is true for all $r \geq r_0$, we obtain from (5.6), (5.7) that (5.3) holds with $c_1 = c^{-2}$. The following claim implies that $|(\psi(r), \psi'(r))| \leq \bar{d}$ does hold for all $r \geq r_0$, provided $|(\psi(r_0), \psi'(r_0))|$ is sufficiently small. This completes the proof of statement (ii).

Claim. Given any $\bar{d} > 0$, there is a constant $d > 0$ such that if $r_0 \geq 1$ and ψ is a solution of (3.3) with $|(\psi(r_0), \psi'(r_0))| \leq d$, then $|(\psi(r), \psi'(r))| \leq \bar{d}$ for all $r \geq r_0$.

The claim is a direct consequence of (5.7) and the fact that $E(r)$ is nonincreasing (cp. (5.5)).

To prove statement (iii), we use arguments of [11, 12] based on the Sturm comparison principle. For any solution ψ of (3.3), the function $v = \psi'$ is a solution of

$$v'' + \frac{N-1}{r} v_r + (f'(\psi(r)) - \frac{N-1}{r^2})v = 0, \quad r > 0.$$

If $\delta \in (0, 1)$ is a sufficiently small constant, $r_0 \geq 1/\delta$, and $|(\psi(r_0), \psi'(r_0))| \leq \delta$, then the above claim implies that the coefficient $f'(\psi(r)) - (N-1)/r^2$ is greater than $\beta^2/2 > 0$ (recall that $f'(0) = \beta^2$). This justifies a Sturmian comparison with the Bessel-type equation

$$v'' + \frac{N-1}{r} v_r + \frac{\beta^2}{2} v = 0.$$

Any nontrivial solution of this equation has an increasing unbounded sequence of zeros, with the distance between any two successive ones bounded by a constant $\ell/2$. By the Sturm comparison principle, between any two such zeros, there is a zero of ψ'. This gives the property in statement (iii). $\square$

Lemma 5.2. *Under the hypotheses of Theorem 2.4, the following statement holds. If ψ, $\tilde{\psi}$ are two distinct solutions of* (3.3), *then*

$$\limsup_{r\to\infty} r^{(N-1)/2} |\psi(r) - \tilde{\psi}(r)| > 0. \tag{5.8}$$

Remark 5.3. A conclusion equivalent to (5.8) is that no two distinct solutions of (3.3) can have the same pair of constants A, B in (5.2).

Proof of Lemma 5.2. Assume that $\psi \not\equiv \tilde{\psi}$ are solutions of (3.3). Let $w = \tilde{\psi} - \psi$. Then w satisfies the equation

$$w_{rr} + \frac{N-1}{r} w_r + (\beta^2 + q(r))w = 0, \qquad r > 0,$$

where $\beta^2 = f'(0)$, as above, and

$$q(r) = \int_0^1 (f'(\psi(r) + \tau w(r)) - f'(0))\, d\tau. \tag{5.9}$$

An elementary computation shows that the function $v(r) := r^{(N-1)/2} w(r)$ satisfies the equation

$$v_{rr} + \left(\beta^2 - \frac{(N-1)(N-3)}{4r^2} + q(r) \right) v = 0, \qquad r > 0. \tag{5.10}$$

Let

$$H := \frac{v_r^2}{2} + \frac{\beta^2 v^2}{2}.$$

Then

$$H' = (v_{rr} + \beta^2 v) v_r = h v v_r, \quad \text{with } h(r) := \frac{(N-1)(N-3)}{4r^2} - q(r). \tag{5.11}$$

Assume for a contradiction that (5.8) is not true, or, in other words, $v(r) \to 0$ as $r \to \infty$. Since v is a solution of (5.10), standard estimates show that then also $v_r(r) \to 0$, and therefore $H(r) \to 0$, as $r \to \infty$. Of course, $H(r) > 0$ for any $r > r_0$, as v is a nontrivial solution of (5.10). From (5.11) we obtain

$$\frac{H'(r)}{H(r)} = h(r) \frac{v(r) v_r(r)}{H(r)},$$

which gives, for any fixed $r_1 > 0$,

$$\log H(r) - \log H(r_1) = \int_{r_1}^{r} h(s) \frac{v(s) v_r(s)}{H(s)} \, ds, \quad (r > r_1).$$

Consequently,

$$\int_{r_1}^{r} h(s) \frac{v(s) v_r(s)}{H(s)} \, ds \to -\infty \quad \text{as } r \to \infty. \tag{5.12}$$

The function $v v_r / H$ is clearly bounded. We next show that the function $q(r)$ given by (5.9) is integrable on (r_1, ∞), therefore also the function $h(r)$ given by (5.11) is integrable on (r_1, ∞). Indeed, Lemma 5.1(i) applied to both ψ and $\tilde{\psi}$, and the hypotheses of Theorem 2.4 give the following estimates. If $N \geq 4$,

$$|q(r)| \leq C \max_{\tau \in [0,1]} |\tau \tilde{\psi}(r) + (1-\tau) \psi(r)|^\sigma \leq \tilde{C} r^{-(N-1)\sigma/2},$$

where $(N-1)\sigma/2 > 1$ and C, $\tilde{C}$ are constants. If $N = 3$ (in which case we are assuming $f''(0) = 0$), then, similarly,

$$|q(r)| \leq C \max_{\tau\in[0,1]} |\tau\tilde{\psi}(r) + (1-\tau)\psi(r)|^{1+\sigma} \leq \tilde{C}r^{-(N-1)(1+\sigma)/2} = \tilde{C}r^{-(1+\sigma)},$$

where $\sigma > 0$. In either case, q and h are integrable, as claimed, and we have a desired contradiction to (5.12). The proof of the lemma is complete. $\square$

We now return to the periodic solution $u(r,t)$. Let $T > 0$ be its period: $u(r, t+T) = u(r,t)$ for all $r \geq 0$, $t \in \mathbb{R}$. Since (2.1) holds with $\zeta = 0$, there is a sequence $\{(r_n, t_n)\}_n$ with $r_n \to \infty$ and $t_n \in [0,T]$ such that

$$\kappa_n := |u(r_n, t_n)| = \|u(r_n, \cdot)\|_{L^\infty(0,T)} = \max_{r\geq r_n,\, t\in\mathbb{R}} |u(r,t)|. \tag{5.13}$$

We will also assume, passing to a subsequence if necessary, that $r_{n+1} > r_n > \varsigma$, where ς is large enough for the following relations to be satisfied with $d > 0$ and $\delta > 0$ as in Lemma 5.1(ii),(iii):

$$\varsigma > \max\{1, 1/\delta\}, \quad |(u(r,t), u_r(r,t))| \leq \min\{d, \delta\} \quad (r \geq \varsigma) \tag{5.14}$$

(in the last relation, we use the fact that $u_r(r,t) \to 0$ as $r \to \infty$, uniformly in t, which follows from (2.1) and standard parabolic regularity estimates).

Our first goal is to find a limit function, as $n \to \infty$, of the sequence $\{u(r_n + r, t_n + t)/\kappa_n\}_n$. For that aim, let

$$v^n(r,t) := u(r_n + r, t_n + t)/\kappa_n \quad (r > -r_n,\ t \in \mathbb{R},\ n = 1, 2, \dots). \tag{5.15}$$

Then v^n is periodic in t with period T and satisfies the equation

$$v^n_t = v^n_{rr} + \frac{N-1}{r_n + r} v^n_r + q^n(r,t)v^n, \qquad r > -r_n,\ t \in \mathbb{R}, \tag{5.16}$$

where

$$q^n(r,t) = \begin{cases} \dfrac{f(u(r_n + r, t_n + t))}{u(r_n + r, t_n + t)} & \text{if } u(r_n + r, t_n + t) \neq 0, \\ f'(0) & \text{if } u(r_n + r, t_n + t) = 0. \end{cases} \tag{5.17}$$

Note that q^n is a continuous function, bounded on $(-r_n, \infty) \times \mathbb{R}$ by a constant independent of n (here we use the Lipschitz continuity of f). We now give estimates, uniform in n, of the functions v_n. First of all, by the definitions of r_n and v^n, we have

$$|v^n(0,0)| = 1 \geq \|v^n(r,\cdot)\|_{L^\infty(\mathbb{R})} \quad (r \geq 0). \tag{5.18}$$

The following lemma yields estimates for $r < 0$.

Lemma 5.4. *For each $\epsilon > 0$ there is a constant C_ϵ such that*

$$e^{\epsilon r}\|v^n(r,\cdot)\|_{L^\infty(\mathbb{R})} \leq C_\epsilon \quad (r \in [-r_n + \varsigma, 0),\ n = 1, 2, \dots). \tag{5.19}$$

Proof. Assume the statement is not true for some $\epsilon > 0$. This means that, possibly after passing to a subsequence, the following holds. There are sequences $\{\rho_n\}$, $\{\vartheta_n\}$, $\{\bar{t}_n\}$ such that $-\rho_n \in [-r_n + \varsigma, 0)$ for all n, $\vartheta_n \to \infty$, and

$$e^{-\epsilon\rho_n}u(r_n - \rho_n, \bar{t}_n) = e^{-\epsilon\rho_n}\kappa_n v^n(-\rho_n, \bar{t}_n - t_n) > \vartheta_n\kappa_n \quad (n = 1, 2, \dots). \tag{5.20}$$

Given any n, let $\psi = \psi^n$ be the solution of (3.3) satisfying the following initial conditions at $r_0 = r_n - \rho_n$

$$\psi(r_n - \rho_n) = u(r_n - \rho_n, \bar{t}_n), \quad \psi'(r_n - \rho_n) = u_r(r_n - \rho_n, \bar{t}_n). \tag{5.21}$$

Applying Lemma 5.1(ii) to ψ (which is justified by (5.14) and the relations $r_n - \rho_n \geq \varsigma > 1$), we obtain

$$\begin{aligned} |(\psi(r_n), \psi'(r_n))| &\geq c_1(1 + \rho_n)^{-(N-1)}|(\psi(r_n - \rho_n), \psi'(r_n - \rho_n))| \\ &\geq c_2 e^{-\epsilon\rho_n}|(\psi(r_n - \rho_n), \psi'(r_n - \rho_n))|, \end{aligned} \tag{5.22}$$

where $c_1 > 0$ is as in (5.3) and $c_2 > 0$ is a small enough constant (such that $c_2 e^{-\epsilon\rho} \leq c_1(1 + \rho)^{-(N-1)}$ for all $\rho > 0$). Combining (5.22) with (5.21) and (5.20), we obtain

$$|(\psi(r_n), \psi'(r_n))| \geq c_2 e^{-\epsilon\rho_n}|(u(r_n - \rho_n, \bar{t}_n), u_r(r_n - \rho_n, \bar{t}_n))| > c_2\vartheta_n\kappa_n. \tag{5.23}$$

By Lemma 5.1(iii), we can pick $\tilde{r}_n \in (r_n, r_n + \ell)$ such that $\psi'(\tilde{r}_n) = 0$ (note that the application of Lemma 5.1(iii) is justified by the relations (5.21), (5.14), $r_n > r_n - \rho_n \geq \varsigma > 1/\delta$). Applying estimate (5.3) again, we obtain

$$\begin{aligned} |\psi(\tilde{r}_n)| = |(\psi(\tilde{r}_n), \psi'(\tilde{r}_n))| &\geq c_1(1 + \tilde{r}_n - r_n)^{-(N-1)}|(\psi(r_n), \psi'(r_n))| \\ &\geq c_1(1 + \ell)^{-(N-1)}|(\psi(r_n, \psi'(r_n))|. \end{aligned}$$

This, in conjunction with (5.23) and (5.13), gives

$$|\psi(\tilde r_n)| > c_1c_2(1+\ell)^{-(N-1)}\vartheta_n\kappa_n \geq c_1c_2(1+\ell)^{-(N-1)}\vartheta_n|u(\tilde r_n,t)| \quad (t\in\mathbb{R}).$$

Since $\vartheta_n \to \infty$, we conclude that if we fix large enough n, the function $\psi=\psi^n$ satisfies

$$|\psi(\tilde r_n)| > |u(\tilde r_n,t)| \quad (t\in\mathbb{R}).$$

Now, by (5.21), the function $u(\cdot,\bar t_n)-\psi$ has a double zero $r_n-\rho_n\in(0,\tilde r_n)$. Therefore, according to Proposition 3.5, u cannot be periodic and we have a desired contradiction. The lemma is proved. $\square$

Estimates in Lemma 5.4 and (5.18) in particular show that for any bounded interval $J\subset\mathbb{R}$ the sequence v^n starting from a large enough n is uniformly bounded on $J\times\mathbb{R}$. Recall also that v_n is a solution of (5.16), with q_n as in (5.17). Using parabolic regularity estimates and passing to a subsequence, we obtain that $v_n\to v$, locally uniformly on $\mathbb{R}^2$, where v is a solution of the following limit equation (with $\beta^2=f'(0)$)

$$v_t = v_{rr}+\beta^2 v, \qquad r\in\mathbb{R},\ t\in\mathbb{R}. \tag{5.24}$$

Also, v is T-periodic in t and, by (5.18) and Lemma 5.4, satisfies the following estimates:

$$v(0,0)=1\geq \|v(r,\cdot)\|_{L^\infty(\mathbb{R})} \quad (r\geq 0) \tag{5.25}$$

$$e^{\epsilon r}\|v(r,\cdot)\|_{L^\infty(\mathbb{R})}\leq C_\epsilon \quad (r<0,\ \epsilon>0). \tag{5.26}$$

We next show that v is independent of t. This follows from a more general classification of (unbounded) periodic solutions of (5.24) given in [23], but since the proof of our more specific claim is simple, we include it here for the reader's convenience.

Multiplying equation (5.24) by $e^{2k\pi it/T}$ and integrating from 0 to T, we obtain equations for the Fourier coefficients of the T-periodic function $v(r,\cdot)$. Namely,

$$\gamma_k(r) := \frac{1}{T}\int_0^T v(r,t)e^{2k\pi it/T}\,dt \tag{5.27}$$

satisfies the following ordinary differential equation on $\mathbb{R}$:

$$-\frac{2k\pi i}{T}\gamma = \gamma''+\beta^2\gamma. \tag{5.28}$$

Also, by (5.25), (5.26),

$$e^{-\epsilon|r|}|\gamma(r)| \leq C_\epsilon \quad (r \in \mathbb{R},\ \epsilon > 0). \tag{5.29}$$

The characteristic equation of (5.28) is

$$\lambda^2 + \beta^2 + \frac{2k\pi i}{T} = 0.$$

Its roots have nonzero real parts if $k \neq 0$. Therefore, for $k \neq 0$, there is no nontrivial solution of (5.28) satisfying (5.29). This shows that $\gamma_k \equiv 0$ for all $k \neq 0$, which means that the function $v(r,t)$ is independent of t.

We have thus proved that, replacing (r_n, t_n) by a subsequence, we have

$$\frac{u(r_n + r, t_n + t)}{\kappa_n} \to \varphi(r), \quad \text{locally uniformly in } (r,t) \in \mathbb{R}^2, \tag{5.30}$$

where φ satisfies $\varphi_{rr} + \beta^2\varphi = 0$ and $|\varphi(0)| = 1$ (thus, $\varphi(r) = \pm\cos\beta r + b\sin\beta r$ for some $b \in \mathbb{R}$). Observe that, due to the periodicity of u in t, the locally uniform converge actually means that the convergence is uniform on any set $J \times \mathbb{R}$, where $J \subset \mathbb{R}$ is any bounded interval.

Having found the limit (5.30), we now compare it to a similar limit when u is replaced by a suitable steady state ψ. Take $\psi = \psi(\cdot, a)$, where a is chosen such that for some $\bar{t}$ the function $u(\cdot, \bar{t}) - \psi$ has a multiple zero ρ (for example, $a = u(0,0)$, in which case 0 is a multiple zero of $u(\cdot, 0) - \psi$). We show that

$$\frac{\psi(r_n + r)}{\kappa_n} \to \varphi(r), \quad \text{locally uniformly in } r \in \mathbb{R}. \tag{5.31}$$

Suppose this is not true. Then, in view of (5.30), we can replace $\{r_n\}$ by a subsequence so as to achieve the following. There exist a bounded sequence $\bar{r}_n$ and a positive constant $\epsilon > 0$ such that

$$\left|\frac{u(r_n + \bar{r}_n, t)}{\kappa_n} - \frac{\psi(r_n + \bar{r}_n)}{\kappa_n}\right| > \epsilon \quad (t \in \mathbb{R},\ n = 1, 2, \dots). \tag{5.32}$$

Since $r_n \to \infty$, we can pick large enough n so that $r_n + \bar{r}_n > \rho$ (ρ being the multiple zero of $u(\cdot, \bar{t}) - \psi$, as above). Using (5.32) and Proposition 3.5, we obtain that u is not periodic in t, which is a contradiction. This contradiction proves (5.31).

We now claim that $u \equiv \psi$. Suppose that $u \not\equiv \psi$. Then, by Lemma 3.4, if we choose $\tilde{a} \neq a$ close enough to a and set $\tilde{\psi} := \psi(\cdot, \tilde{a})$, then the function $u(\cdot, t) - \tilde{\psi}$ has also a multiple zero for some t. Therefore, (5.31) is valid equally well when ψ is replaced by $\tilde{\psi}$. Consequently,

$$\frac{\psi(r_n + r)}{\tilde{\psi}(r_n + r)} \to 1, \tag{5.33}$$

uniformly on any compact set $K \subset \mathbb{R}$ not containing any zero of the function $\varphi(r)$. This implies that if A, B are as in Lemma 5.1(i), and $\tilde{A}$, $\tilde{B}$ are the constants in a similar expression for $\tilde{\psi}$, then $A = \tilde{A}$ and $B = \tilde{B}$. To see this, replace r_n by a subsequence such that the sequence $\{r_n \mod 2\pi\}_n$ converges to some $\bar{r} \in [0, 2\pi]$. Take any compact interval K (of positive length) not containing any zero of the function $\varphi(r) = \pm \cos r + b \sin r$. For any $\eta \in \bar{r} + K$, use (5.33) and the expressions from Lemma 5.1(i) for ψ, $\tilde{\psi}$ to obtain the identity

$$\frac{A \cos \beta\eta + B \sin \beta\eta}{\tilde{A} \cos \beta\eta + \tilde{B} \sin \beta\eta} = 1 \quad (\eta \in \bar{r} + K).$$

The relations $A = \tilde{A}$ and $B = \tilde{B}$ now follow immediately from the linear independence of the functions $\cos \beta\eta$, $\sin \beta\eta$.

Relations $A = \tilde{A}$, $B = \tilde{B}$ and Lemma 5.1(i) in particular imply that $r^{(N-1)/2}|\psi(r) - \tilde{\psi}(r)| \to 0$ as $r \to \infty$. This, however, is a contradiction to Lemma 5.2 because $\tilde{\psi} \not\equiv \psi$ (we have chosen $\tilde{a} \neq a$). By this contradiction we have proved that $u \equiv \psi$, completing the proof of Theorem 2.4.

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