Planar propagating terraces and the asymptotic one-dimensional symmetry of solutions of semilinear parabolic equations

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Abstract. We consider the equation $u_t = \Delta u + f(u)$ on $\mathbb{R}^N$. Under suitable conditions on f and the initial value $u_0 = u(\cdot, 0)$, we show that as $t \to \infty$ the solution $u(\cdot, t)$ approaches a planar propagating terrace, or a stacked family of planar traveling fronts. Using this result, we show the asymptotic one-dimensional symmetry of $u(\cdot, t)$ as well as its quasi-convergence in $L^\infty_{loc}(\mathbb{R}^N)$.

Key words: Parabolic equations, large-time behavior, asymptotic one-dimensional symmetry, quasi-convergence, traveling fronts, propagating terraces, Liouville theorems.

AMS Classification: 35K15, 35B40, 35B35, 35B05

Contents

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1 Introduction

Consider the Cauchy problem

$$u_t = \Delta u + f(u), \qquad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where $N \geq 2$, $f \in C^1(\mathbb{R})$, and u_0 is a bounded continuous function. We assume that for some $\gamma > 0$ one has $f(\gamma) = f(0) = 0$. Writing the spatial variable as $x = (x_1, x')$ with $x_1 \in \mathbb{R}$, we are interested in the behavior of the solutions of (1.1), (1.2) for a class of initial data including in particular the functions $u_0 \in C(\mathbb{R}^N)$ satisfying the conditions

(uL) $0 \leq u_0 \leq \gamma$, $\lim_{x_1 \to -\infty} u_0(x_1, x') = \gamma$, and $\lim_{x_1 \to \infty} u_0(x_1, x') = 0$, where both limits are uniform in $x' \in \mathbb{R}^{N-1}$.

(Our actual hypotheses on u_0 are a bit weaker and do not require the existence of the limits as $x_1 \to \pm\infty$, see Sections 2.2, 2.3). Under minor additional hypotheses, our main conclusions regarding the unique solution u of (1.1), (1.2) can roughly be summarized as follows:

(I) $u(x,t)$ has the asymptotic one-dimensional symmetry: all its "generalized limit profiles" as $t \to \infty$ are functions of x_1 only.

(II) As $t \to \infty$, $u(\cdot, t)$ is attracted by the minimal $[0,\gamma]$-propagating terrace of the one dimensional equation

$$u_t = u_{x_1 x_1} + f(u), \quad x_1 \in \mathbb{R},\ t > 0. \tag{1.3}$$

(III) u is quasi-convergent: with respect to the locally uniform convergence, all limit profiles of $u(\cdot, t)$ as $t \to \infty$ are steady states of (1.1).

The results are stated formally in Section 2. Here we discuss them on a more intuitive level and put them in context with various existing theorems.

To explain statements (I) and (III), we introduce two notions of limit sets of the solution u of (1.1), (1.2). The first one is a standard ω-limit set of u, denoted by $\omega(u)$ or $\omega(u_0)$, with respect to the locally uniform convergence:

$$\omega(u) := \{\varphi : u(\cdot, t_n) \to \varphi \text{ for some sequence } t_n \to \infty\}, \qquad (1.4)$$

where the convergence is in $L^\infty_{loc}(\mathbb{R}^N)$. In (III), the "limit profiles of u" refers to the elements of $\omega(u)$. In (I), by the "generalized limit profiles of u," we mean elements of the Ω-limit set of u, which is defined as follows:

$$\Omega(u) = \Omega(u_0) := \{\varphi : u(\cdot + x_n, t_n, u_0) \to \varphi \\ \text{for some sequences } t_n \to \infty \text{ and } x_n \in \mathbb{R}^N\}. \qquad (1.5)$$

The convergence here is again in $L^\infty_{loc}(\mathbb{R}^N)$. Obviously, $\omega(u_0) \subset \Omega(u_0)$, but the opposite inclusion is not true in general. Both these limit sets provide a useful information on the solution: $\Omega(u_0)$ gives a picture of the global shape of $u(\cdot, t)$ for large times. Indeed, the asymptotic shape of any bounded part of the graph of $u(\cdot, t)$ is captured in $\Omega(u)$. This notion is also useful when one wants to examine the behavior of the solution in various moving coordinate frames. The set $\omega(u_0)$, on the other hand, is more relevant for the investigation of the large-time behavior of $u(\cdot, t)$ in fixed compact regions, as a "stationary observer" would see it. Thus, $\omega(u_0)$ gives a specific information on the solution not encoded in $\Omega(u_0)$.

The meaning of conclusion (I) is that any element φ of $\Omega(u)$ is a function of x_1 only: $\varphi = \varphi(x_1)$. This result is related to one-dimensional symmetry properties of solutions of elliptic equations

$$\Delta v + f(v) = 0, \quad x \in \mathbb{R}^N.$$

Assuming that v is a solution satisfying the same conditions as u_0 in (uL), several authors have proved that under suitable assumptions on f, v is necessarily a function of x_1 only. Proofs of this result, often referred to as the Gibbons conjecture, can be found in [3, 6, 10, 14, 15, 16, 17, 21, 36, 42] (see also [43] and references therein for related results on the De Giorgi conjecture, in which $f(u) = u(1-u^2)$ and the solution v is assumed to be monotone in x_1, but the uniformity requirement in (uL) is dropped). When considering solutions of the evolution problem (1.1), (1.2), we cannot in general expect the one-dimensional symmetry of $u(\cdot, t)$ at any finite time (unless u_0 is already one-dimensional). Conclusion (I) tells us that the symmetry occurs asymptotically. Of course, if $u_0 = v$ is a steady state of (1.1), (1.2), then the asymptotic symmetry reduces to the symmetry of v. The situation here is similar to the reflectional or radial symmetry of solutions of elliptic and parabolic problems (for an overview see [33]). We remark that Conclusion I is not valid without the requirement of uniform convergence in (uL). Counter-examples can be found in various non-planar traveling waves with conical or paraboloidal interfaces (see, for example, [9, 22, 44] and references therein).

While Conclusion (I) concerns the asymptotic spatial profiles of the solution, in (III) we deal with the temporal behavior of u. We say that the solution u is convergent if $\omega(u)$ consists of a single limit profile φ, necessarily a steady state of (1.1), (1.2). If $\omega(u)$ consists entirely of steady states, u is said to be *quasi-convergent*. Thus, the behavior of quasi-convergent solutions in compact spatial regions is governed by steady states. This is the behavior seen in solutions of gradient-like systems, such as equation (1.1) considered on a bounded domain with Dirichlet or Neumann boundary conditions. However, when the spatial domain is $\mathbb{R}^N$, quasi-convergence of bounded solutions of (1.1) is not to be expected in general even in one-space dimension (see [34, 35] for examples and a discussion of this problem). In dimensions $N \geq 3$, there are even examples of radial bounded localized solutions which are not quasi-convergent [38]. On the other hand, several classes of quasi-convergent solutions have been identified. In addition to solutions contained in a suitable energy space, these include nonnegative solutions on $\mathbb{R}$ with $u_0 \in C_0(\mathbb{R})$ [32], front-like solutions on $\mathbb{R}$ [37], and nonnegative solutions with compact initial support in any dimension [11, 12]. Conclusion (III) exposes another class of quasi-convergent solutions, namely, solutions with initial data satisfying (uL).

Both Conclusions (I) and (III) are rather straightforward consequences of Conclusion (II), which is really the main result of this paper. To explain it, we need to define the concept of the minimal propagating terrace of the one-dimensional equation (1.3). Intuitively, the minimal propagating terrace for the given interval $[0, \gamma]$ is a collection of traveling fronts of (1.3) characterized by a certain minimality property. Under some generic conditions on f, the minimal propagating terrace is given by a finite system of solutions of (1.3) of the form

$$U_I(x_1, t) = \phi_I(x_1 - c_I t), \quad I \in \mathcal{N}. \tag{1.6}$$

Here $\mathcal{N}$ is a system of mutually disjoint open intervals $I \subset (0, \gamma)$ whose closures cover $[0, \gamma]$, and for each $I \in \mathcal{N}$ the function ϕ_I is a decreasing solution of the equation

$$\phi'' + c_I \phi' + f(\phi) = 0, \quad x_1 \in \mathbb{R}, \tag{1.7}$$

whose range is equal to the interval I. Note that (1.7) means that U_I is a solution of (1.3); it is a *traveling front* of (1.3) with the *profile function* ϕ_I and *speed* c_I. To describe the minimality property of system (1.6), consider the trajectories

$$\tau(\varphi_I) := \{(\phi_I(x_1), \phi_I'(x_1)) : x_1 \in \mathbb{R}\}, \quad I \in \mathcal{N}, \tag{1.8}$$

in the plane $\{(u, v) : u, v \in \mathbb{R}\}$. Since the ϕ_I are decreasing, from the properties of the intervals $I \in \mathcal{N}$ it follows that the closures of these trajectories form the graph of a continuous function $u \mapsto R(u) : [0, \gamma] \to (-\infty, 0]$. One can now compare different functions on $[0, \gamma]$ obtained this way in the pointwise ordering of $C[0, \gamma]$. The minimal propagating terrace is characterized by the minimality of the corresponding function R. This is a uniquely defined system (for a given nonlinearity f and interval $[0, \gamma]$), up to translations of the profile functions.

We give the definition of the minimal propagating terrace for a general nonlinearity in the next section. The minimal propagating terrace always exists, but, unlike in the generic case discussed above, the set $\mathcal{N}$ may be infinite and the closures of the intervals $I \in \mathcal{N}$ may not cover the whole interval $[0, \gamma]$.

In several extensively studied cases, including the monostable and bistable nonlinearities, the minimal propagating terrace consists of a single traveling front [46]. The attractivity properties of traveling fronts with respect to the semiflow of the parabolic one-dimensional Cauchy problem have been well understood (see [1, 2, 7, 8, 18, 19, 20, 27, 25, 28, 31, 39, 45, 46] and references therein). Now, traveling fronts of (1.3) can also be viewed as special solutions of the multidimensional equation (1.1); as such, they are usually referred to as planar traveling fonts. Their local and global stability properties relative to the multidimensional problem have also been studied by several authors (see [26, 29, 30, 40, 50, 52]).

In case equation (1.3) possesses no traveling front with range $(0, \gamma)$, the large time behavior of a class of bounded solutions of (1.3) can often be described in terms of the minimal propagating terrace. For early results of this form we refer the reader to [18, 19, 46, 48]; [41], [13] contain related theorems for monotone systems and spatially periodic equations. For general equations (1.3), a global attractivity property of the minimal propagating terrace was recently proved in [37].

In view of these results, it is natural to ask if the minimal propagating terrace of (1.3) also attracts the solution of the multidimensional problem (1.1), (1.2) with u_0 as in (uL). Conclusion (II) is to say that this is indeed true, under natural additional conditions on u_0. In the generic case, the conclusion can be phrased in terms of $\Omega(u)$ as follows. For any $I \in \mathcal{N}$, let $a_I < b_I$ be the end points of I: $I = (a_I, b_I)$. Then

$$\Omega(u) := \{\phi_I(\cdot - \xi) : I \in \mathcal{N}, \xi \in \mathbb{R}\} \cup \{0\} \cup \{b_I : I \in \mathcal{N}\}. \qquad (1.9)$$

Since $I = (a_I, b_I)$ is the range of ϕ_I – a decreasing solution of (1.7), both a_I and b_I are zeros of f, hence they are steady states of (1.1). Thus (1.9) says that at large times the solution $u(\cdot, t)$ has the shape whose parts are roughly given by the decreasing profile functions ϕ_I and flat parts given by constant steady states. This can be expressed, somewhat more tangibly, as follows.

(II)' There exist $s > 0$ and C^1-functions ζ_I, $I \in \mathcal{N}$, defined on $\mathbb{R}^{N-1} \times (s, \infty)$ such that for each $I \in \mathcal{N}$

$$\begin{aligned} &\lim_{t\to\infty} \sup_{x' \in \mathbb{R}^{N-1}} \left(|\nabla_{x'} \zeta_I(x', t)| + \frac{|\zeta_I(x', t)|}{t} \right) = 0, \\ &\sup_{x' \in \mathbb{R}^{N-1},\, t > s} |\zeta_I(x', t) - \zeta_I(0, t)| < \infty, \end{aligned} \qquad (1.10)$$

and

$$\lim_{t\to\infty} \big(u(x_1, x', t) - \big(\sum_{I \in \mathcal{N}} \phi_I(x_1 - c_I t - \zeta_I(x', t)) - \sum_{I \in \mathcal{N}} a_I \big)\big) = 0, \quad (1.11)$$

where the convergence is uniform with respect to $(x_1, x') \in \mathbb{R}^N$.

In the generic case discussed here, one has $c_I > c_J$ whenever the interval I is to the left of the interval J. Since also $a_I = 0$ for the left-most interval $I \in \mathcal{N}$, (1.11) says that for large t the graph of $u(\cdot, t)$ looks like the sketch in Figure 1. According to (1.10), the transitions between the flat levels of the terrace move in the x_1-direction with the asymptotic speeds c_I, $I \in \mathcal{N}$.

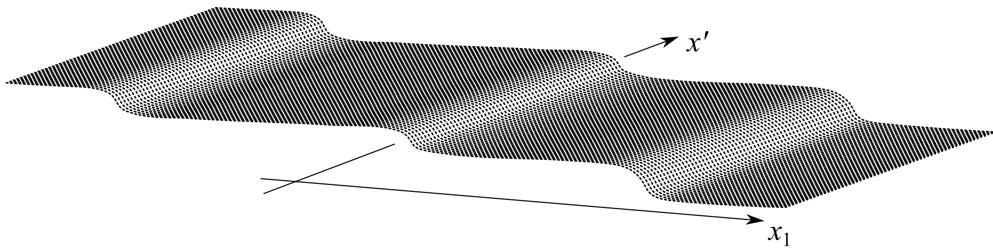

Figure 1: A propagating-terrace asymptotics of the solution u

The proofs of our main theorems are based on general results on the approach to a propagating terrace of one-dimensional problems [37] and Liouville theorems for entire solutions of (1.1) [5]. An entire solution refers to a solution defined for all $t \in \mathbb{R}$, not just for $t \geq 0$. It is well-known that the limit sets $\Omega(u)$ and $\omega(u)$ of a bounded solution u of (1.1) consist of entire solutions (see Section 3.1). We employ the results of [37] in order to show that each entire solution v contained in $\Omega(u)$ is either a constant steady state, or else it is trapped between two shifts of a planar traveling front, a member of the minimal propagating terrace. By a Liouville theorem (see Section 3.2), v itself must be a shift of that same planar traveling front, which leads to the desired conclusion.

The rest of the paper is organized as follows. In the next section, we first define the minimal propagating terrace for a general one-dimensional problem and recall some of its basic properties. Then we state our main theorems on the the approach to planar propagating terraces for solutions of (1.1) and derive their corollaries on the one-dimensional symmetry and quasi-convergence. Section 4 contains the proofs of the main results. In the preliminary Section 3, we recall properties of the Ω-limit set and a Liouville theorem for entire solutions.

2 Main results

Throughout the paper, our *standing hypotheses* are as follows:

(H) f is a C^1 function on $\mathbb{R}$ with bounded derivative, $\gamma > 0$, and $f(0) = f(\gamma) = 0$.

We assume the global Lipschitz continuity just for convenience. This is at no cost to generality: since all our results concern a bounded solution, if $f(u)$

is merely locally Lipschitz, we can always modify it outside the range of the solution to make it globally Lipschitz.

In the next subsection, we recall the definition and basic properties of the minimal propagating terrace. Then we formulate our main results, first for generic f under minimal assumptions on u_0, then for general f under stronger conditions on u_0. In the last subsection, we discuss the asymptotic one-dimensional symmetry and quasi-convergence of solutions of (1.1).

2.1 Minimal systems of waves and propagating terraces

In this subsection, we consider the one-dimensional problem

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t > 0. \tag{2.1}$$

We first define, following [46], the notion of a minimal $[0, \gamma]$-system of waves. We use the following notation. If ϕ is a C^1 function on $\mathbb{R}$, we set

$$\tau(\phi) = \{(\phi(x), \phi_x(x)) : x \in \mathbb{R}\}. \tag{2.2}$$

Recall that a traveling front of (2.1) with speed c and profile ϕ is a solution of (2.1) of the form $U(x,t) = \phi(x - ct)$, where ϕ is a decreasing solution of

$$\phi'' + c\phi' + f(\phi) = 0, \quad x \in \mathbb{R}. \tag{2.3}$$

Definition 2.1. A $[0, \gamma]$-system of waves, or simply a *system of waves* if there is no danger of confusion, of (2.1) is a continuous function R on $[0, \gamma]$ with the following properties:

(i) $R(0) = R(\gamma) = 0, \quad R(u) \le 0\ \ (u \in [0, \gamma])$;

(ii) If $I = (a, b) \subset [0, \gamma]$ is a nodal interval of R, that is, a connected component of the set $R^{-1}(-\infty, 0)$, then there is $c \in \mathbb{R}$ and a decreasing solution ϕ of (2.3) such that $\phi(-\infty) = b$, $\phi(\infty) = a$, and

$$\{(u, R(u)) : u \in (a, b)\} = \tau(\phi). \tag{2.4}$$

Thus the graph of R between its successive zeros is given by the trajectory of the profile of a traveling front.

Definition 2.2. A system of waves R_0 is said to be *minimal* if for an arbitrary system of waves R one has

$$R_0(u) \le R(u) \quad (u \in [0,\gamma]).$$

By definition, the minimal system of waves is unique. As shown in [46, Theorem 1.3.2], for any f satisfying (H), a minimal system of waves exists and can be found as follows. For each $u \in [0,\gamma]$, set

$$R_0(u) = \inf_{\phi} \phi'(0), \tag{2.5}$$

where ϕ is the decreasing profile function of a traveling front with the range in $[0,\gamma]$ such that $\phi(0) = u$. The infimum is taken over all such ϕ; if no such ϕ exists, one puts $R_0(u) = 0$.

Additional properties of R_0 are stated in the next theorem (see [46, Sect. 1.3] for the proofs; related results can be found in [18, 49, 47]).

Theorem 2.3. *For any f satisfying* (H)*, there exists a unique minimal system of waves R_0. Moreover, R_0 has the following properties:*

(i) $R_0^{-1}(0) \subset f^{-1}(0)$*;*

(ii) *If $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$ are nodal intervals of R_0 with $b_1 \le a_2$, and if c_1, c_2 are the speeds of the traveling fronts from Definition 2.1 corresponding to I_1, I_2, respectively, then $c_1 \ge c_2$.*

Let R_0 be the minimal system of waves. We denote by $\mathcal{N}$ the (countable) set of all nodal intervals of R_0. Since R_0 is single valued, for each $I \in \mathcal{N}$ the speed $c = c_I$ and the solution $\phi = \phi_I$ in Definition 2.1(ii) are determined uniquely if we postulate

$$\phi(0) = \frac{a+b}{2}. \tag{2.6}$$

This way we obtain the *families of speeds and profile functions* corresponding to R_0:

$$\{c_I : I \in \mathcal{N}\},\ \{\phi_I : I \in \mathcal{N}\}. \tag{2.7}$$

We define a natural ordering on $\mathcal{N}$:

$$I_1 < I_2 \text{ if } I_1 = (a_1, b_1),\ I_2 = (a_2, b_2) \text{ and } b_1 \le a_2, \tag{2.8}$$

and write $I_1 \le I_2$, if $I_1 = I_2$ or $I_1 < I_2$. Since two different nodal intervals of R_0 cannot overlap, $\mathcal{N}$ in simply ordered by this relation. By Theorem 2.3(ii),

$$\text{if } I_1 < I_2, \text{ then } c_{I_1} \ge c_{I_2}. \tag{2.9}$$

Also, by the definition of R_0 and Theorem 2.3(i), the boundary points a, b of any interval $(a,b) \in \mathcal{N}$ are in $R_0^{-1}(0) \subset f^{-1}(0)$.

Consider now the family of traveling fronts $U_I(x,t) = \phi_I(x - c_I t)$, $I \in \mathcal{N}$. As in [13, 37], we refer to this family as the $[0,\gamma]$-*minimal propagating terrace* or simply the *minimal propagating terrace*, of (2.1).

We remark that in general the set $\mathcal{N}$ may be infinite; and positive, negative, and zero speeds may be included in $\{c_I : I \in \mathcal{N}\}$. To provide the reader with more information on what R_0 and the minimal propagating terrace can look like, we recall some results from [37]. Set

$$\begin{aligned} \mathcal{N}^+ &:= \{I \in \mathcal{N} : c_I > 0\}, \\ \mathcal{N}^- &:= \{I \in \mathcal{N} : c_I < 0\}, \\ \mathcal{N}^0 &:= \{I \in \mathcal{N} : c_I = 0\}. \end{aligned} \tag{2.10}$$

Of course, some of these sets may be empty.

If $\mathcal{N}^+ \neq \emptyset$, we further define

$$\begin{aligned} \gamma_* &:= \sup \bigcup_{I \in \mathcal{N}^+} I \\ &= \sup\{b \in (0,\gamma] : \mathcal{N}^+ \text{ contains the interval } (a,b) \text{ for some } a \in [0,b)\}. \end{aligned} \tag{2.11}$$

If $\mathcal{N}^+ = \emptyset$, we set $\gamma_* = 0$. Similarly, we set $\gamma^* = \gamma$ if $\mathcal{N}^- = \emptyset$. If $\mathcal{N}^- \neq \emptyset$, we define

$$\begin{aligned} \gamma^* &:= \inf \bigcup_{I \in \mathcal{N}^-} I \\ &= \inf\{a \in [0,\gamma) : \mathcal{N}^- \text{ contains the interval } (a,b) \text{ for some } b \in (a,\gamma]\}. \end{aligned} \tag{2.12}$$

By the continuity of R_0, we have $R_0(\gamma_*) = R_0(\gamma^*) = 0$. Consequently, γ_*, γ^* are zeros of f (cp. Theorem 2.3).

Define

$$F(u) = \int_0^u f(s)\,ds. \tag{2.13}$$

We say that a critical point $\xi \in [0,\gamma]$ of F is a *left-global maximizer* of F in $[0,\gamma]$ (or, simply, a left-global maximizer) if

$$F(v) \leq F(\xi) \quad (0 \leq v < \xi). \tag{2.14}$$

If the first inequality in (2.14) is strict, we say ξ is a *strict left-global maximizer*. Similarly we define (strict) right-global maximizers. Note that we count 0 as a strict left-global maximizer and γ as a strict right-global maximizer.

We denote by Γ^-, Γ^+, Γ^0 the sets of strict left-global, strict right-global, and global maximizers, respectively. Obviously, $\Gamma^- \cup \Gamma^0$, $\Gamma^+ \cup \Gamma^0$ are sets of left-global and right-global maximizers, respectively.

Proposition 2.4. *The following statements are valid:*

(i) $0 \leq \gamma_* \leq \gamma^* \leq \gamma$;

(ii) one has

$$R_0^{-1}\{0\} \cap [\gamma_*, \gamma^*] \subset \Gamma^0, \tag{2.15}$$
$$R_0^{-1}\{0\} \cap [0, \gamma_*] \ \ \subset \Gamma^-, \tag{2.16}$$
$$R_0^{-1}\{0\} \cap [\gamma^*, \gamma] \ \ \subset \Gamma^+; \tag{2.17}$$

(iii) for each $I = (a,b) \in \mathcal{N}$ *one has*

$$c_I > 0 \text{ if and only if } b \leq \gamma_*, \tag{2.18}$$
$$c_I < 0 \text{ if and only if } \gamma^* \leq a, \tag{2.19}$$
$$c_I = 0 \text{ if and only if } \gamma_* \leq a < b \leq \gamma^*; \tag{2.20}$$

(iv) each of the sets $R_0^{-1}\{0\} \cap [0,\gamma_*]$, $R_0^{-1}\{0\} \cap [\gamma^*,\gamma]$ *is finite or countable,* γ_* *is the only possible accumulation point of the set* $R_0^{-1}\{0\} \cap [0,\gamma_*]$*, and* γ^* *is the only possible accumulation point of* $R_0^{-1}\{0\} \cap [\gamma^*,\gamma]$;

(v) If $\{I_j\}_{j=1}^{\infty}$ *is a strictly monotone sequence in* $\mathcal{N}$ *(recall that the ordering on* $\mathcal{N}$ *is defined in* (2.8)*), then* $c_{I_j} \to 0$.

This is proved in [37, Proposition 3.11]. Figure 2 illustrates some possibilities of what R_0 can look like. Note that the complexities in the graph of R_0 are always due to the presence of zero or arbitrarily small speeds in $\{c_I : I \in \mathcal{N}\}$.

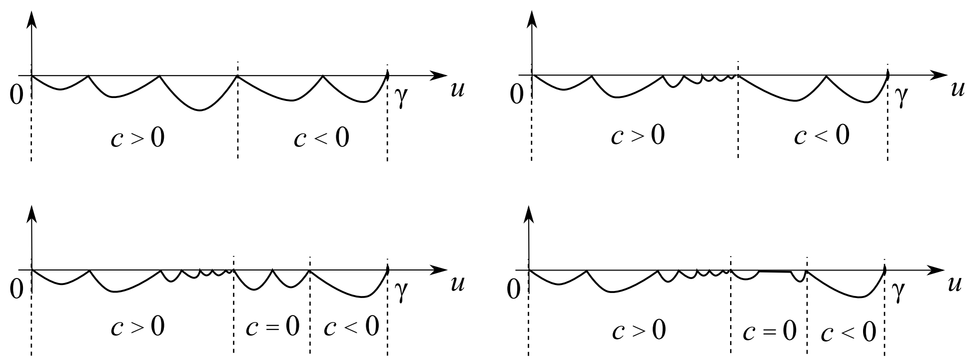

Figure 2: Possible graphs of R_0, with the signs of the speeds of the corresponding traveling fronts indicated. The first figure corresponds to a generic case—finitely many fronts with nonzero speeds; the other figures depict some "degenerate" cases.

We introduce a few more pieces of notation. First, we define a value $\gamma_0 \in [0, \gamma)$. If 0 is unstable from above for the equation $\dot{\xi} = f(\xi)$, that is, $f > 0$ on some interval $(0, \delta)$, then $R_0 < 0$ on this interval (see Theorem 2.3(i)). Hence $\mathcal{N}$ contains an interval $I = (0, b)$ with $b > 0$ and in this case we set $\gamma_0 := b$. Otherwise, that is, if 0 is stable from above, we define $\gamma_0 := 0$. Similarly, if γ is unstable from below, then $\mathcal{N}$ contains an interval $I := (a, \gamma)$ with $a < \gamma$. We define γ_1 to be this value a if γ is unstable from below; otherwise we set $\gamma_1 = \gamma$. Further, we denote

$$\tilde{\mathcal{N}} := \{I \in \mathcal{N} : I \subset (\gamma_0, \gamma_1)\}. \tag{2.21}$$

We add a few comments pertaining to the notation just introduced. In case $\gamma_0 > 0$, $I = (0, \gamma_0)$ is the minimal element of $\mathcal{N}$ in the ordering (2.8). The corresponding traveling front $U_I = \phi_I(x - c_I t)$ connects the positive steady state γ_0 to 0. Necessarily, γ_0 is stable from below, $c_I > 0$, and c_I is the minimal speed for all traveling fronts connecting γ_0 and 0 (see [46, Theorem 1.3.14]). Similar comments apply if $\gamma_1 < \gamma$. It follows from the definition of a minimal system of waves that $\tilde{R}_0 := R_0 \big|_{[\gamma_0, \gamma_1]}$ is the minimal $[\gamma_0, \gamma_1]$-system of waves. Its families of speeds and profile functions are $\{c_I : I \in \tilde{\mathcal{N}}\}$ and $\{\varphi_I : I \in \tilde{\mathcal{N}}\}$, respectively.

2.2 Generic f

Let R_0, $\mathcal{N}$, $\tilde{\mathcal{N}}$, ϕ_I, c_I be as in the previous subsection.

In this subsection, we assume that, in addition to the standing hypothesis (H), f satisfies the following conditions (the maximizers are relative to the function F as in (2.13) and the interval $[0, \gamma]$):

(G1) Each left-global maximizer is strict and each right-global maximizer is strict. In particular, F has a unique maximizer in $[0, \gamma]$.

(G2) If $\xi \in [0, \gamma]$ is a left-global or right-global maximizer of F in $[0, \gamma]$, then it is a nondegenerate critical point of F: $f'(\xi) \neq 0$.

(G3) For any two distinct $I, J \in \mathcal{N}$ one has $c_I \neq c_J$.

Note that Condition (G2) in particular implies $f'(0) \neq 0$ and $f'(\gamma) \neq 0$ (for any interior left-global or right-global maximizer ξ, condition (G2) of course means that $f'(\xi) < 0$).

Remark 2.5. We use the term "generic" rather loosely here, but it can be made precise. Namely, denote by $\mathcal{F}$ the subspace of $C^1[0, \gamma]$ consisting of functions satisfying $f(0) = f(\gamma) = 0$. We equip $C^1[0, \gamma]$ with a standard norm and $\mathcal{F}$ with the induced norm. Then the set $\mathcal{G} := \{f \in \mathcal{F} : \text{(G1)–(G3) hold}\}$ is open and dense in $\mathcal{F}$. While it is a simple exercise to prove that $\mathcal{G}_0 := \{f \in \mathcal{F} : \text{(G1), (G2) hold}\}$ is open and dense, to prove the same for $\mathcal{G}$ requires some work. The proof of the openness amounts to showing that the intervals $I \in \mathcal{N}$ and the corresponding speeds c_I perturb only slightly under small perturbation of f. For the proof of density, one can show, for example, that a given speed c_I can always be decreased a little by perturbing f in I only. The detailed proofs are not really difficult—variational and mini-max characterizations of minimal speeds of traveling fronts, as found in [4, 46], for example, can be used effectively—they would take us too far aside our main points and we do not include them.

To formulate our hypotheses on u_0, let D_0 and D_γ denote the sets of attraction of the equilibria 0 and γ with respect to the equation $\dot{\xi} = f(\xi)$. Recall that the set, or domain, of attraction of an equilibrium η is the set of all initial values from which the solution converges to η. Specifically,

$$D_\eta = \{\eta\} \cup (\eta, \eta^+) \cup (\eta^-, \eta), \tag{2.22}$$

where (η, η^+) is the maximal interval of this form on which $f < 0$ if such an interval exists, otherwise $(\eta, \eta^+) = \emptyset$. The set (η^-, η) is defined in an analogous way (with $f > 0$ on (η^-, η)). Of course, if $f'(\eta) < 0$, then D_η is an open interval containing η; and if $f'(\eta) > 0$, then $D_\eta = \{\eta\}$.

We will assume that u_0 satisfies the following conditions:

$$\liminf_{x' \in \mathbb{R}^{N-1},\ x_1 \to -\infty} u_0(x_1, x') \in D_\gamma, \qquad \sup_{x \in \mathbb{R}^N} u_0(x) \in D_\gamma; \tag{2.23}$$

$$\limsup_{x' \in \mathbb{R}^{N-1},\ x_1 \to \infty} u_0(x_1, x') \in D_0, \qquad \inf_{x \in \mathbb{R}^N} u_0(x) \in D_0. \tag{2.24}$$

Obviously, the conditions are weaker than condition (uL) in the introduction.

Recall that γ_0, γ_1, and $\tilde{\mathcal{N}}$ were introduced at the end of the previous subsection. The following theorem is one of our main results.

Theorem 2.6. *Assume that f satisfies conditions* (H), (G1)–(G3)*, and $u_0 \in C(\mathbb{R}^N)$ satisfies* (2.23)*,* (2.24)*. Then*

$$\Omega(u_0) = R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \tilde{\mathcal{N}},\ \xi \in \mathbb{R}\} \cup \Omega_0(u_0) \cup \Omega_1(u_0), \tag{2.25}$$

where $\Omega_0(u_0)$ is a set of functions with range in $(0, \gamma_0)$, and $\Omega_1(u_0)$ is a set of functions with range in (γ_1, γ).

Needless to say, in (2.25) the elements of $R_0^{-1}\{0\}$ are viewed as constant functions and the ϕ_I are viewed as function on $\mathbb{R}^N$ independent of x'. If $f'(0) < 0$ and $f'(\gamma) < 0$, then $\gamma_0 = 0$, $\gamma_1 = \gamma$, $(0, \gamma_0) = (\gamma_1, \gamma) = \emptyset$, hence $\Omega_0(u_0) = \Omega_1(u_0) = \emptyset$. In this case, Theorem 2.6 gives a complete description of $\Omega(u_0)$. If at least one of the instabilities $\gamma_0 > 0$, $\gamma_1 < \gamma$ occurs, it is impossible to determine the parts $\Omega_0(u_0)$ and $\Omega_1(u_0)$ of $\Omega(u_0)$ without additional assumptions regarding the behavior of $u_0(x_1, x')$ as $x_1 \to \pm\infty$. Even in the simplest situation when $N = 1$ and $f > 0$ in $[0, \gamma]$ (in which case $\Omega(u)$ reduces to $\Omega_0(u_0)$), it is known that the solution does not in general approach any traveling front. It may oscillate between fronts with different speeds [23, 51] or it may even propagate faster than any traveling front [24]. Similar remarks apply to Theorem 2.7 below. In the next section, we give a complete description of $\Omega(u_0)$ under additional assumptions on u_0.

For $I \in \mathcal{N}$, let $a_I < b_I$ be the end points of I: $I = (a_I, b_I)$. Condition (G2) and Proposition 2.4 imply that

$$R_0^{-1}\{0\} = \{a_I : I \in \mathcal{N}\} \cup \{\gamma\} = \{0\} \cup \{b_I : I \in \mathcal{N}\} \tag{2.26}$$

and this set is finite. Consequently, the sets $\tilde{\mathcal{N}} \subset \mathcal{N}$ are finite. Therefore, for some k we have

$$R_0^{-1}\{0\} \cap [\gamma_0, \gamma_1] = \{b_1, \dots, b_{k+1}\}, \text{ with } \gamma_0 = b_1 < b_2 < \dots < b_{k+1} = \gamma_1, \tag{2.27}$$

and $\tilde{\mathcal{N}} = \{I_1, \dots, I_k\}$ with $I_j = (b_j, b_{j+1})$, $j = 1, \dots, k$.

In our next theorem, we also use quantities $c_{I_1}^+, c_{I_k}^- \in [-\infty, \infty]$ defined as follows. If $\gamma_0 = 0$ (0 is stable from above for the equation $\dot{\xi} = f(\xi)$), set $c_{I_1}^+ := \infty$. If $\gamma_0 > 0$, then $I_0 := (0, \gamma_0) \in \mathcal{N} \setminus \tilde{\mathcal{N}}$ and $c_{I_0} > c_{I_1}$ (cp. (2.9) and (G3)). In this case, we set

$$c_{I_1}^+ := \frac{c_{I_0} + c_{I_1}}{2} > c_{I_1}. \tag{2.28}$$

Similarly, if $\gamma_1 = \gamma$ we set $c_{I_k} := -\infty$, and if $\gamma_1 < \gamma$,

$$c_{I_k}^- := \frac{c_{I_{k+1}} + c_{I_k}}{2} < c_{I_k}, \tag{2.29}$$

where $I_{k+1} = (\gamma_1, \gamma) \in \mathcal{N} \setminus \tilde{\mathcal{N}}$.

Theorem 2.7. *Assume that f satisfies conditions* (H), (G1)–(G3)*, and let b_j, $j = 1, \dots, k+1$, I_j, $j = 1, \dots, k$, $c_{I_1}^+$, $c_{I_k}^-$ be as above. Then for each $u_0 \in C(\mathbb{R}^N)$ satisfying* (2.23), (2.24)*, the solution u of* (1.1), (1.2) *has the following property. There exist $s > 0$ and C^1-functions ζ_j, $j = 1, \dots, k$, defined on $\mathbb{R}^{N-1} \times (s, \infty)$ such that for each $I \in \mathcal{N}$*

$$\begin{aligned} &\lim_{t\to\infty} \sup_{x' \in \mathbb{R}^{N-1}} \left(|\nabla_{x'} \zeta_j(x', t)| + \left| \frac{\partial \zeta_j(x', t)}{\partial t} \right| \right) = 0, \\ &\sup_{x' \in \mathbb{R}^{N-1},\, t > s} |\zeta_j(x', t) - \zeta_j(0, t)| < \infty, \end{aligned} \tag{2.30}$$

and as $t \to \infty$ one has

$$\sup_{\substack{c_{I_k}^- t < x_1 < c_{I_1}^+ t \\ x' \in \mathbb{R}^{N-1}}} \Big| u(x_1, x', t) - \Big(\sum_{j=1}^{k} \phi_{I_j}(x_1 - c_{I_j} t - \zeta_j(x', t)) - \sum_{j=1}^{k-1} b_{j+1} \Big) \Big| \to 0. \tag{2.31}$$

Notice that if $\gamma_0 = 0$ and $\gamma_1 = \gamma$, that is, the equilibria 0, γ are stable for the equation $\dot{\theta} = f(\theta)$, then $\mathcal{N} = \tilde{\mathcal{N}}$, $c_{I_k}^- = -\infty$, and $c_{I_1}^+ = \infty$. Thus (2.31) gives the uniform convergence on $\mathbb{R}^N$. In dimension $N = 1$, similar results are proved in [41, Theorem 2.2], [37, Theorem 2.23], where in addition each of the functions $\zeta_j(t)$ has a limit $\zeta_j(\infty)$ (thus, in (2.31) one can replace $\zeta_j(t)$ by $\zeta_j(\infty)$). In higher dimensions, the $\zeta_j(x', t)$ do not necessarily have limits as $t \to \infty$, not even pointwise in x'. This follows from results of [30, 40] concerning a bistable nonlinearity. In the bistable case, the propagating terrace reduces to a single traveling front and there is just one function $\zeta_1(x', t)$. If the limit of $\zeta_1(x', t)$ as $t \to \infty$ existed, then, by (2.30), it would be independent of x'. Thus, (2.31) would give the locally uniform approach of $u(\cdot, t)$ to a single traveling front. As shown in [30, Proposition 1.9] and [40, Section 2.2.1], this does not hold in general. On the other hand, it was also proved in [30, Proposition 1.9] that one does get the approach to a single bistable traveling front if u_0 is almost periodic in x'. In the setting of Theorem 2.7, assuming periodicity or almost periodicity of u_0 in x', one may be able to prove the convergence of the functions $\zeta_j(x', t)$ to some (constant) limits as $t \to \infty$, but this is not pursued in this paper.

Remark 2.8. If $\gamma_0 > 0$ or $\gamma_1 < \gamma$, Theorem 2.7 gives no information on the shape of $u(x_1, x', t)$ for $x_1 > c^+_{I_1} t$ or $x_1 < c^-_{I_k} t$. As already mentioned above, there is no simple general global description of the behavior of $u(x_1, x', t)$ in these intervals if no additional conditions on u_0 are made. However, the following modest statements are easy to prove by comparison with the traveling fronts in $(0, \gamma_0)$ and (γ_1, γ) (or see Remark 4.1 below) and are valid without the generic assumptions (G1)–(G3):

(i) If $\gamma_0 > 0$, then, denoting $I_0 := (0, \gamma_0) \in \mathcal{N}$, one has $c_{I_0} > 0$ and, for any $c \in [0, c_{I_0})$,
$$\liminf_{\substack{t\to\infty,\\ x_1 \le ct,\, x' \in \mathbb{R}^{N-1}}} u(x_1, x', t) \ge \gamma_0. \tag{2.32}$$

(ii) If $\gamma_1 < \gamma$, then, denoting $I_\gamma := (\gamma_1, \gamma) \in \mathcal{N}$, one has $c_{I_\gamma} < 0$ and, for any $c \in (c_{I_\gamma}, 0]$,
$$\limsup_{\substack{t\to\infty,\\ x_1 \ge ct,\, x' \in \mathbb{R}^{N-1}}} u(x_1, x', t) \le \gamma_1. \tag{2.33}$$

2.3 More general f

In this subsection, the conditions on the nonlinearity are much weaker than in the previous subsection, but the assumptions on u_0 are stronger. The formulation of the results is more complicated here, as the minimal propagating terrace may involve infinitely many traveling fronts.

We use the notation $\mathcal{N}^+$, $\mathcal{N}^-$, $\mathcal{N}^0$, and Γ^-, Γ^+, Γ^0 introduced in Section 2.1 (see (2.10), and the paragraph preceding Proposition 2.4).

Our hypotheses on f are the standing hypothesis (H) and the following ones:

(M+) For each $I = (a, b) \in \mathcal{N}^+$ there is $\epsilon > 0$ such that $f' \leq 0$ in $(b - \epsilon, b)$.

(M−) For each $I = (a, b) \in \mathcal{N}^-$ there is $\epsilon > 0$ such that $f' \leq 0$ in $(a, a + \epsilon)$.

(M0) For each $I = (a, b) \in \mathcal{N}^0$ there is $\epsilon > 0$ such that $f' \leq 0$ in $(a, a + \epsilon) \cup (b - \epsilon, b)$.

Note that these conditions concern only those $\xi \in R_0^{-1}\{0\}$ which are end points of intervals $I \in \mathcal{N}$. It clearly allows for $R_0^{-1}\{0\}$ and $\mathcal{N}$ to have infinitely many elements, and for $R_0^{-1}\{0\}$ to contain continua of elements; cp. Figure 2.

Sufficient conditions for (M+), (M−), (M0) in terms of left-global, right-global, global maximizers of F are given in the following proposition They show in particular that (M+), (M−), (M0) hold if f satisfies the generic conditions (G1)–(G3).

Proposition 2.9. *(i) Condition* (M−) *is satisfied if for each* $\xi \in \Gamma^-$ *with* $\xi > 0$ *there is* $\epsilon > 0$ *such that* $f' \leq 0$ *in* $(\xi - \epsilon, \xi)$.

(ii) Condition (M+) *is satisfied if for each* $\xi \in \Gamma^+$ *with* $\xi < \gamma$ *there is* $\epsilon > 0$ *such that* $f' \leq 0$ *in* $(\xi, \xi + \epsilon)$.

(iii) Condition (M0) *is satisfied if for any two elements* $\xi_1, \xi_2 \in \Gamma^0$ *satisfying* $\xi_1 < \xi_2$ *and* $(\xi_1, \xi_2) \cap \Gamma^0 = \emptyset$ *there is* $\epsilon > 0$ *such that* $f' \leq 0$ *in* $(\xi_1, \xi_1 + \epsilon) \cup (\xi_2 - \epsilon, \xi_2)$.

Proof. Statements (i) and (ii) follow directly from Proposition 2.4(ii),(iii). Statement (iii) follows from Proposition 2.4(ii),(iii) and the well known fact that if there is a standing front (that is, a traveling front with speed zero) connecting b and a, then $F(u) < F(a) = F(b)$ for all $u \in (a, b)$. □

Under the present weaker assumptions on f, we need to strengthen assumptions on u_0. We will assume the following:

(uS) There is $\bar{u}_0 \in C(\mathbb{R})$ satisfying (2.23), (2.24) ($\bar{u}_0$ is viewed here as a function on $\mathbb{R}^N$ independent of x') such that for some $\eta_0 > 0$ one has

$$\bar{u}_0(x_1 + \eta_0) \le u_0(x_1, x') \le \bar{u}_0(x_1) \quad ((x_1, x') \in \mathbb{R}^N). \tag{2.34}$$

Thus, not only is u_0 supposed to satisfy (2.23), (2.24), it is required that it be sandwiched between two functions of x_1 which satisfy (2.23), (2.24) and are shifts of one another. The last property is to be emphasized here; without it, such functions of x_1 can always be found.

For the strongest result in the cases $\gamma_0 > 0$ (0 is unstable from above) and $\gamma_1 < \gamma$ (γ is unstable from below), we shall also assume the following conditions:

(u0) In the case $\gamma_0 > 0$, there is $m > 0$ such that $\bar{u}_0 \equiv 0$ on (m, ∞).

(u1) In the case $\gamma_1 < \gamma$, there is $m > 0$ such that $\bar{u}_0 \equiv 0$ on $(-\infty, -m)$.

Note that the relations in (2.34) imply that

$$\bar{u}_0(x_1 + k\eta_0) \le \bar{u}_0(x_1) \le \bar{u}_0(x_1 - k\eta_0) \quad (x_1 \in \mathbb{R},\ k = 1, 2, \dots).$$

Therefore, if (u0) holds, then $\bar{u}_0 \ge 0$; and if (u1) holds, then $\bar{u}_0 \le \gamma$.

Theorem 2.10. *Assume that f satisfies conditions* (H), (M+), (M−), (M0), *and $u_0 \in C(\mathbb{R}^N)$ satisfies* (uS). *Then the same conclusion as in Theorem 2.6 holds:*

$$\Omega(u_0) = R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \tilde{\mathcal{N}},\, \xi \in \mathbb{R}\} \cup \Omega_0(u_0) \cup \Omega_1(u_0), \tag{2.35}$$

where $\Omega_0(u_0)$ is a set of functions with range in $(0, \gamma_0)$, and $\Omega_1(u_0)$ is a set of functions with range in (γ_1, γ).

If, in addition, conditions (u0), (u1) *are satisfied, then*

$$\Omega(u_0) = R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \mathcal{N},\, \xi \in \mathbb{R}\}. \tag{2.36}$$

Under the stability conditions $\gamma_0 = 0$ and $\gamma_1 = \gamma$, relations (2.35) and (2.36) are the same; see the discussion following Theorem 2.6. As remarked there, if $\gamma_0 > 0$ (or $\gamma_1 < \gamma$), statement (2.36) is not valid without additional assumptions on u_0. Our assumption (u0) can be relaxed—a sufficiently fast (depending on f) exponential decay of $\bar{u}_0$ at ∞ would be sufficient—and similarly for (u1). For simplicity, we just work with (u0), (u1).

Theorem 2.11. *Assume that f satisfies conditions* (H), (M+), (M−), (M0), *and $u_0 \in C(\mathbb{R}^N)$ satisfies* (uS). *Let u be the solution of* (1.1), (1.2). *Then the following statements are valid:*

(i) *For each $I \in \tilde{\mathcal{N}}$ there exist $s_I > 0$ and a C^1 function ζ_I defined on $(s_I, \infty) \times \mathbb{R}^{N-1}$ such that the following relations hold:*

(a)

$$\lim_{t\to\infty} \sup_{x'\in\mathbb{R}^{N-1}} \left(|\nabla_{x'}\zeta_I(x',t)| + \left| \frac{\partial \zeta_I(x',t)}{\partial t} \right| \right) = 0, \tag{2.37}$$

$$\sup_{x'\in\mathbb{R}^{N-1},\, t>s_I} |\zeta_I(x',t) - \zeta_I(0,t)| < \infty; \tag{2.38}$$

(b) $\big((a+b)/2 - u(x_1 + c_I t + \zeta_I(x',t), x', t)\big)x_1 > 0 \quad (x_1 \in \mathbb{R}\setminus\{0\},\ t > s_I)$*;*

(c) $\lim_{t\to\infty} \sup_{x'\in\mathbb{R}^{N-1}} |u(x_1 + c_I t + \zeta_I(x',t), x', t) - \phi_I(x_1)| = 0$*, locally uniformly with respect to $x_1 \in \mathbb{R}$;*

(d) *if $I_1, I_2 \in \tilde{\mathcal{N}}$, $I_1 < I_2$, and $c_{I_1} = c_{I_2}$, then $\inf_{x'\in\mathbb{R}^{N-1}} \big(\zeta_{I_1}(x',t) - \zeta_{I_2}(x',t)\big) \to \infty$ as $t \to \infty$.*

(ii) *If the additional hypotheses* (u0) *and* (u1) *are satisfied, then statement (i) remains valid with $\tilde{\mathcal{N}}$ replaced by $\mathcal{N}$. Moreover, the following statement holds as well:*

(e) *if $\{(x_n, t_n)\} = \{(x_{1,n}, x'_n, t_n)\}$ is any sequence in $\mathbb{R}^{N+1}$ such that $t_n \to \infty$ and for each $I \in \mathcal{N}$ one has*

$$\lim_{n\to\infty} |c_I t_n + \zeta(x'_n, t_n) - x_{1,n}| = \infty, \tag{2.39}$$

then there exist a subsequence $\{(x_{n_k}, t_{n_k})\}$ and $\xi \in R_0^{-1}\{0\}$ such that

$$\lim_{k\to\infty} u(\cdot + x_{n_k}, t_{n_k}) = \xi,$$

locally uniformly on $\mathbb{R}^N$.

(iii) *If the additional hypotheses* (u0) *and* (u1) *are satisfied and the set $R_0^{-1}\{0\}$ is finite, say*

$$R_0^{-1}\{0\} = \{a_1, \dots, a_{k+1}\}, \text{ with } 0 = a_1 < a_2 < \cdots < a_{k+1} = \gamma, \tag{2.40}$$

so that $\mathcal{N} = \{I_1, \dots, I_k\}$ *with* $I_j = (a_j, a_{j+1})$, $j = 1, \dots, k,$, *then as* $t \to \infty$ *one has*

$$\sup_{(x_1,x')\in\mathbb{R}^N} \Big|u(x_1, x', t) - \Big(\sum_{j=1}^{k} \phi_{I_j}(x_1 - c_{I_j}t - \zeta_{I_j}(x', t)) - \sum_{j=1}^{k-1} a_{j+1}\Big)\Big| \to 0. \quad (2.41)$$

As seen above, the set $\mathcal{N}$ is finite in the generic case. Also it is finite, for example, if F has a unique maximizer ξ_{max} in $[0, \gamma]$ and ξ_{max} is an isolated critical point of F in $[0, \gamma]$ (see [37] for the proof and other sufficient conditions).

2.4 Asymptotic one-dimensional symmetry and quasi-convergence

As mentioned in the introduction, our results from the previous subsection imply the asymptotic one-dimensional symmetry and quasi-convergence of solutions of (1.1). Here we spell these properties out in detail and give the proofs.

Throughout this subsection, we assume that f satisfies the standing hypothesis (H) and $u_0 \in C(\mathbb{R}^N)$.

We start with the asymptotic one-dimensional symmetry.

Corollary 2.12. *Assume that either conditions* (G1)–(G3) *together with* (2.23), (2.24) *are satisfied, or conditions* (M+), (M−), (M0)*, together with* (uS), (u0), (u1) *are satisfied. Then the solution* u *of* (1.1), (1.2) *has the following properties:*

(i) *For each* $\varphi \in \Omega(u)$ *one has* $\nabla_{x'}\varphi \equiv 0$*, that is,* φ *is a function of* x_1 *alone.*

(ii) $\lim_{t\to\infty} \sup_{(x_1,x')\in\mathbb{R}^N} |\nabla_{x'}u(x_1, x', t)| = 0.$

Proof. Statement (i) follows immediately from Theorems 2.6, 2.10. To prove statement (ii), suppose it is not valid. Then there exist $\delta > 0$ and $x_n = (x_{1,n}, x'_n) \in \mathbb{R}^N$, $t_n > 0$, $n = 1, 2, \dots$, such that $t_n \to \infty$ and $|\nabla_{x'}u(x_n, t_n)| \geq$

δ for $n = 1, 2, \ldots$. Using standard parabolic estimates one shows easily (see Section 3 for details) that after passing to a subsequence one has

$$u(\cdot + x_n, t_n) \to \psi, \tag{2.42}$$

where $\psi \in \Omega(u)$ and the convergence is in $C^1_{loc}(\mathbb{R}^N)$. In particular, $|\nabla_{x'}\psi(0)| \geq \delta > 0$, which contradicts statement (i). This contradiction completes the proof. $\square$

Next, we show the quasi-convergence of the solutions. Note that hypotheses (u0), (u1) are not needed in this result even if $\gamma_0 > 0$ or $\gamma_1 < \gamma$. Recall that $\omega(u_0)$ is the ω-limit set of the solution of (1.1), (1.2) with respect to the locally uniform convergence, see (1.4).

Theorem 2.13. *Assume that either conditions* (G1)–(G3) *together with* (2.23), (2.24) *are satisfied, or conditions* (M+), (M−), (M0) *together with* (uS) *are satisfied. Then* $\omega(u_0)$ *consists of steady states of* (1.1)*. More specifically, it consists of constant steady states and planar standing fronts.*

Here, a planar standing front refers to a traveling front of (1.3) with speed $c = 0$, that is, a solution of (2.3) with $c = 0$. Of course, all standing fronts are steady states of (1.1).

Proof of Theorem 2.13. Since $\omega(u_0) \subset \Omega(u_0)$, by Theorems 2.6, 2.10, we have

$$\omega(u_0) \subset R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \tilde{\mathcal{N}},\, \xi \in \mathbb{R}\} \cup \Omega_0(u_0) \cup \Omega_1(u_0), \tag{2.43}$$

where $\Omega_0(u_0)$ is a set of functions with range in $(0, \gamma_0)$, and $\Omega_1(u_0)$ is a set of functions with range in (γ_1, γ). In view of Remark 2.8, we can delete $\Omega_0(u_0)$, $\Omega_1(u_0)$ in (2.43). In other words, $\omega(u_0)$ consists of constant steady states from $R_0^{-1}\{0\}$, planar standing fronts, and, possibly, translates of ϕ_I, for some $I \in \tilde{\mathcal{N}}$ with $c_I \neq 0$. In order to complete the proof, we just need to show that the last possibility does not occur.

This follows easily from Theorem 2.7 if (G1)–(G3) hold; in this case we are done. In the rest of the proof we assume that (M+), (M−), (M0) and (uS) are satisfied.

Suppose that, to the contrary, $\phi_I(\cdot + \xi) \in \omega(u_0)$ for some $\xi \in \mathbb{R}$ and some $I \in \tilde{\mathcal{N}}$ with $c_I < 0$ (the case $c_I > 0$ can be ruled out in a similar way). Then, for some sequence $t_n \to \infty$ one has $u(\cdot, t_n) \to \phi_I(\cdot + \xi)$ in $L^\infty_{loc}(\mathbb{R}^N)$. In particular,

$$u(0, t_n) \to \phi_I(\xi). \tag{2.44}$$

Fixing any $\xi_0 > \xi$, we have by Theorem 2.11(i)(c) that

$$u(\xi_0 + c_I t_n + \zeta_I(0, t_n), 0, t_n) \to \phi_I(\xi_0) < \phi_I(\xi). \tag{2.45}$$

Since $c_I < 0$ and $\zeta_I(0,t)/t \to 0$ as $t \to \infty$ (cp. (2.37)), from (2.44) and (2.45) we infer that for each large enough n there is $x_{1,n}$ such that $\xi_0 + c_I t_n + \zeta_I(t_n, 0) < x_{1,n} < 0$ and

$$\phi_I(\xi_0) < u(x_{1,n}, 0, t_n) < \phi_I(\xi), \quad u_{x_1}(x_{1,n}, 0, t_n) \geq 0. \tag{2.46}$$

Passing to subsequences, we may assume that $u(x_{1,n}+\cdot, \cdot, t_n) \to \psi$ in $C^1_{loc}(\mathbb{R}^N)$ for some $\psi \in \Omega(u_0)$ (the compactness we are using here follows from standard parabolic estimates; we recall these in detail in Section 3). By (2.46),

$$\phi_I(\xi_0) \leq \psi(0) \leq \phi_I(\xi), \quad \psi_{x_1}(0) \geq 0. \tag{2.47}$$

However, by (2.43), ψ is a translate of ϕ_I and this contradicts the relation $\psi_{x_1}(0) \geq 0$. This contradiction completes the proof. □

Remark 2.14. A slight modification of the above proof yields the following stronger result. Theorem 2.13 remains valid if $\omega(u_0)$ is replaced by the set $\Omega_{N-1}(u_0)$ defined as follows:

$$\Omega_{N-1}(u_0) := \{\varphi : u(\cdot, \cdot + x_n', t_n, u_0) \to \varphi \\ \text{for some sequences } t_n \to \infty \text{ and } x_n' \in \mathbb{R}^{N-1}\}. \tag{2.48}$$

Thus, here, similarly as in $\Omega(u_0)$, one looks at the large-time profiles of the solutions in bounded regions which can be shifted around arbitrarily, but now the shifts are allowed in directions x' only.

3 Preliminaries

3.1 The Ω-limit sets and entire solutions

In this section, we take a closer look at the Ω-limit set of bounded solutions u of (1.1). It is sometimes useful to consider the solutions in a moving coordinate frame; thus we also consider the problem

$$u_t = \Delta u + c u_{x_1} + f(u), \qquad x \in \mathbb{R}^N, \ t > 0, \tag{3.1}$$

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^N. \tag{3.2}$$

We assume here that f is a locally Lipschitz function on $\mathbb{R}$, $c \in \mathbb{R}$, and $u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

The Ω-limit set of a bounded solution u of (3.1), (3.2) is defined as in (1.5) and denoted by $\Omega(u)$ or $\Omega(u_0)$. Note that if u is a bounded solution of (1.1), then the function $\tilde{u}(x_1, x', t) := u(x_1 + ct, x', t)$ is a bounded solution of (3.1). Clearly, u and $\tilde{u}$ have the same initial value at $t = 0$ and $\Omega(u) = \Omega(\tilde{u})$. In other words, if u_0 is given, then $\Omega(u_0)$ is independent of the choice of c in problem (3.1), (3.2).

Assume that the solution u of (3.1), (3.2) is bounded. Then, standard parabolic regularity estimates imply that u_t, $\nabla_x u$, $D_x^2 u$ are bounded on $\mathbb{R} \times [1, \infty)$ and are globally α-Hölder on this set for each $\alpha \in (0, 1)$. The following results are standard consequences of this regularity property: $\Omega(u_0)$ is a nonempty, compact, connected subset of $L^\infty_{loc}(\mathbb{R}^N)$. Moreover, in (1.5) one can take the convergence in $C^1_{loc}(\mathbb{R}^N)$, and $\Omega(u_0)$ is compact and connected in that space as well. Recall that $L^\infty_{loc}(\mathbb{R})$ (and similarly $C^1_{loc}(\mathbb{R}^N)$) is a metrizable locally convex space with the system of seminorms

$$p_k := \| \cdot \|_{L^\infty((-k,k)^N)}, \quad k = 1, 2, \dots .$$

We now recall the invariance property of $\Omega(u_0)$. Let $\varphi \in \Omega(u)$, so that $u(\cdot + x_n, t_n) \to \varphi$ for some sequence $\{(x_n, t_n)\}$ in $\mathbb{R}^N \times (0, \infty)$ with $t_n \to \infty$. Then, passing to a subsequence if necessary, one shows easily that the sequence $u(x_n + \cdot, t_n + \cdot)$ converges in $C^1_{loc}(\mathbb{R}^{N+1})$ to a function U which is an entire solution of (3.1) (that is, a solution of (3.1) on $\mathbb{R}^{N+1}$). Obviously, $U(\cdot, 0) = \varphi$ and $U(\cdot, t) \in \Omega(u_0)$ for all $t \in \mathbb{R}$.

Finally, we note that $\Omega(u_0)$ is also translation-invariant: with each $\varphi \in \Omega(u_0)$, $\Omega(u_0)$ contains the whole translation group orbit of φ, $\{\varphi(\cdot + z) : z \in \mathbb{R}^N\}$. This follows directly from the definition of $\Omega(u_0)$.

3.2 A Liouville theorem

In this section, we assume that conditions (M+), (M−), (M0), in addition to the standing hypothesis (H), are satisfied.

The Liouville theorem which is used in the proofs of our main results says that any entire solution of (1.1) which is sandwiched between two shifts of the planar wave $\phi_I(x_1 - c_I t)$, for some $I \in \mathcal{N}$, is also a shift of the same planar wave. We state this formally using a moving coordinate frame.

Theorem 3.1. *Given any $I \in \mathcal{N}$, assume that U is an entire solution of* (3.1) *with $c = c_I$ such that for some $\eta_1, \eta_2 \in \mathbb{R}$ one has*

$$\phi_I(x_1 + \eta_1) \le U(x_1, x', t) \le \phi_I(x_1 + \eta_2) \quad ((x_1, x') \in \mathbb{R}^N,\ t \in \mathbb{R}). \tag{3.3}$$

Then there is η such that

$$U(x_1, x', t) = \phi_I(x_1 + \eta) \quad ((x_1, x') \in \mathbb{R}^N,\ t \in \mathbb{R}). \tag{3.4}$$

Proof. The theorem is essentially proved by Berestycki and Hamel in [5] although their results and proofs need to be modified for this purpose. We explain how.

Fix any $I = (a, b) \in \mathcal{N}$ and let U be an entire solution satisfying (3.3).

Assume first that $c_I = 0$. Then, by (M0), we have $f' \le 0$ in $(a, a+\delta) \cup (b-\delta, b)$ for some $\delta > 0$. This is the setting of Theorem 3.1 in [5] and this theorem gives (3.4). Strictly speaking, [5, Theorem 3.1] deals with a different class of entire solutions, namely, almost planar waves, but it is easy to verify that solutions satisfying (3.3) fall in that class.

Now assume that $c_I > 0$ (the case, $c_I < 0$ is completely analogous and will not be discussed here). In this case, (M+) gives $f' \le 0$ in $(b-\delta, b)$. If also $f' \le 0$ in $(a, a+\tilde{\delta})$ for some $\tilde{\delta} > 0$, in particular if $f'(a) < 0$, we can still use [5, Theorem 3.1] to obtain the conclusion (3.4). If $f'(a) = 0$ or $f'(a) > 0$ (which is possible if $a = 0$) we have to argue differently. Using an asymptotic property of $\phi_I(x)$ as $x \to \infty$, we adapt the proof of [5, Theorem 3.5]. Although that theorem is formulated for a generic monostable case, an inspection of the proof shows the following. First, the strict relation $f'(b) < 0$ assumed in [5, Theorem 3.5] is not needed, the proof works the same under the present condition $f' \le 0$ in $(b-\delta, b)$. Second (cp. [5, Remark 3.6]), the conditions $f > 0$ in (a, b) and $f'(a) > 0$ assumed in [5, Theorem 3.5] can be removed if it is known that, up to translations, $\phi = \phi_I$ is the unique decreasing solution of

$$v'' + c_I v' + f(v) = 0 \ \text{ in } \mathbb{R}, \tag{3.5}$$

with $v(-\infty) = b$, $v(\infty) = a$; and that ϕ satisfies

$$\liminf_{s \to \infty} \frac{\phi(s-\tau)}{\phi(s)} > 0 \tag{3.6}$$

for some for $\tau > 0$. Both these requirements are satisfied in our case. The proof of the former (the uniqueness) can be found in [46, Lemma 1.3.1] or

[37, Lemma 3.2.(piii)], for example. We now verify, assuming $f'(a) \geq 0$, that (3.6) is satisfied as well. (Note that since we are considering the case $c_I > 0$, Proposition 2.4(ii),(iii) implies that a is a left-global maximizer of F, hence $f'(a) > 0$ is possible only if $a = 0$). As noted and used in [5], if $f'(a) > 0$, then $\phi(x)$ has necessarily exponential asymptotics as $x \to \infty$, which yields (3.6). This is related to the linear stability of $(a, 0)$ as an equilibrium of the planar system associated with (3.5). In contrast, if $f'(a) = 0$, then the linearization at $(a, 0)$ has zero as an eigenvalue, in addition to a negative eigenvalue. In this case, traveling fronts connecting b to a do not always have the exponential asymptotics. However, the fronts coming from the minimal propagating terrace are special; the solution $\phi = \phi_I$ does have exponential asymptotics and, in particular, it satisfies (3.6) (see [37, Lemma 3.15] and [46, Proposition 1.5.6]). With these additional arguments, the proofs of [5] apply in our situation. □

We conclude this subsection with the following simple but useful result. It follows from standard uniqueness and backward uniqueness properties of (1.1).

Lemma 3.2. *If U is a bounded entire solution of* (1.1) *such that*

$$U(x_1, x', 0) = \phi_I(x_1 - \eta) \quad ((x_1, x') \in \mathbb{R}^N)$$

for some $I \in \mathcal{N}$ and $\eta \in \mathbb{R}$, then (3.4) *holds.*

4 Proofs of the main results

Throughout this section we assume that f satisfies the standing hypothesis (H) and $u_0 \in C(\mathbb{R}^N)$. For the proofs of our results, we need to consider two cases:

Case A. Conditions (G1)–(G3) together with (2.23), (2.24) are satisfied.

Case B. Conditions (M+), (M−), (M0) together with (uS) are satisfied.

These cases can be treated simultaneously to an extent. Our main theorems are derived from results on one-dimensional equations (1.3) via Liouville theorems. First, we define suitable functions $u_0^+, u_0^- \in C(\mathbb{R})$ to be used as initial data for (1.3). In Case B, with $\bar{u}_0$, η_0 as in (uS), we set:

$$u_0^+(x_1) = \bar{u}_0(x_1), \quad u_0^-(x_1) = \bar{u}_0(x_1 + \eta_0) \quad (x_1 \in \mathbb{R}). \tag{4.1}$$

By (uS) we have

$$u_0^-(x_1) \le u_0(x_1, x') \le u_0^+(x_1) \quad ((x_1, x') \in \mathbb{R}^N). \tag{4.2}$$

In Case A, we choose any *monotone nonincreasing* functions $u_0^+, u_0^- \in C(\mathbb{R})$ which, when viewed as functions on $\mathbb{R}^N$ independent of x', satisfy conditions (2.23), (2.24), and are such that (4.2) holds. It is easy to show that such functions exist.

We take $u_0^\pm$ as initial data for solutions of (1.3). Of course, these solutions can also be viewed as solutions of (1.1) independent of x'. Let u be the solution of (1.1), (1.2), and u^-, u^+ the solutions of (1.3) with the initial conditions $u^-(\cdot, 0) = u_0^-$, $u^+(\cdot, 0) = u_0^+$, respectively. By the comparison principle, we have

$$u^-(x_1, t) \le u(x_1, x', t) \le u^+(x_1, t) \quad ((x_1, x') \in \mathbb{R}^N,\ t > 0). \tag{4.3}$$

Remark 4.1. Using relations (4.3) and properties of the solutions u^-, u^+, one can immediately establish some properties of u_0. For example, as proved in [37, Theorem 2.19(i),(ii)], statements (i) and (ii) in Remark 2.8 hold if u_0 is replaced by any of the functions u_0^+, u_0^-. This and (4.3) imply that these statements hold for u_0 as well. Also, one has

$$\lim_{t\to\infty} \big(\liminf_{\substack{x_1\to-\infty \\ x'\in\mathbb{R}^{N-1}}} u(x_1, x', t)\big) = \lim_{t\to\infty} \big(\sup_{x\in\mathbb{R}^N} u(x,t)\big) = \gamma, \tag{4.4}$$

$$\lim_{t\to\infty} \big(\limsup_{\substack{x_1\to\infty \\ x'\in\mathbb{R}^{N-1}}} u(x_1, x', t)\big) = \lim_{t\to\infty} \big(\inf_{x\in\mathbb{R}^N} u(x,t)\big) = 0 \tag{4.5}$$

This follows by similar estimates for u^+, u^- (see [37, Lemma 6.1]) and relations (4.3)

Below we will use results from [37] concerning the asymptotics of the solutions $u^\pm$ of the one-dimensional problem. We summarize them here for reference.

Theorem 4.2. *With u_0^+, u_0^- defined as above, the following statements are valid.*

(a) In Case A, the conclusion of Theorems 2.6 and 2.7 are valid if u_0 is replaced by any of the functions u_0^+, u_0^-, and, moreover, in Theorem 2.7 the functions ζ_j can then be taken constant (independent of x' and t).

(b) *In Case B, the conclusion of Theorems 2.10 and 2.11 are valid if u_0 is replaced by any of the functions u_0^+, u_0^-, and, moreover, in Theorem 2.11 the functions ζ_I are then independent of x'.*

Proof. In the generic Case A, [37, Theorem 2.23] shows that the conclusion of Theorem 2.7 holds for $u_0^\pm$ (with constant ζ_j), and the conclusion of Theorem 2.6 is an easy consequence of this and Remark 4.1.

In Case B, Theorem 2.10 for the one-dimensional problem combines the statements of Theorem 2.13 and Corollary 2.16 of [37]. We remark that in the most general setting, the conclusion of [37, Theorem 2.13] is a little weaker than the conclusion in Theorem 2.10 of the present paper. However, as explained in [37, Remark 2.14], the present statement is valid if, for example, the initial data $u_0^\pm$ satisfy

$$u_0^\pm(x_1+\eta_0) \le u_0^\pm(x_1) \quad (x_1 \in \mathbb{R}) \tag{4.6}$$

for some fixed $\eta_0 > 0$. In our case, these relations are satisfied as a direct consequence of (4.1), (4.2).

The statements of Theorem 2.11 in the one-dimensional case (with the functions ζ_I independent of x') are contained in [37, Theorem 2.19]. Again, as shown in [37, Remark 2.14], relations (4.6) verify an extra assumption in [37, Theorem 2.19(v)]. □

4.1 Proofs of Theorems 2.6, 2.10

Pick any element $\varphi \in \Omega(u)$. Then for some $x_n = (x_{1,n}, x_n') \in \mathbb{R}^N$, $t_n > 0$, $n = 1, 2, \dots$, we have $t_n \to \infty$ and $u(\cdot + x_n, t_n) \to \varphi$ in $L^\infty_{loc}(\mathbb{R}^N)$. The results in Section 3.1 show that, after passing to subsequences, the following limits exist (the first one in $C^1_{loc}(\mathbb{R}^N \times \mathbb{R})$, the last two in $C^1_{loc}(\mathbb{R} \times \mathbb{R})$)

$$u(\cdot + x_n, \cdot + t_n) \to U, \quad u^-(\cdot + x_{1,n}, \cdot + t_n) \to U^-, \quad u^+(\cdot + x_{1,n}, \cdot + t_n) \to U^+, \tag{4.7}$$

U being an entire solution of (1.1), and U^-, U^+ entire solutions of (1.3). Clearly, (4.3) yields

$$U^-(x_1,t) \le U(x_1,x',t) \le U^+(x_1,t) \quad ((x_1,x') \in \mathbb{R}^N,\ t \in \mathbb{R}). \tag{4.8}$$

In particular,

$$\varphi^-(x_1) := U^-(x_1,0) \le \varphi(x_1,x') \le \varphi^+(x_1) := U^+(x_1,0) \qquad ((x_1,x') \in \mathbb{R}^N,\ t \in \mathbb{R}). \tag{4.9}$$

The proofs of Theorem 2.10, 2.6 can now be completed as follows.

Completion of the proof of Theorem 2.10. With the hypotheses of Theorem 2.10, we are in Case B, whence $u_0^- = u_0^+(\cdot - \eta_0)$. By the uniqueness for the Cauchy problem,

$$u^- = u^+(\cdot - \eta_0, \cdot). \tag{4.10}$$

Consequently,

$$U^- = U^+(\cdot - \eta_0, \cdot), \quad \varphi^- = \varphi^+(\cdot - \eta_0). \tag{4.11}$$

Using this, relations (4.9), and Theorem 4.2(b), we obtain that φ is either a function with range in $(0, \gamma_0)$, or a function with range in (γ_1, γ), or identical to a constant in $R_0^{-1}\{0\}$, or else $\varphi^+ = \phi_I(\cdot + \xi)$ for some $I \in \tilde{\mathcal{N}}$, $\xi \in \mathbb{R}$. In the last case, Lemma 3.2 implies that U^+ is the traveling front with the profile function $\phi_I(\cdot + \xi)$. Using Theorem 3.1, we conclude that U is also a traveling front whose profile function is a shift of ϕ_I. In particular, φ is a shift of ϕ_I. By this we have proved that (2.35) holds with the equality sign replaced by the inclusion "$\subset$."

To prove the opposite inclusion, take any

$$\phi \in R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \tilde{\mathcal{N}}, \xi \in \mathbb{R}\}.$$

If $\phi \equiv 0$ or $\phi \equiv \gamma$, then $\phi \in \Omega(u)$ due to (4.4), (4.5). If $\phi \notin \{0, \gamma\}$, then $\theta := \phi(0) \in (0, \gamma) \cap [\gamma_0, \gamma_1]$. By (4.4), (4.5), and the continuity of $x_1 \mapsto u(x_1, 0, t)$, for each sufficiently large t there is $x_1 \in \mathbb{R}$ such that $u(x_1, 0, t) = \theta$. It follows that $\Omega(u)$ contains an element φ with $\varphi(0) = \theta = \phi(0)$. By the inclusion proved above, necessarily $\varphi \equiv \phi$, proving that $\phi \in \Omega(u)$. This proves the inclusion "$\supset$" in (2.35) and completes the proof of (2.35).

Assume now that conditions (u0), (u1) are satisfied. Then the second conclusion of Theorem 2.10 applies to both u_0^- and u_0^+. Thus, similar arguments as above show that the stronger conclusion (2.36) holds for u_0 as well. □

Completion of the proof of Theorem 2.6. Assume the hypotheses of Theorem 2.6 (Case A). Recall that in this generic case we have (2.27) for some k, and that $c_{I_1}^+$, $c_{I_k}^-$ are defined in (2.28), (2.29). According to Theorem 4.2(a), for each of the functions u^-, u^+ relation (2.31) holds with some *constants* ζ_j (which depend on u^-, u^+).

With the sequence $\{(x_n, t_n)\}$ as in (4.7), we first pass to a subsequence so as to achieve that one of the following possibilities occurs:

(p1) $c^-_{I_k} t_n < x_{1,n} < c^+_{I_1} t_n \quad (n = 1, 2, \dots)$,

(p2) $x_{1,n} \le c^-_{I_k} t_n \quad (n = 1, 2, \dots)$,

(p3) $c^+_{I_1} t_n \le x_{1,n} \quad (n = 1, 2, \dots)$.

Here, possibility (p2) can occur only if $\gamma_1 < 0$ (so that $c^-_{I_k} > -\infty$) and (p3) can occur only if $\gamma_0 > 0$ (so that $c^+_{I_1} < \infty$).

Consider possibility (p1). Passing to another subsequence, we may further assume that there is $\ell \in \{1, \dots, k+1\}$ such that for $n = 1, 2, \dots$ the point $x_{1,n}$ belongs to the ℓth interval in the sequence

$$(c^-_{I_k} t_n, c_{I_k} t_n],\ (c_{I_k} t_n, c_{I_{k-1}} t_n], \dots, (c_{I_1} t_n, c^+_{I_1} t_n); \tag{4.12}$$

and either its distance to exactly one of the boundary points of that interval remains bounded, or the distance to both boundary points diverges to ∞ as $t \to \infty$ (here we use condition (G3)). One then verifies easily, using relations (2.31) for $u^\pm$, that either the limits φ^-, φ^+ are both identical to one of the constants b_j (this occurs in the latter case) or for some $I \in \tilde{\mathcal{N}}$, and η_1, η_2 one has

$$\varphi^- = \phi_I(\cdot - \eta_1), \quad \varphi^+ = \phi_I(\cdot - \eta_2). \tag{4.13}$$

This and Lemma 3.2 give

$$U^-(x_1, x', t) = \phi_I(x_1 - c_I t - \eta_1), \quad U^+(x_1, x', t) = \phi_I(x_1 - c_I t - \eta_2) \qquad ((x_1, x') \in \mathbb{R}^N,\ t \in \mathbb{R}).$$

As in the previous proof, applying Theorem 3.1, we obtain that $\varphi = \phi_I(\cdot - \xi)$ for some $\xi \in \mathbb{R}$, as desired.

Next, consider the possibility (p3). Using relation (2.31) for u^+ and the monotonicity of $u^+(\cdot, t)$ (this follows from our choice of monotone nonincreasing u^+_0), we obtain that $U^+ \le \gamma_0$ everywhere. Also, $U^- \ge 0$, by Theorem 2.6 applied to u^-. Consequently, $0 \le U \le \gamma_0$. The strong comparison principle implies that either $0 < U < \gamma_0$ everywhere or else U is identical to one of the constants 0, γ_0. Thus, either φ is identical to one of these constants, both elements of $R_0^{-1}\{0\}$, or its range is contained in the interval $(0, \gamma_0)$.

Analogous considerations show that if (p2) occurs, then either φ is identical to one of the constants $\gamma, \gamma_1 \in R_0^{-1}\{0\}$, or its range is contained in the interval (γ_1, γ).

We have thus proved that (2.25) holds with the equality sign replaced by the inclusion "$\subset$." The other inclusion is proved by the same argument as in the proof of Theorem 2.10. □

4.2 Proofs of Theorems 2.7, 2.11

We will use Theorems 2.6, 2.10 already proved above. The proofs of Theorems 2.7, 2.11 will be carried out in several steps, some being common to both proofs.

STEP 1. Here we show that in both Cases A and B, statement (i) of Theorem 2.11 holds. For this end, fix an arbitrary interval $I = (a, b) \in \tilde{\mathcal{N}}$. Given any value $\theta \in I$, we claim that if t is sufficiently large, then

$$u_{x_1}(x_1, x', t) < 0 \text{ for each } x = (x_1, x') \in \mathbb{R}^N \text{ with } u(x_1, x', t) = \theta. \qquad (4.14)$$

Indeed, suppose our claim is not true. Then there exist points $x_n = (x_{1,n}, x'_n) \in \mathbb{R}^N$ and times t_n $(n = 1, 2, \dots)$, such that $t_n \to \infty$ and

$$u(x_n, t_n) = \theta, \quad u_{x_1}(x_n, t_n) \geq 0. \qquad (4.15)$$

Upon extracting subsequences, we obtain that for some $\varphi \in \Omega(u_0)$

$$u(\cdot + x_n, t_n) \to \varphi,$$

with the convergence in $C^1_{loc}(\mathbb{R}^N)$. By (4.15), $\varphi(0) = \theta$ and $\varphi_{x_1}(0) \geq 0$. However, since $\theta \in I$, Theorems 2.6, 2.10 imply that φ is a shift of ϕ_I, which contradicts the relation $\varphi_{x_1}(0) \geq 0$. Thus the claim is true. It follows that for all sufficiently large t and for any $x' \in \mathbb{R}^{N-1}$, there is at most one value $x_1 \in \mathbb{R}$ with $u(x_1, x', t) = \theta$. We also know, by Remark 4.1, that such a value does exist.

Applying the above with $\theta := (a + b)/2$, we find $s_I > 0$ such that for all $t > s_I$ and $x' \in \mathbb{R}^{N-1}$ the equation $u(c_I t + y_1, x', t) = \theta$ has a unique solution y_1; we define this solution as the value of a function ζ_I at (x', t). By the claim, $u_{x_1}(c_I t + \zeta_I(x', t), x', t) < 0$. This and the uniqueness of the solution y_1 imply that relation (b) in Theorem 2.11(i) holds. The implicit function theorem further implies that ζ_I is a C^1 function on $\mathbb{R}^{N-1} \times (s_I, \infty)$.

Next we prove that statement (c) in Theorem 2.11(i) holds. Suppose it does not. Then there exist $q > 0$, $\delta > 0$ and a sequence $\{(x'_n, t_n)\}$ in $\mathbb{R}^{N-1} \times (s_I, \infty)$ such that $t_n \to \infty$ and

$$\sup_{x_1 \in (-q,q)} |u(x_1 + c_I t_n + \zeta_I(x'_n, t), x'_n, t_n) - \phi_I(x_1)| \geq \delta. \qquad (4.16)$$

On the other hand, after passing to a subsequence, for some $\varphi \in \Omega(u_0)$ we have

$$u(\cdot + c_I t_n + \zeta_I(x'_n, t_n), \cdot + x'_n, t_n) \to \varphi,$$

with the convergence in $L^\infty_{loc}(\mathbb{R}^N)$. The definition of ζ implies that $\varphi(0,0) = \theta = \phi_I(0)$ (cp. (2.6)). Therefore, by Theorems 2.6, 2.10, $\varphi \equiv \phi_I$ and the convergence in $L^\infty_{loc}(\mathbb{R}^N)$ contradicts (4.16). Statement (c) is proved.

We now show that (2.37) holds. To simplify the notation, set $\tilde{u}(x,t) := u(x+ct,t)$. By the results from Section 3.1, any sequence $\{(x'_n,t_n)\}$ in $\mathbb{R}^{N-1}\times(0,\infty)$ with $t_n\to\infty$ can be replaced by a subsequence such that $\tilde{u}(\cdot+\zeta_I(x'_n,t_n),\cdot+x'_n,\cdot+t_n)$ converges in $C^1_{loc}(\mathbb{R}^{N+1})$ to an entire solution U of equation (3.1). By statement (c), we have $U(\cdot,\cdot,0)=\phi_I$. Therefore, by Lemma 3.2, $U\equiv\phi_I$. Since ϕ_I is independent of x' and t, the convergence in $C^1_{loc}(\mathbb{R}^{N+1})$ yields

$$\begin{aligned}&\big(\tilde{u}(\cdot+\zeta_I(x'_n,t_n),x'_n,\cdot+t_n),\tilde{u}_{x_1}(\cdot+\zeta_I(x'_n,t_n),x'_n,\cdot+t_n),\\ &\nabla_{x'}\tilde{u}(\cdot+\zeta_I(x'_n,t_n),x'_n,\cdot+t_n),\tilde{u}_t(\cdot+\zeta_I(x'_n,t_n),x'_n,\cdot+t_n)\big)\to\big(\phi_I,\phi'_I,0,0\big).\end{aligned}$$

Since this is true for any sequence $\{(x'_n,t_n)\}$ in $\mathbb{R}^{N-1}\times(0,\infty)$ with $t_n\to\infty$, the convergence takes place with x'_n replaced by an arbitrary $x'\in\mathbb{R}^{N-1}$, t_n replaced by t, with $t\to\infty$, and it is uniform in $x'\in\mathbb{R}^{N-1}$. In particular, at $x_1=0$ we have

$$\begin{aligned}\big(\tilde{u}(\zeta_I(x',t),x',t),\tilde{u}_{x_1}(\zeta_I(x',t),x',t),\nabla_{x'}\tilde{u}(\zeta_I(x',t),x',t),\tilde{u}_t(\zeta_I(x',t),x',t)\big)&\\ \to(\theta,\phi'_I(0),0,0),\quad&(4.17)\end{aligned}$$

as $t\to\infty$, uniformly in $x'\in\mathbb{R}^N$. Here, we have used the relation

$$\tilde{u}(\zeta_I(x',t),x',t)=\theta,$$

which follows from the definition of ζ_I. Differentiating this relation, we obtain

$$\begin{aligned}\tilde{u}_{x_1}(\zeta_I(x',t),x',t)\nabla_{(x',t)}\zeta(x',t)+\nabla_{x'}\tilde{u}(\zeta_I(x',t),x',t)&\\ +\tilde{u}_t(\zeta_I(x',t),t)=0.\quad&(4.18)\end{aligned}$$

Since $\phi'(0)<0$, from the uniform convergence in (4.17) we conclude that $\nabla_{(x',t)}\zeta(x',t)\to0$ as $t\to\infty$, uniformly in $x'\in\mathbb{R}^N$. This is equivalent to (2.37).

To complete the proof of statement (a), we verify that the function $\zeta_I(x',t)-\zeta_I(0,t)$ is bounded, that is, (2.38) holds. For this we use (4.3). It implies that for each $x'\in\mathbb{R}^{N-1}$ and $t>s_I$, $\zeta_I(x',t)$ is bounded above by a zero of $u^+(\cdot+c_It,t)-\theta$ and below by a zero of $u^-(\cdot+c_It,t)-\theta$. The distance of these two zeros is bounded. Indeed, in Case B, the definition of $u_0^{\pm}$ gives (4.10) and the boundedness is immediate. In Case A, the boundedness follows from the asymptotics of the solutions $u^{\pm}$ given by Theorem 4.2(a) (see (2.31) and remember that the ζ_j associated with u_+ or u^- are constant).

For the completion of the proof of statement (i) of Theorem 2.11, it remains to prove that (d) holds. There is nothing to be proved in Case A: the assumption in (d) is ruled out by (G3). Consider Case B and assume that for some $I_1, I_2 \in \tilde{\mathcal{N}}$ with $I_1 < I_2$ we have $c_{I_1} = c_{I_2}$. By Theorem 4.2(b), statement (d) holds for $u^{\pm}$ with some functions $\zeta_{I_1}^{\pm}$, $\zeta_{I_2}^{\pm}$ independent of x'. Due to (4.10), we have $\zeta_I^-(t) = \zeta_I^+(t) - \eta_0$ for all t and $I \in \{I_1, I_2\}$. Since $\zeta_I(x', t)$ is between ζ_I^-, ζ_I^+, statement (d) holds for u as well.

STEP 2. In this step, we assume that the hypotheses of Case B and the extra hypotheses (u0), (u1) are satisfied. We prove statement (ii) of Theorem 2.11. The fact that statement (i) remains valid with $\tilde{\mathcal{N}}$ replaced by $\mathcal{N}$ can be proved by the same arguments as in Step 1. One simply takes $I \in \mathcal{N}$, rather than just $I \in \tilde{\mathcal{N}}$, and the arguments go through thanks to the second statement of Theorem 2.10.

We now prove statement (ii)(e). Let $\{(x_{1,n}, x'_n, t_n)|\}$ be as in that statement. We first pass to a subsequence such that, for some $\varphi \in \Omega(u_0)$,

$$\lim_{n\to\infty} u(\cdot + x_{1,n}, \cdot + x'_n, t_n) = \varphi \tag{4.19}$$

in $C^1_{loc}(\mathbb{R}^N)$. We need to prove that φ is identical to a constant in $R_0^{-1}\{0\}$. According to Theorem 2.10,

$$\varphi \in R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \mathcal{N},\, \xi \in \mathbb{R}\}. \tag{4.20}$$

Thus, all we need to prove is that φ is not a shift of ϕ_I for any $I \in \mathcal{N}$. We show this by contradiction. Suppose that for some $I \in \mathcal{N}$ and $\xi \in \mathbb{R}$, one has $\varphi \equiv \phi_I(\cdot - \xi)$. Then $\varphi(\xi) = \phi_I(0) = (a+b)/2$ and $\varphi'(\xi) > 0$. Therefore, (4.19) implies that for all large enough n, the function $u(\cdot + x_{1,n}, x'_n, t_n) - (a+b)/2$ has a zero near ξ. By statement (i) proved above, this zero is equal to $ct_n + \zeta_I(x'_n, t_n) - x_{1,n}$. However, in view of (2.39), this point cannot be located near ξ for all n. This contradiction completes the proof of statement (e).

STEP 3. Here we assume that the hypotheses of Case A are satisfied. We claim that statement (e) is valid here as well if the following modifications are made: $\mathcal{N}$ is replaced by $\tilde{\mathcal{N}}$ and the sequence $\{(x_n, t_n)\} = \{(x_{1,n}, x'_n, t_n)\}$

is required to satisfy the extra requirement that $c^-_{I_k} t_n < x_{1,n} < c^+_{I_1} t_n$, where $c^\pm_{I_k}$ are as in Theorem 2.7. In fact, the conclusion in (e) then holds with $\xi \in R_0^{-1}(0) \cap [\gamma_0, \gamma_1]$. The proof can be carried out in much the same way as in the previous step. One just needs to show that for the function φ in (4.19) one still has (4.20), that is, the range of φ is not contained in $(0, \gamma_0)$ or (γ_1, γ) (cp. Theorem 2.6). But this follows immediately from Remark 2.8 and relations (2.28), (2.29).

STEP 4. Here we complete the proof Theorem 2.11 by showing that statement (iii) is true. We assume that the hypotheses of Case B together with (u0), (u1) are satisfied and the set $R_0^{-1}\{0\}$ is finite: (2.40) holds. Here we adapt arguments used in the proof of Theorem 2.19 of [37].

To simplify the notation, we denote $\zeta_j := \zeta_{I_j}$. Note that the relations $c_{I_j} \geq c_{I_{j+1}}$ (cp. Theorem 2.3(ii)) and the properties of the functions ζ_j established in statements (a) and (d) imply that, as $t \to \infty$,

$$\inf_{x' \in \mathbb{R}^{N-1}} \big(c_{I_j} t + \zeta_j(x', t) - (c_{I_{j+1}} t + \zeta_{j+1}(x', t)\big) \to \infty \quad (j = 1, \dots, k). \quad (4.21)$$

We prove (2.41) by contradiction. Suppose that it does not hold, that is, there exist $\delta > 0$, $(x_{1,n}, x'_n) \in \mathbb{R}^N$, $t_n > 0$ $(n = 1, 2, \dots)$, such that $t_n \to \infty$ and

$$\Big|u(x_{1,n}, x'_n, t_n) - \big(\sum_{j=1,\dots,k} \phi_{I_j}(x_{1,n} - c_{I_j} t_n - \zeta_{I_j}(x'_n, t_n)) - \sum_{1 \leq j \leq k-1} a_{j+1}\big)\Big| \geq \delta. \quad (4.22)$$

Obviously, passing to subsequences, we may assume that there is ℓ such that for each $n = 1, 2, \dots$ the point $x_{1,n}$ belongs to the ℓth interval in the following sequence of mutually disjoint intervals covering $\mathbb{R}$:

$$\begin{aligned} &(-\infty,\, c_{I_k} t_n + \zeta_k(x'_n, t_n)], \\ &(c_{I_{j+1}} t_n + \zeta_{j+1}(x'_n, t_n),\, c_{I_j} t_n + \zeta_j(x'_n, t_n)], \quad j = k-1, k-2, \dots, 1, \\ &(c_{I_1} t_n + \zeta_1(x'_n, t_n),\, \infty). \end{aligned} \quad (4.23)$$

Passing to a further subsequence, we obtain that either there is exactly one j such $c_{I_j} t_n + \zeta_j(x'_n, t_n) - x_{1,n}$ converges to a finite value, or for all j one has $|c_{I_j} t_n + \zeta_j(x'_n, t_n) - x_{1,n}| \to \infty$. Finally, passing to a yet another subsequence, we may assume that for some $\varphi \in \Omega(u)$ one has $u(\cdot + x_{1,n}, x'_n, t_n) \to \varphi$ in $C^1_{loc}(\mathbb{R})$.

Consider the case when each $x_{1,n}$ belongs to the first of the intervals (4.23), and let

$$\rho := \lim_{n\to\infty} \big(x_{1,n} - (c_{I_k} t_n + \zeta_k(x_n', t_n))\big) \in [-\infty, 0].$$

Then for $j < k$ one has

$$\phi_{I_j}(x_{1,n} - c_{I_j} t_n - \zeta_j(x_n', t_n)) \to \phi_{I_j}(-\infty) = a_{j+1}. \tag{4.24}$$

If now $\rho > -\infty$, then, by statement (i)(c) proved in Step 1, $u(x_{1,n}, x_n', t_n) - \phi_{I_k}(x_{1,n} - c_{I_k} t_n - \zeta_k(x_n', t_n)) \to 0$. Thus, the limit of the left-hand side of (4.22) is 0, and we have a contradiction. If $\rho = -\infty$, then (4.24) also holds for $j = k$ (and $a_{k+1} = \gamma$). Also, by statement (e) proved in Step 2, for the limit function φ we have $\varphi \equiv \xi \in R_0^{-1}\{0\}$. Moreover, since $x_{1,n}$ belongs to the first interval in (4.23), statement (i)(b) of Theorem 2.11 implies that $\xi \geq (a_{k+1} + a_k)/2 > a_k$. Therefore, $\xi \geq a_{k+1} = \gamma$, thus, necessarily, $\xi = \gamma$. In this case, again, the limit of the left-hand side of (4.22) is 0, and we have a contradiction.

The proof in the cases where the $x_{1,n}$ belong to any other interval in (4.23) can be done in a similar way and is omitted. The proof of Theorem 2.11 is now complete.

STEP 5. Assume the hypotheses of Case A. With what has been done in Steps 1 and 3, to complete the proof Theorem 2.7 one can use essentially the same arguments as in the previous step. Indeed, relation (2.31) is similar to (2.41). The difference is that in the latter all intervals $I \in \mathcal{N}$ involved and the supremum is taken over $x_1 \in \mathbb{R}$, whereas in the former we only take $I \in \tilde{\mathcal{N}}$, which is a smaller set if $\gamma_0 > 0$ or $\gamma_1 < 0$, and, accordingly, we take the supremum over x_1 in an interval which may be bounded on the left or right. Therefore, in the above proof one needs to replace γ by γ_1, 0 by γ_0 and the first and last intervals in the sequence (4.23) need to be replaced by $(c_{I_k}^- t_n, c_{I_k} t_n]$ and $(c_{I_1} t_n, c_{I_1}^+ t_n)$, respectively. All the other modifications are straightforward and we omit further details.

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