Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part I: Elliptic equations and systems

Peter Poláčik*
School of Mathematics, University of Minnesota
Minneapolis, MN 55455, USA
e-mail: polacik@math.umn.edu

Pavol Quittner†
Department of Applied Mathematics and Statistics, Comenius University,
Mlynská dolina, 84248 Bratislava, Slovakia
e-mail: quittner@fmph.uniba.sk

Philippe Souplet
Analyse, Géométrie et Applications, Institut Galilée, Université Paris-Nord,
93430 Villetaneuse, France
e-mail: souplet@math.univ-paris13.fr

Abstract

In this paper, we study some new connections between Liouville-type theorems and local properties of nonnegative solutions to superlinear elliptic problems. Namely, we develop a general method for derivation of universal, pointwise a priori estimates of *local* solutions from Liouville-type theorems, which provides a simpler and unified treatment for such questions. The method is based on rescaling arguments combined with a key "doubling" property, and it is different from the classical rescaling method of Gidas and Spruck. As an important heuristic consequence of our approach, it turns out that universal boundedness theorems for local solutions and Liouville-type theorems are essentially equivalent.

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The method enables us to obtain new results on universal estimates of spatial singularities for elliptic equations and systems, under optimal growth assumptions which, unlike previous results, involve only the behavior of the nonlinearity at infinity. For semilinear systems of Lane-Emden type, these seem to be the first results on singularity estimates to cover the full subcritical range. In addition, we give an affirmative answer to the so-called *Lane-Emden conjecture* in three dimensions.

1 Introduction

The aim of this paper is to study some new connections between Liouville-type theorems and local properties of nonnegative solutions to superlinear elliptic problems. In all the paper, the word "solution" always refers to "nonnegative solution", regardless of whether it is specifically mentioned. By a (nonlinear) Liouville-type theorem, we here mean the statement of nonexistence of nontrivial bounded solutions on the whole space or on a half-space.

In the last two decades, following the seminal paper [17], Liouville-type theorems have been widely used, in conjunction with rescaling arguments, in derivation of pointwise a priori estimates of solutions of boundary value problems (see [12] for a survey, for more recent results and references see e.g. [25, 15, 7, 31, 14, 32]).

In this paper, we develop a general method for derivation of pointwise a priori estimates of *local* solutions (i.e., on an arbitrary domain and without any boundary conditions), from Liouville-type theorems. This method enables us to obtain new results on universal estimates of spatial singularities for elliptic problems. At the same time, it gives somewhat simpler proofs of several known results and provides a unified treatment for such questions.

As a further motivation to our approach, let us mention that in an important recent paper [29] on quasilinear elliptic equations, Serrin and Zou have observed that Liouville theorems can be seen as a consequence and a limiting case of universal boundedness theorems. For instance, for the model equation

$$-\Delta u = u^p \tag{1.1}$$

with subcritical $p > 1$, the nonexistence of nontrivial solutions in $\mathbb{R}^n$ is a direct corollary to the universal boundedness result [11, Lemma 1], which states that any classical solution of (1.1) in an arbitrary domain $\Omega \subset \mathbb{R}^n$ satisfies

$$u(x) \leq C(n,p)\,\mathrm{dist}^{-\frac{2}{p-1}}(x,\partial\Omega), \quad x \in \Omega. \tag{1.2}$$

In other words, citing [29, p. 82]: "they both provide *upper bounds* for nonnegative solutions", with the Liouville theorem "being the extreme case where the domain is all of $\mathbb{R}^n$ and the upper bound becomes zero", and universal boundedness theorems provide "a continuous embedding of the Liouville theorem in a family of results for an expanding sequence of bounded domains". From this point of view, a remarkable consequence of the approach in the present paper is that *the converse is also true,* so that *Liouville theorems and universal boundedness theorems are in fact equivalent* (for problems with homogeneous nonlinearities such as (1.1), or (3.2) and (4.1) below).

As a consequence of our method, we improve a number of known results on semilinear and quasilinear elliptic equations. In particular, although it was natural to expect that singularity estimates should depend only on the *behavior of the nonlinearity at infinity*, all previous results [16, 5, 29, 4] required global assumptions. In contrast, our method does not rely on any global assumptions, thus our results confirm the above expectation. We also treat semilinear systems of Lane-Emden type; the singularity estimates that we obtain seem to be the first results of this type to cover the full subcritical range (below the so-called Sobolev hyperbola).

Other by-products of the method are *strong* Liouville-type theorems, that is statements on nonexistence of nontrivial solutions (bounded or not) on the whole space or on a half-space. In particular, we give an affirmative answer to the so-called *Lane-Emden conjecture* for elliptic systems in three dimensions.

The method in this paper is based on rescaling arguments combined with a key "doubling" property (see Lemma 5.1 below). A heuristic explanation of our approach, and of the differences with the "classical" rescaling method [17] (where *global* a priori estimates are derived from Liouville-type results) is given at the beginning of Section 5.

The doubling property is an extension of an idea of [18]. In that work (see also the references in [24]), a similar doubling property in time was used to estimate blow-up rates of nonglobal solutions of certain nonlinear parabolic problems. However, this powerful idea does not seem to have been fully exploited up to now, nor its wide applicability in singularity estimates has been noticed. In particular, it has not been applied to elliptic problems so far. Even for parabolic equations, the link with (parabolic) Liouville-type theorems and universal bounds of solutions has not been made completely clear yet. These parabolic aspects are developed in the second part of our work [24].

Remark 1.1. There are two completely different types of proofs for nonlinear Liouville theorems:

- they can be a consequence of integral a priori estimates of solutions obtained by, often quite sophisticated, energy techniques. Sometimes these estimates are derived for local solutions as well and they can be directly used to establish pointwise singularity estimates (see [16, 5, 29] and [2], respectively for elliptic and parabolic problems). But in other cases they apply only to global solutions on the whole space, cf. [27];

- they can be proved directly by monotonicity techniques based on the maximum principle (such as moving planes or spheres [17, 9, 10, 6, 25, 7], or intersection-comparison [23] for certain parabolic problems). In this case, the proof does not depend on the knowledge of any estimate of local solutions.

Therefore, our approach for deriving pointwise singularity estimates from Liouville theorems is of particular interest in this second case (and in the global instance of the first case).

The outline of the paper is as follows. Estimates of singularities for semilinear and quasilinear scalar equations are stated in Sections 2 and 3, respectively. In Section 4, we turn to semilinear elliptic systems, for which we give strong Liouville theorems, as well as singularity estimates. The key doubling lemma is stated and proved in Section 5. Sections 6 and 7 are then devoted to the proofs and to some more general results, for scalar equations and systems, respectively.

2 Estimates of singularities for semilinear elliptic equations

In this section, we consider equations of the form

$$-\Delta u = f(u). \tag{2.1}$$

The function $f : [0,\infty) \to \mathbb{R}$ is only assumed to be continuous. By a solution $u \geq 0$ of (2.1) in an (arbitrary) domain Ω, we mean a strong local solution, i.e. $u \in W^{2,q}_{\mathrm{loc}}(\Omega)$, for all $q \in (1,\infty)$ (or a local classical solution $u \in C^2(\Omega)$ if f is locally Hölder continuous). By p_S we denote the Sobolev critical exponent:

$$p_S := \begin{cases} \dfrac{n+2}{n-2}, & \text{if } n \geq 3, \\ \infty, & \text{if } n = 1, 2. \end{cases}$$

Our first result is an a priori estimate of possible singularities of local solutions to (2.1).

Theorem 2.1. *Let $1 < p < p_S$ and assume that*

$$\lim_{u\to\infty} u^{-p} f(u) = \ell \in (0,\infty). \tag{2.2}$$

Let Ω be an arbitrary domain of $\mathbb{R}^n$. Then there exists $C = C(n,f) > 0$ (independent of Ω and u) such that for any (nonnegative) solution u of (2.1) in Ω, there holds

$$u + |\nabla u|^{\frac{2}{p+1}} \leq C\bigl(1 + \mathrm{dist}^{-\frac{2}{p-1}}(x, \partial\Omega)\bigr), \quad x \in \Omega. \tag{2.3}$$

In particular if $\Omega = B_R \setminus \{0\}$ for some $R > 0$, then

$$u + |\nabla u|^{\frac{2}{p+1}} \leq C\bigl(1 + |x|^{-\frac{2}{p-1}}\bigr), \quad 0 < |x| \leq R/2.$$

Theorem 2.1 extends results from [16, 29] (see also [8, 5, 11]) by covering a significantly different class of nonlinearities. Indeed, all the previous results require global lower bound and regularity on f, along with some global monotonicity assumption (see (2.4) below). As one naturally expects, and as our theorem confirms, such global assumptions are not necessary and singularity estimates for solutions of (2.1) depend only on the behavior of $f(u)$ for large u (note that f is not even supposed to be nonnegative away from ∞ in the theorem). Note, however, that the improvement is at the expense of somewhat stronger assumption on the asymptotic behavior of f. The estimate of $|\nabla u|$ (which appears in [16] for $x \to 0$) is provided by our method at almost no additional cost.

Remarks 2.2. (a) In the corresponding results of [16, 29], it is assumed that for some $1 < p \le \alpha < p_S$ and $C_2 \ge C_1 > 0$, f verifies

$$\left.\begin{aligned} &f \in C([0,\infty)) \cap C^1((0,\infty)),\\ &u \mapsto u^{-\alpha} f(u) \ \text{ is nonincreasing for } u > 0,\\ &C_1 u^p \le f(u) \le C_2(1+u^p), \quad u \ge 0. \end{aligned}\right\} \tag{2.4}$$

(b) Theorem 2.1 can be extended to more general nonlinearities of the form $f(x, u, \nabla u)$, see Theorem 6.1 below.

(c) The constant C in (2.3) depends on f through the modulus of convergence in (2.2) only (i.e., it is uniform for a family of nonlinearities f_k for which (2.2) holds uniformly w.r.t. k). A similar remark applies to Theorem 3.1 below.

(d) Recall that the powers $2/(p-1)$ and $(p+1)/(p-1)$ in (2.3) (for u and $|\nabla u|$, respectively) are sharp for $n/(n-2) < p < p_S$ and $n \ge 3$, as can be seen from the explicit solution of (1.1), of the form $u(x) = c(n,p)|x|^{-2/(p-1)}$, which exists in that range. On the other hand, $2/(p-1)$ is not sharp for (1.1) with $p < n/(n-2)$, since the sharp power is then known to be $n-2$ ($n \ge 3$), see [26, 19]. Recall also that, once upper bounds of the form (2.3) are known, a more precise description of isolated singularities can be obtained by dynamical systems techniques (see e.g. [5]).

(e) It is well-known that the condition $p < p_S$ is sharp for the existence of a universal upper estimate of local solutions of (1.1). Indeed, if $p \ge p_S$, then there exists a (bounded) classical solution v of the model equation (1.1) on $\mathbb{R}^n$. Since $v_k := k^{2/(p-1)} v(kx)$ is also a solution, say on B_1, and $v_k(0) \to \infty$, as $k \to \infty$, it follows that no universal estimate can hold.

In the special case $f(u) = u^p$, by the same method, we recover the following more precise result, which is essentially known (see [16, Theorem 3.6], [29, Theorem IV (b)]), and which in particular provides sharp decay estimates in the case of exterior domains.

Theorem 2.3. *Let $1 < p < p_S$ and let $\Omega \ne \mathbb{R}^n$ be a domain of $\mathbb{R}^n$. There exists $C = C(n,p) > 0$ (independent of Ω and u) such that any (nonnegative) solution u of*

(1.1) *in* Ω *satisfies*

$$u+|\nabla u|^{\frac{2}{p+1}} \leq C\mathrm{dist}^{-\frac{2}{p-1}}(x,\partial\Omega), \quad x\in\Omega. \tag{2.5}$$

In particular if Ω *is an exterior domain, i.e.* $\Omega \supset \{x\in\mathbb{R}^n;\ |x|>R\}$ *for some* $R>0$, *then*

$$u+|\nabla u|^{\frac{2}{p+1}} \leq C|x|^{-\frac{2}{p-1}}, \quad |x|\geq 2R.$$

3 Estimates of singularities for quasilinear elliptic equations

In this section, we consider quasilinear equations of the form

$$-\Delta_m u = f(u). \tag{3.1}$$

where $\Delta_m u := \mathrm{div}(|\nabla u|^{m-2}\nabla u)$, $1<m<\infty$, is the m–Laplacian operator and the function $f:[0,\infty)\to\mathbb{R}$ is continuous. By a solution $u\geq 0$ of (3.1) in Ω, we mean a weak solution, i.e. a distribution solution with $u\in C^1(\Omega)$ (cf. [29]). The corresponding Sobolev critical exponent is now given by

$$p_S(m) := \begin{cases} \dfrac{n(m-1)+m}{n-m}, & \text{if } n>m,\\ \infty, & \text{if } n\leq m. \end{cases}$$

The first results on Liouville theorems and singularity estimates for solutions of

$$-\Delta_m u = u^p \tag{3.2}$$

in the full subcritical range $m-1<p<p_S(m)$ have been obtained recently in [29]. We shall prove the following a priori estimate of possible singularities of local solutions to (3.1).

Theorem 3.1. *Let* $0<m-1<p<p_S(m)$ *and assume that* (2.2) *holds true. Let* Ω *be an arbitrary domain of* $\mathbb{R}^n$. *Then there exists* $C=C(m,n,f)>0$ *(independent of* Ω *and* u*) such that for any (nonnegative) solution* u *of* (3.1) *in* Ω, *there holds*

$$u+|\nabla u|^{\frac{m}{p+1}} \leq C(1+\mathrm{dist}^{-\frac{m}{p+1-m}}(x,\partial\Omega)), \quad x\in\Omega. \tag{3.3}$$

In particular if $\Omega = B_R\setminus\{0\}$ *for some* $R>0$, *then*

$$u+|\nabla u|^{\frac{m}{p+1}} \leq C(1+|x|^{-\frac{m}{p+1-m}}), \quad 0<|x|\leq R/2.$$

Theorem 3.1 extends results from [29] (see Theorem IV), by covering a significantly different class of nonlinearities, in a similar way as in Section 2. Again, the results from [29] required global assumptions on f. Also the estimate of $|\nabla u|$ is new. Note that the results of this section include those of Section 2 as a special case. However, due to the special importance of the semilinear case, we have given separate statements (and proofs).

Remark 3.2. The restriction $p < p_S(m)$ in Theorem 3.1 is sharp, and so is the power $m/(p+1-m)$ in (3.3) for $p_*(m) < p < p_S(m)$, where $p_*(m) := n(m-1)/(n-m)$ and $n > m$ (see [29, Section 5]). For $p < p_*(m)$ and $n > m$ the sharp power is $(n-m)/(m-1)$ [1]. On the other hand, Theorem 3.1 could be extended to more general nonlinearities of the form $f(x, u, \nabla u)$ by the same method.

In the special case $f(u) = u^p$, we have the following more precise result.

Theorem 3.3. *Let $0 < m-1 < p < p_S(m)$ and let $\Omega \neq \mathbb{R}^n$ be a domain of $\mathbb{R}^n$. There exists $C = C(m, n, p) > 0$ (independent of Ω and u) such that any (nonnegative) solution u of (3.2) in Ω satisfies*

$$u + |\nabla u|^{\frac{m}{p+1}} \leq C \mathrm{dist}^{-\frac{m}{p+1-m}}(x, \partial\Omega), \quad x \in \Omega. \tag{3.4}$$

In particular if Ω is an exterior domain, i.e. $\Omega \supset \{x \in \mathbb{R}^n;\ |x| > R\}$ for some $R > 0$, then

$$u + |\nabla u|^{\frac{m}{p+1}} \leq C|x|^{-\frac{m}{p+1-m}}, \quad |x| \geq 2R.$$

4 Singularity estimates and strong Liouville theorems for elliptic systems

In this section, we consider semilinear elliptic systems. An important model-case is the Lane-Emden system, which has been intensively studied:

$$\left.\begin{aligned} -\Delta u &= v^p, \\ -\Delta v &= u^q, \end{aligned}\right\} \tag{4.1}$$

with $p, q > 0$ and $pq > 1$. It turns out that our approach can be applied to this system whenever the corresponding Liouville-type theorems (in the whole space and/or in the half-space) are known. Since some of such theorems require

$$p, q > 1, \tag{4.2}$$

throughout this section we will assume (4.2).

In what follows we shall denote by α, β the scaling exponents associated with (4.1):

$$\alpha = \frac{2(p+1)}{pq-1}, \quad \beta = \frac{2(q+1)}{pq-1}.$$

For (4.1), it is expected that the critical role (similar to that of the exponent p_S in the scalar equation (1.1)) should be played by the Sobolev hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}.$$

Indeed it is known that if (p,q) lies on or above the Sobolev hyperbola, i.e.

$$\frac{1}{p+1}+\frac{1}{q+1}\le\frac{n-2}{n},$$

then (4.1) admits some positive (radial, bounded) classical solutions on the whole of $\mathbb{R}^n$ [28]. The so-called *Lane-Emden conjecture* states that if

$$\frac{1}{p+1}+\frac{1}{q+1}>\frac{n-2}{n}, \tag{4.3}$$

or equivalently $\alpha+\beta>n-2$, then there are no nontrivial classical solutions of (4.1) on $\mathbb{R}^n$. (Here classical solution means that $0\le u,v\in C^2(\mathbb{R}^n)$, without any assumptions at infinity.) The interesting case is $n\ge 3$. For $n\le 2$, the conjecture is a consequence of a relatively easier, known result: the system of inequalities $-\Delta u\ge v^p$, $-\Delta v\ge u^q$ admits no nontrivial solutions whenever

$$\max(\alpha,\beta)\ge n-2, \tag{4.4}$$

see [21, 22].

The conjecture is known to be true for radial solutions in all dimensions [20, 21]. In the nonradial case, partial results are known. Nonexistence was proved for (p,q) in certain subregions of (4.3): for example, when both p and q are subcritical [13, 25] (i.e., $p,q\le p_S$, $(p,q)\ne(p_S,p_S)$), or when

$$\alpha,\beta\ge(n-2)/2,\qquad \alpha+\beta>n-2 \tag{4.5}$$

(this follows from [7] and the sufficient condition (4.4); notice that if both p and q are subcritical then (4.5) holds true). For $n=3$, it was proved in the full range (4.3), but under the additional assumption that u,v have at most polynomial growth at ∞ [27], and it remained an open problem whether or not this growth assumption was necessary (cf. also [12, p. 142]).

Here we obtain the following result. It in particular establishes the full Lane-Emden conjecture in dimension $n=3$, by removing the polynomial growth assumption in [27].

Theorem 4.1. *Let $n\ge 3$, $p,q>1$ be fixed and assume that* (4.1) *does not admit any bounded nontrivial (nonnegative) solution in $\mathbb{R}^n$. Then it does not admit any nontrivial (nonnegative) solution in $\mathbb{R}^n$, bounded or not. In particular, the conclusion holds if*

$$n=3,\quad \frac{1}{p+1}+\frac{1}{q+1}>\frac{n-2}{n}=\frac{1}{3}. \tag{4.6}$$

Turning to the case of a half-space $\mathbb{R}^n_+=\{x\in\mathbb{R}^n;\ x_1>0\}$, we have the following result.

Theorem 4.2. *Let $n \geq 1$, $p, q > 1$ be fixed and assume that* (4.1) *does not admit any bounded nontrivial (nonnegative) solution in $\mathbb{R}^n$. Then the problem*

$$\left.\begin{aligned} -\Delta u &= v^p, \quad x \in \mathbb{R}^n_+, \\ -\Delta v &= u^q, \quad x \in \mathbb{R}^n_+, \\ u = v &= 0, \quad x \in \partial\mathbb{R}^n_+, \\ u, v &\in C(\overline{\mathbb{R}^n_+}) \cap C^2(\mathbb{R}^n_+) \end{aligned}\right\} \tag{4.7}$$

does not admit any nontrivial (nonnegative) solution, bounded or not. In particular, the conclusion holds under assumption (4.6)*.*

Using an argument of [10] (for scalar equations), it was proved in [6] that nonexistence of bounded nontrivial solutions of (4.1) in $\mathbb{R}^{n-1}$ for given p, q implies nonexistence of *bounded* nontrivial solutions of (4.7) in $\mathbb{R}^n_+$. Note the difference between that result and ours (our assumption is stronger but so is our conclusion). Also, nonexistence of nontrivial (nonnegative) solution of (4.7) (bounded or not) was proved in [25] for $p, q < p_S$, $(p, q) \neq (p_S, p_S)$, via Kelvin transform. Theorem 4.2 partially improves on [25].

Our next result gives a priori estimate of possible singularities of local solutions to the Lane-Emden system (4.1), and also decay estimates in the case of exterior domains.

Theorem 4.3. *Let $p, q > 1$. Assume that* (4.1) *does not admit any bounded nontrivial (nonnegative) solution in $\mathbb{R}^n$. Let $\Omega \neq \mathbb{R}^n$ be a domain of $\mathbb{R}^n$. There exists $C = C(n, p, q) > 0$ (independent of Ω and (u, v)) such that any (nonnegative) solution (u, v) of* (4.1) *in Ω satisfies*

$$u(x) \leq C \mathrm{dist}^{-\alpha}(x, \partial\Omega), \quad x \in \Omega$$

and

$$v(x) \leq C \mathrm{dist}^{-\beta}(x, \partial\Omega), \quad x \in \Omega.$$

If Ω is an exterior domain, i.e. $\Omega \supset \{x \in \mathbb{R}^n;\ |x| > R\}$ for some $R > 0$, then it follows that

$$u(x) \leq C|x|^{-\alpha}, \quad |x| \geq 2R$$

and

$$v(x) \leq C|x|^{-\beta}, \quad |x| \geq 2R.$$

In particular, the above conclusions hold under assumption (4.6) *or, if $n \neq 3$, under assumption* (4.5) *or* (4.4)*.*

Theorem 4.3 seems to be the first result on estimates of singularities for system (4.1) in the full Sobolev subcritical range (4.3). These estimates are optimal if the pair (p, q) moreover satisfies $\max(\alpha, \beta) > n - 2$. Indeed, in this case, there exist explicit solutions of the form $u(x) = c_1|x|^{-\alpha}$, $v(x) = c_2|x|^{-\beta}$. In the (lower) complementary range $\max(\alpha, \beta) < n-2$, a classification of singularities has been obtained in [3]. For the different, Hamiltonian system $-\Delta u = u^q\, v^{p+1}$, $-\Delta v = v^p\, u^{q+1}$, singularity estimates have been given in the Sobolev subcritical range $p+q+1 < p_S$ [4].

Theorem 4.3 can be extended to more general nonlinearities $f(v), g(u)$, see Theorem 7.3 below. On the other hand, we can also treat elliptic systems of more than two equations, provided suitable Liouville-type theorems are available, see Theorem 7.5 below.

5 Main technical tool: a doubling lemma

Most of our results use Lemma 5.1 below. In the Euclidean case, it roughly states the following. Consider a real function M, defined on a domain Ω of $\mathbb{R}^n$ and locally bounded. Assume that M takes at some point y a value larger than the inverse of the distance of y to $\partial\Omega$. Then, for at least one point x where M is similarly large, M cannot double its size in the ball centered at x and of radius the inverse of $M(x)$.

Based on this doubling property, one can then start the rescaling procedure to prove local estimates of solutions of superlinear problems. The idea, by contradiction, is that if an estimate (in terms of the distance to $\partial\Omega$) fails, then the violating sequence of solutions u_k will be increasingly large along a sequence of points x_k, such that each x_k has a suitable neighborhood where the relative growth of u_k remains controlled. After appropriate rescaling, one can blow up the sequence of neighborhoods and pass to the limit to obtain a bounded solution of a limiting problem in the whole of $\mathbb{R}^n$.

Let us stress that, although the two approaches are closely related, there is a key difference with the "classical" case of global a priori estimates obtained by the rescaling method [17]. Indeed in that case, since u is smooth up to the boundary (where given boundary conditions are fulfilled), it is possible to rescale directly about a sequence of points of global maxima, the size of the solution being then automatically controlled around.

In fact, in view of certain applications, it is useful to state the Lemma in a more general and abstract framework. First, it may be useful to use the distance to a part of $\partial\Omega$ (instead of the whole $\partial\Omega$) – cf. the proof of Theorem 4.2 below. On the other hand, more general (non-Euclidean) metric spaces naturally arise when, for instance, one considers parabolic -instead of elliptic- equations (leading to the use of the parabolic distance – see [24]).

Lemma 5.1. *Let (X, d) be a complete metric space and let $\emptyset \neq D \subset \Sigma \subset X$, with Σ closed. Set $\Gamma = \Sigma \setminus D$. Finally let $M : D \to (0, \infty)$ be bounded on compact subsets of D and fix a real $k > 0$. If $y \in D$ is such that*

$$M(y)\,\mathrm{dist}(y, \Gamma) > 2k, \tag{5.1}$$

then there exists $x \in D$ such that

$$M(x)\,\mathrm{dist}(x, \Gamma) > 2k, \qquad M(x) \geq M(y), \tag{5.2}$$

and

$$M(z) \le 2M(x) \quad \textit{for all } z \in D \cap \overline{B}_X\big(x, k\,M^{-1}(x)\big).$$

Remarks 5.2. (a) If $\Gamma = \emptyset$ then $\mathrm{dist}(x,\Gamma) := +\infty$.

(b) Take $X = \mathbb{R}^n$, Ω an open subset of $\mathbb{R}^n$, and put $D = \Omega$, $\Sigma = \overline{D}$, hence $\Gamma = \partial\Omega$. Then we have $\overline{B}\big(x, k\,M^{-1}(x)\big) \subset D$. Indeed, since D is open, (5.2) implies that $\mathrm{dist}(x, D^c) = \mathrm{dist}(x,\Gamma) > 2k\,M^{-1}(x)$.

Proof of Lemma 5.1. Assume that the Lemma is not true. Then we claim that there exists a sequence (x_j) in D such that

$$M(x_j)\,\mathrm{dist}(x_j,\Gamma) > 2k, \tag{5.3}$$

$$M(x_{j+1}) > 2M(x_j), \tag{5.4}$$

and

$$d(x_j, x_{j+1}) \le kM^{-1}(x_j) \tag{5.5}$$

for all $j \in \mathbb{N}$. We choose $x_0 = y$. By our contradiction assumption, there exists $x_1 \in D$ such that

$$M(x_1) > 2M(x_0)$$

and

$$d(x_0, x_1) \le k\,M^{-1}(x_0).$$

Fix some $i \ge 1$ and assume that we have already constructed $x_0, \cdots, x_i$ such that (5.3)–(5.5) hold for $j = 0, \cdots, i-1$. We have

$$\mathrm{dist}(x_i,\Gamma) \ge \mathrm{dist}(x_{i-1},\Gamma) - d(x_{i-1}, x_i) > (2k-k)\,M^{-1}(x_{i-1}) > 2k\,M^{-1}(x_i),$$

hence

$$M(x_i)\,\mathrm{dist}(x_i,\Gamma) > 2k.$$

Since we also have $M(x_i) \ge M(y)$, our contradiction assumption implies that there exists $x_{i+1} \in D$ such that

$$M(x_{i+1}) > 2M(x_i)$$

and

$$d(x_i, x_{i+1}) \le k\,M^{-1}(x_i).$$

We have thus proved the claim by induction.

Now, we have

$$M(x_i) \ge 2^i M(x_0) \quad \text{and} \quad d(x_i, x_{i+1}) \le k\,2^{-i} M^{-1}(x_0), \qquad i \in \mathbb{N}. \tag{5.6}$$

In particular, (x_i) is a Cauchy sequence, hence it converges to some $a \in \overline{D} \subset \Sigma$. Moreover,

$$d(x_0, x_i) \le \sum_{j=0}^{i-1} d(x_j, x_{j+1}) \le k\,M^{-1}(x_0) \sum_{j=0}^{i-1} 2^{-j} \le 2k\,M^{-1}(x_0),$$

hence

$$\operatorname{dist}(x_i,\Gamma) \geq \operatorname{dist}(x_0,\Gamma) - 2k\,M^{-1}(x_0) =: \delta > 0.$$

Therefore, $K := \{x_i;\ i \in \mathbb{N}\} \cup \{a\}$ is a compact subset of $\Sigma \setminus \Gamma = D$. Since $M(x_i) \to \infty$ as $i \to \infty$ by (5.6), this contradicts the assumption that M is bounded on compact subsets of D. The Lemma is proved. □

6 Proofs and additional results for scalar equations

In order to illustrate the basic method, we start with the easier case of Theorem 2.3 and the model equation (1.1). Note that although Theorems 2.1 and 2.3 are special cases of Theorems 3.1 and 3.3, we give separate proofs of them for simplicity and for the convenience of those readers mainly interested in the semilinear case.

Proof of Theorem 2.3. Assume that estimate (2.5) fails. Then, there exist sequences Ω_k, u_k, $y_k \in \Omega_k$, such that u_k solves (1.1) on Ω_k and the functions

$$M_k := u_k^{\frac{p-1}{2}} + |\nabla u_k|^{\frac{p-1}{p+1}}, \quad k = 1, 2, \cdots \tag{6.1}$$

satisfy

$$M_k(y_k) > 2k \operatorname{dist}^{-1}(y_k, \partial\Omega_k). \tag{6.2}$$

By Lemma 5.1 and Remark 5.2 (b), it follows that there exists $x_k \in \Omega_k$ such that

$$M_k(x_k) \geq M_k(y_k), \quad M_k(x_k) > 2k \operatorname{dist}^{-1}(x_k, \partial\Omega_k) \tag{6.3}$$

and

$$M_k(z) \leq 2M_k(x_k), \qquad |z - x_k| \leq k\, M_k^{-1}(x_k). \tag{6.4}$$

Now we rescale u_k by setting

$$v_k(y) := \lambda_k^{2/(p-1)} u_k(x_k + \lambda_k y), \quad |y| \leq k, \qquad \text{with } \lambda_k = M_k^{-1}(x_k). \tag{6.5}$$

The function v_k solves

$$-\Delta_y v_k(y) = v_k^p(y), \quad |y| \leq k. \tag{6.6}$$

Moreover,

$$\big[v_k^{\frac{p-1}{2}} + |\nabla v_k|^{\frac{p-1}{p+1}}\big](0) = \lambda_k M_k(x_k) = 1 \tag{6.7}$$

and

$$\big[v_k^{\frac{p-1}{2}} + |\nabla v_k|^{\frac{p-1}{p+1}}\big](y) \leq 2, \quad |y| \leq k. \tag{6.8}$$

By using elliptic L^q estimates and standard imbeddings, we deduce that some subsequence of v_k converges in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a (classical) solution $v \geq 0$ of (1.1) in $\mathbb{R}^n$. Moreover, $[v^{\frac{p-1}{2}} + |\nabla v|^{\frac{p-1}{p+1}}](0) = 1$ by (6.7), so that v is nontrivial, and v, ∇v are bounded, due to (6.8). This contradicts the Liouville-type theorem from [16, 9] and proves Theorem 2.3. □

Proof of Theorem 2.1. Assume that estimate (2.3) fails. Keeping the same notation as in the proof of Theorem 2.3, we have sequences Ω_k, u_k, $y_k \in \Omega_k$, such that u_k solves (2.1) on Ω_k and

$$M_k(y_k) > 2k\big(1 + \mathrm{dist}^{-1}(y_k, \partial\Omega_k)\big) > 2k\,\mathrm{dist}^{-1}(y_k, \partial\Omega_k).$$

Then, formulae (6.1)–(6.8) are unchanged except that the function v_k now solves

$$-\Delta_y v_k(y) = f_k(v_k(y)) := \lambda_k^{\frac{2p}{p-1}} f(\lambda_k^{\frac{-2}{p-1}} v_k(y)), \quad |y| \le k \tag{6.9}$$

instead of (6.6), and that (since $M_k(x_k) \ge M_k(y_k) > 2k$) we also have

$$\lambda_k \to 0, \quad k \to \infty.$$

Since $-C \le f(s) \le C(1+s^p)$, $s \ge 0$, due to (2.2) (and f being continuous), it follows that

$$-C\lambda_k^{2p/(p-1)} \le f_k(v_k(y)) \le C', \quad |y| \le k, \quad k = 1, 2, \cdots .$$

By using elliptic L^q estimates, standard imbeddings, and (2.2), we deduce that some subsequence of v_k converges in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a function $0 \le v \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^n)$, $1 < q < \infty$, which satisfies $-\Delta v \ge 0$. Moreover $[v^{\frac{p-1}{2}} + |\nabla v|^{\frac{p-1}{p+1}}](0) = 1$ by (6.7). Therefore, v is nontrivial, hence $v(y) > 0$, $y \in \mathbb{R}^n$, by the strong maximum principle. Using assumption (2.2) again, we deduce that for each $y \in \mathbb{R}^n$, $f_k(v_k(y)) \to \ell\, v^p(y)$ as $k \to \infty$. Consequently, v is a solution of

$$-\Delta v = \ell\, v^p, \quad y \in \mathbb{R}^n$$

(and furthermore, v and ∇v are bounded due to (6.8)). This contradicts the Liouville-type theorem from [16, 9]. □

Theorem 2.1 can be extended to more general equations of the form

$$-\Delta u = f(x, u, \nabla u). \tag{6.10}$$

Let Ω be an arbitrary domain of $\mathbb{R}^n$ and set $q = 2p/(p+1)$. Assume that $f : \Omega \times [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a Caratheodory function and there exist $p_1 \in (0, p)$, $q_1 \in (0, q)$ and $C_1 > 0$ such that

$$-C_1(1+s^{p_1}+|\xi|^{q_1}) \le f(x, s, \xi) \le C_1(1+s^p+|\xi|^q), \quad x \in \Omega, \quad s \ge 0, \quad \xi \in \mathbb{R}^n. \tag{6.11}$$

Theorem 6.1. *Let $1 < p < p_S$, Ω be an arbitrary domain of $\mathbb{R}^n$ and let f satisfy* (6.11)*. We assume that for all $x \in \overline{\Omega}$,*

$$\lim_{s\to\infty,\ \Omega\ni z\to x} s^{-p} f(z, s, s^{(p+1)/2}\xi) = \ell(x) \in (0, \infty), \tag{6.12}$$

uniformly for ξ bounded. Moreover, if Ω is unbounded, then we assume that (6.12) *also holds for $x = \infty$. Then there exists $C = C(n, f) > 0$ (independent of Ω and u) such that any (nonnegative) solution u of* (6.10) *satisfies estimate* (2.3)*.*

A typical example of f satisfying the requirements of Theorem 6.1 is given by $f(x,u,\nabla u) = a(x)u^p - b(x)|\nabla u|^q$, where $1 < p < p_S$, $0 < q < 2p/(p+1)$, $b \in L^\infty(\Omega)$, and $0 < a \in C(\overline{\Omega})$, with $\lim_{\Omega \ni x \to \infty} a(x) =: \ell \in (0,\infty)$ if Ω is unbounded.

The condition $q < 2p/(p+1)$ in the above example is optimal (up to the equality case), as shown by the following counterexample. For any $q > 2p/(p+1)$, we may choose $\alpha > 2/(p-1)$ such that $(p-q)\alpha < q$. Direct computation then shows that $u(x) = |x|^{-\alpha}$ is a solution of

$$-\Delta u = u^p + b(x)|\nabla u|^q, \quad 0 < |x| < 1,$$

with

$$b(x) = (n-2-\alpha)\alpha^{1-q}|x|^{(\alpha+1)q-\alpha-2} - \alpha^{-q}|x|^{(\alpha+1)q-\alpha p} \in C(\overline{B}_1).$$

Proof of Theorem 6.1. The proof is completely similar to that of Theorem 2.1. The only difference is that v_k solves

$$-\Delta_y v_k(y) = f_k(v_k(y)) := \lambda_k^{\frac{2p}{p-1}} f(x_k + \lambda_k y, \lambda_k^{\frac{-2}{p-1}} v_k(y), \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k(y)), \quad |y| \le k$$

instead of (6.9). Since the assumption (6.11) on f implies that

$$-C\lambda_k^\varepsilon \le f_k(v_k(y)) \le C', \quad |y| \le k,$$

for some $\varepsilon > 0$ and all k large, we deduce similarly as before that some subsequence of v_k converges in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a function $v(y) > 0$. Fixing $y \in \mathbb{R}^n$ and denoting $\mu_k = \lambda_k^{\frac{-2}{p-1}} v_k(y)$, $\xi_k = v_k^{-\frac{p+1}{2}}(y)\, \nabla v_k(y)$, we may write

$$f_k(v_k(y)) = v_k^p(y)\, \mu_k^{-p}\, f(x_k + \lambda_k y, \mu_k, \mu_k^{\frac{p+1}{2}} \xi_k).$$

Note that, as $k \to \infty$, we have $\mu_k \to \infty$ and ξ_k bounded. If (x_k) is bounded, then we may assume that $x_k \to \overline{x} \in \overline{\Omega}$ by extracting a further subsequence, and assumption (6.12) implies that

$$f_k(v_k(y)) \to \ell(\overline{x})\, v^p(y), \quad k \to \infty. \tag{6.13}$$

Otherwise, if Ω is unbounded and $x_k \to \infty$ (along some subsequence), then the additional assumption on f implies that (6.13) still holds with $\overline{x} = \infty$. The rest of the proof is then unchanged. □

Turning to the quasilinear case, we now give the

Proof of Theorem 3.3. Put $\alpha = m/(p+1-m)$. Assume that estimate (3.4) fails. Then, there exist sequences Ω_k, u_k, $y_k \in \Omega_k$, such that u_k solves (3.2) on Ω_k and the functions

$$M_k := u_k^{1/\alpha} + |\nabla u_k|^{1/(\alpha+1)}, \quad k = 1, 2, \cdots \tag{6.14}$$

satisfy

$$M_k(y_k) > 2k \operatorname{dist}^{-1}(y_k, \partial\Omega_k). \tag{6.15}$$

By Lemma 5.1 and Remark 5.2 (b), it follows that there exists $x_k \in \Omega_k$ such that

$$M_k(x_k) > 2k\,\mathrm{dist}^{-1}(x_k, \partial\Omega_k) \tag{6.16}$$

and

$$M_k(z) \le 2M_k(x_k), \qquad |z - x_k| \le k\,M_k^{-1}(x_k). \tag{6.17}$$

Now we rescale u_k by setting

$$v_k(y) := \lambda_k^\alpha u_k(x_k + \lambda_k y), \quad |y| \le k, \qquad \text{with } \lambda_k = M_k^{-1}(x_k). \tag{6.18}$$

Since $(m-1)(\alpha+1)+1 = (m-1)(p+1)/(p+1-m)+1 = p\alpha$, we deduce that the function v_k solves

$$-\Delta_m v_k(y) = v_k^p(y), \quad |y| \le k. \tag{6.19}$$

Moreover,

$$\big[v_k^{1/\alpha} + |\nabla v_k|^{1/(\alpha+1)}\big](0) = 1 \tag{6.20}$$

and

$$\big[v_k^{1/\alpha} + |\nabla v_k|^{1/(\alpha+1)}\big](y) \le 2, \quad |y| \le k. \tag{6.21}$$

By using the estimate in [30], we deduce that there exists $\beta \in (0,1)$, such that v_k is bounded in $C^{1+\beta}_{\mathrm{loc}}(\mathbb{R}^n)$. We deduce that some subsequence of v_k converges in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a solution $v \ge 0$ of (3.2) in $\mathbb{R}^n$. Moreover $[v^{1/\alpha} + |\nabla v|^{1/(\alpha+1)}](0) = 1$ by (6.20), so that v is nontrivial (and furthermore v and ∇v are bounded, due to (6.21)). This contradicts the Liouville-type theorem [29, Theorem II (a) and (c)] and proves Theorem 3.3. □

Proof of Theorem 3.1. Assume that estimate (3.3) fails. Keeping the same notation as in the proof of Theorem 3.3, we have sequences Ω_k, u_k, $y_k \in \Omega_k$, such that u_k solves (3.1) on Ω_k and

$$M_k(y_k) > 2k\big(1 + \mathrm{dist}^{-1}(y_k, \partial\Omega_k)\big) > 2k\,\mathrm{dist}^{-1}(y_k, \partial\Omega_k).$$

Then, formulae (6.14)–(6.21) are unchanged except that the function v_k now solves

$$-\Delta_m v_k(y) = f_k(v_k(y)) := \lambda_k^{(\alpha+1)(m-1)+1} f(\lambda_k^{-\alpha} v_k(y)), \quad |y| \le k$$

instead of (6.19), and that (since $M_k(x_k) \ge M_k(y_k) > 2k$) we also have

$$\lambda_k \to 0, \quad k \to \infty.$$

Since $-C \le f(s) \le C(1+s^p)$, $s \ge 0$, due to (2.2) (and f continuous), it follows that

$$-C\lambda_k^{(\alpha+1)(m-1)+1} \le f_k(v_k(y)) \le C', \quad |y| \le k, \quad k = 1, 2, \cdots.$$

By using the estimate in [30], we deduce that there exists $\beta \in (0,1)$, such that v_k is bounded in $C^{1+\beta}_{\mathrm{loc}}(\mathbb{R}^n)$. We deduce that some subsequence of v_k converges in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a function $v \ge 0$ which satisfies $-\Delta_m v \ge 0$ in $\mathbb{R}^n$. Moreover $[v^{1/\alpha} + |\nabla v|^{1/(\alpha+1)}](0) = 1$ by (6.20). Therefore, v is nontrivial, hence $v(y) > 0$, $y \in \mathbb{R}^n$, by the strong maximum principle for the m–Laplacian (see e.g. [29, Lemma 2.1]). Using assumption (2.2) again, we deduce that for each $y \in \mathbb{R}^n$, $f_k(v_k(y)) \to \ell\, v^p(y)$ as $k \to \infty$. Consequently, v is a solution of

$$-\Delta_m v = \ell\, v^p, \quad y \in \mathbb{R}^n$$

(and furthermore, v and ∇v are bounded due to (6.21)). This contradicts the Liouville-type theorem [29, Theorem II (a) and (c)]. □

7 Proofs and additional results for systems

Proof of Theorem 4.3. Assume that the Theorem fails. Then, there exist sequences Ω_k, (u_k, v_k), $y_k \in \Omega_k$, such that (u_k, v_k) solves (4.1) on Ω_k and

$$M_k := u_k^{1/\alpha} + v_k^{1/\beta}, \quad k = 1, 2, \cdots \tag{7.1}$$

satisfies

$$M_k(y_k) > 2k\, \mathrm{dist}^{-1}(y_k, \partial\Omega_k).$$

By Lemma 5.1 and Remark 5.2 (b), it follows that there exists $x_k \in \Omega_k$ such that

$$M_k(x_k) > 2k\, \mathrm{dist}^{-1}(x_k, \partial\Omega_k)$$

and

$$M_k(z) \le 2M_k(x_k), \qquad |z - x_k| \le k\, M_k^{-1}(x_k).$$

Now we rescale (u_k, v_k) by setting

$$\left.\begin{aligned} \lambda_k &= M_k^{-1}(x_k) \\ \tilde{u}_k(y) := \lambda_k^\alpha u_k(x_k + \lambda_k y), &\quad \tilde{v}_k(y) := \lambda_k^\beta v_k(x_k + \lambda_k y), \qquad |y| \le k. \end{aligned}\right\} \tag{7.2}$$

Since $\alpha + 2 = \beta p$ and $\beta + 2 = \alpha q$, we see that $(\tilde{u}_k, \tilde{v}_k)$ still solves

$$\left.\begin{aligned} -\Delta_y \tilde{u}_k(y) &= \tilde{v}_k^p(y), \\ -\Delta_y \tilde{v}_k(y) &= \tilde{u}_k^q(y), \end{aligned}\right\}$$

for $|y| \le k$. Moreover,

$$\big[\tilde{u}_k^{1/\alpha} + \tilde{v}_k^{1/\beta}\big](0) = 1 \tag{7.3}$$

and

$$\big[\tilde{u}_k^{1/\alpha} + \tilde{v}_k^{1/\beta}\big](y) \le 2, \quad |y| \le k. \tag{7.4}$$

By using elliptic L^q estimates and standard imbeddings, we deduce that some subsequence of $(\tilde{u}_k, \tilde{v}_k)$ converges in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a (classical) solution $(\tilde{u}, \tilde{v})$ of (4.1) in $\mathbb{R}^n$. Moreover $[\tilde{u}^{1/\alpha} + \tilde{v}^{1/\beta}](0) = 1$ by (7.3), hence $(\tilde{u}, \tilde{v})$ is nontrivial, and moreover, u, v are bounded due to (7.4). This contradicts the assumption of Theorem 4.3. □

Theorem 4.1 is an easy consequence of Theorem 4.3.

Proof of Theorem 4.1. Assume that (u, v) is a solution of (4.1) on $\mathbb{R}^n$ (bounded or not). Then for each $x_0 \in \mathbb{R}^n$ and $R > 0$, by applying Theorem 4.3 in $\Omega = B(x_0, R)$, we obtain

$$u(x_0) \le C(n,p,q)R^{-\alpha}, \quad v(x_0) \le C(n,p,q)R^{-\beta}.$$

Upon letting $R \to \infty$, we obtain $u(x_0) = v(x_0) = 0$, hence $u \equiv v \equiv 0$. □

Let us now consider the case of half-spaces. Theorem 4.2 is a consequence of the following result, of possible independent interest.

Proposition 7.1. *Let $R > 0$ and denote $B_R^+ = B_R \cap \{x_1 > 0\}$ and $\Pi_R = B_R \cap \{x_1 = 0\}$. Under the assumption of Theorem 4.2, there exists $C = C(n,p,q) > 0$ such that any (nonnegative) solution of*

$$\left.\begin{aligned} -\Delta u &= v^p, \quad x \in B_R^+, \\ -\Delta v &= u^q, \quad x \in B_R^+, \\ u = v &= 0, \quad x \in \Pi_R \end{aligned}\right\} \tag{7.5}$$

with $u, v \in C(B_R^+ \cup \Pi_R)$, satisfies

$$u(x) \le C(R-|x|)^{-\alpha}, \quad v(x) \le C(R-|x|)^{-\beta}, \qquad x \in B_R^+.$$

Remark 7.2. It is likely that Proposition 7.1 could be generalized to more general domains, with boundary conditions prescribed on a part of the boundary, with an estimate involving the distance to the "free" part of the boundary (where no conditions are prescribed).

Proof of Proposition 7.1. By scaling, it is easily seen that it is sufficient to establish the result for $R = 1$. If the Proposition fails, then there exists a sequence of solutions (u_k, v_k) of (7.5) and points $y_k \in B_1^+$ such that M_k, defined in (7.1), satisfies

$$M_k(y_k) > 2k\,(1-|y_k|)^{-1}.$$

We apply Lemma 5.1 with $X = \mathbb{R}^n$, $\Sigma = \overline{B_1^+}$, $\Gamma = \partial B_1 \cap \{x_1 \ge 0\}$, $D = \Sigma \setminus \Gamma = B_1^+ \cup \Pi_1$. (Notice that $1 - |y| = \mathrm{dist}(y, \Gamma)$ for all $y \in D$). Then there exists $x_k \in D$ such that

$$M_k(x_k)\,\mathrm{dist}(x_k, \Gamma) > 2k \tag{7.6}$$

and

$$M_k(z) \le 2M_k(x_k), \qquad \text{for all } z \in D_k := D \cap \overline{B}\big(x_k, k\,M^{-1}(x_k)\big).$$

We note right away that

$$D_k = \{z \in \mathbb{R}^n;\ |z - x_k| \le k\,M^{-1}(x_k) \text{ and } z_1 \ge 0\}.$$

(Indeed, due to (7.6) and $|x_k| < 1$, $|z - x_k| \le k\,M^{-1}(x_k)$ implies that

$$|z| \le |x_k| + |z - x_k| \le |x_k| + (1 - |x_k|)/2 < 1.)$$

Now we rescale (u_k, v_k) according to (7.2). Then $(\tilde u_k, \tilde v_k)$ solves the system

$$\left.\begin{aligned} -\Delta_y \tilde u_k(y) &= \tilde v_k^p(y), \qquad |y|<k, \quad y_1 > -\lambda_k^{-1}x_{k,1}, \\ -\Delta_y \tilde v_k(y) &= \tilde u_k^q(y), \qquad |y|<k, \quad y_1 > -\lambda_k^{-1}x_{k,1}, \\ \tilde u_k(y) = \tilde v_k(y) &= 0, \qquad |y|<k, \quad y_1 = -\lambda_k^{-1}x_{k,1}, \end{aligned}\right\}$$

with

$$\big[\tilde u_k^{1/\alpha} + \tilde v_k^{1/\beta}\big](0) = 1 \qquad \text{and} \qquad \big[\tilde u_k^{1/\alpha} + \tilde v_k^{1/\beta}\big](y) \le 2, \quad |y| \le k, \quad y_1 > -\lambda_k^{-1}x_{k,1}.$$

According to whether $\lambda_k^{-1}x_{k,1} \to \infty$ or $\lambda_k^{-1}x_{k,1} \to c \ge 0$ (along a subsequence), it follows from interior (resp., interior-boundary) elliptic estimates that some subsequence of $(\tilde u_k, \tilde v_k)$ converges to a bounded, nontrivial (classical) solution $(\tilde u, \tilde v)$ of either (4.1) in $\mathbb{R}^n$, or (4.7) (with $\mathbb{R}^n_+$ replaced by $\{x;\ x_1 > -c\}$).

But it was proved in [6] (cf. also [10]) that the existence of a bounded, nontrivial solution of (4.7) in a half-space of $\mathbb{R}^n$ implies the existence of a bounded, nontrivial solution $(\hat u, \hat v)$ of (4.1) in $\mathbb{R}^{n-1}$ if $n \ge 2$. Then it follows that the same is true in $\mathbb{R}^n$ (just consider $\overline{u}(x_1, \cdots, x_n) := \hat u(x_1, \cdots, x_{n-1})$, $\overline{v}(x_1, \cdots, x_n) := \hat v(x_1, \cdots, x_{n-1})$). If $n = 1$ then the nonexistence of a nontrivial solution (u, v) of (4.7) on a half-line is trivial: such functions u, v would be concave and positive on R^+, hence monotone increasing in x, hence u'', v'' would converge to negative limits as $x \to \infty$, which is impossible.

Therefore, in either case, we obtain a contradiction with our assumption. □

Proof of Theorem 4.2. By Proposition 7.1, we have in particular

$$u(x) \le CR^{-\alpha}, \quad v(x) \le CR^{-\beta}, \qquad x \in B^+_{R/2}.$$

Fix $x_0 \in \mathbb{R}^n_+$. Then $x_0 \in B^+_{R/2}$ for all R sufficiently large and we deduce that $u(x_0) = v(x_0) = 0$, by letting $R \to \infty$. Therefore $u \equiv v \equiv 0$. □

Similarly as in Section 2, we can obtain singularity estimates for local solutions of the more general system:

$$\left.\begin{aligned} -\Delta u &= f(v), \\ -\Delta v &= g(u), \end{aligned}\right\} \tag{7.7}$$

where the functions $f, g : [0, \infty) \to \mathbb{R}$ are assumed to be continuous.

Theorem 7.3. *Let $p, q > 1$, let Ω be an arbitrary domain of $\mathbb{R}^n$, and assume that*

$$\lim_{v\to\infty} v^{-p} f(v) = \ell_1, \quad \lim_{u\to\infty} u^{-q} g(u) = \ell_2, \quad \ell_1, \ell_2 \in (0, \infty). \tag{7.8}$$

Assume that (4.1) *does not admit any bounded nontrivial (nonnegative) solution in* $\mathbb{R}^n$. *Then there exists $C = C(n, f, g) > 0$ (independent of Ω, u and v) such that any (nonnegative) solution (u, v) of* (7.7) *in Ω satisfies*

$$u(x) \le C\big(1 + \mathrm{dist}^{-\alpha}(x, \partial\Omega)\big), \quad x \in \Omega$$

and

$$v(x) \le C\big(1 + \mathrm{dist}^{-\beta}(x, \partial\Omega)\big), \quad x \in \Omega.$$

In particular, the above conclusions hold under assumption (4.6).

The proof follows by modifying the proof of Theorem 4.3 similarly as in the proof of Theorem 2.1. We omit the details.

Remark 7.4. One could also obtain gradient estimates for systems and treat problems of the form (7.7) with more general nonlinearities $f(x, u, v, \nabla u, \nabla v)$, $g(x, u, v, \nabla u, \nabla v)$, under suitable assumptions on f and g.

Finally, Theorem 4.3 can be extended to elliptic systems of more than two equations, provided suitable Liouville-type theorems are available. Let us consider for instance the following system:

$$-\Delta u_i = u_{i+1}^{p_i}, \quad i = 1, \cdots, r, \tag{7.9}$$

under the conditions

$$r \ge 3, \qquad p_i > 1, \; i = 1, \cdots, r. \tag{7.10}$$

(with the convention $u_{r+1} = u_1$). The optimal conditions for the validity of Liouville-type theorems for (7.9) are presently unknown. However, such a theorem has been proved in [25] under the assumption

$$\max_{1\le i\le r} p_i \le p_S, \qquad \min_{1\le i\le r} p_i < p_S. \tag{7.11}$$

In addition, a straightforward generalization of [21, Remark 2.2] shows that such a theorem is also true under the assumption

$$\max_{1\le i\le r} \alpha_i \ge n - 2, \tag{7.12}$$

where α_i is defined in (7.13). To state our result on singularity estimates, let us introduce some notation. Let $\alpha = (\alpha_1, \cdots, \alpha_r)$ be the solution of the linear system

$$p_i\, \alpha_{i+1} = \alpha_i + 2, \quad i = 1, \cdots, r \qquad (\text{with } \alpha_{r+1} = \alpha_1). \tag{7.13}$$

Note that α exists and is unique, since one easily computes that system (7.13) has determinant $(-1)^r(1-A) \ne 0$, where $A := \prod_{i=1}^r p_i$. Moreover, we have $\alpha_i > 0$. Indeed, an easy computation (substituting the i-th equation into the $(i+1)$-th for $i = 1$ to $r-1$) gives $\alpha_1 = 2(1 + p_1 + \cdots + p_1 \cdots p_{r-1})(A-1)^{-1} > 0$ and (7.13) then implies positivity of all α_i's.

Theorem 7.5. *Let* (7.10) *hold and assume that* (7.9) *does not admit any bounded nontrivial (nonnegative) solution in* $\mathbb{R}^n$. *Let* Ω *be a domain of* $\mathbb{R}^n$. *There exists* $C = C(n,p) > 0$ *(independent of* Ω *and* u*) such that any solution* $u = (u_1, \cdots, u_r)$ *of* (7.9) *in* Ω *satisfies*

$$u_i(x) \le C\, \mathrm{dist}^{-\alpha_i}(x, \partial\Omega), \quad x \in \Omega, \quad i = 1, \cdots, r,$$

where $\alpha_i > 0$ *are defined by* (7.13)*. It follows in particular that* (7.9) *does not admit any- nontrivial solution in* $\mathbb{R}^n$ *(bounded or not). If* Ω *is an exterior domain, i.e.* $\Omega \supset \{x \in \mathbb{R}^n;\ |x| > R\}$ *for some* $R > 0$*, then it follows that*

$$u_i(x) \le C|x|^{-\alpha_i}, \quad |x| \ge 2R, \quad i = 1, \cdots, r.$$

In particular, the above conclusions hold under assumption (7.11).

Proof of Theorem 7.5. The proof is similar to that of Theorem 4.3. The main changes are that, for the sequence of solutions $u_k = (u_{k,1}, \cdots, u_{k,r})$, we apply Lemma 5.1 and Remark 5.2 (b) to the functions

$$M_k := \sum_{i=1}^{r} u_{k,i}^{1/\alpha_i}$$

and that we rescale u_k by setting $\lambda_k = M_k^{-1}(x_k)$ and $\tilde{u}_{k,i}(y) := \lambda_k^{\alpha_i}\, u_{k,i}(x_k + \lambda_k y)$. The relations (7.13) guarantee that the rescaled functions satisfy the same system (in a rescaled domain). We omit further details. □

References

[1] M.-F. Bidaut-Véron, Local and global behaviour of solutions of quasilinear elliptic equations of Emden-Fowler type, *Arch. Rat. Mech. Anal.* **107** (1989), 293–324.

[2] M.-F. Bidaut-Véron, Initial blow-up for the solutions of a semilinear parabolic equation with source term. Equations aux dérivées partielles et applications, articles dédiés à Jacques-Louis Lions, Gauthier-Villars, Paris, 1998, pp. 189–198.

[3] M.-F. Bidaut-Véron, Local behaviour of the solutions of a class of nonlinear elliptic systems, *Advances Differ. Equations* **5** (2000), 147–192.

[4] M.-F. Bidaut-Véron and Th. Raoux, Asymptotics of solutions of some nonlinear elliptic systems, *Commun. Partial. Differ. Equations* **21** (1996), 1035-1086.

[5] M.-F. Bidaut-Véron and L. Véron, Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations, *Invent. Math.* **106** (1991), 489–539.

[6] I. Birindelli and E. Mitidieri, Liouville theorems for elliptic inequalities and applications, *Proc. Roy. Soc. Edinburgh* **128A** (1998), 1217–1247.

[7] J. Busca and R. Manasevich, A Liouville-type theorem for Lane-Emden system, *Indiana Univ. Math. J.* **51** (2002), 37–51.

[8] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Commun. Pure Appl. Math.* **42** (1989), 271–297.

[9] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.* **63** (1991), 615–622.

[10] N. Dancer, Some notes on the method of moving planes, *Bull. Austral. Math. Soc.* **46** (1992), 425–434.

[11] N. Dancer, Superlinear problems on domains with holes of asymptotic shape and exterior problems, *Math. Z.* **229** (1998), 475–491.

[12] D. G. de Figueiredo, Semilinear elliptic systems, *in:* Nonlinear Functional Analysis and Applications to Differential Equations (Trieste 1997), 122–152, World Sci. Publishing, River Edge, N.J., 1998.

[13] D. G. de Figueiredo and P. Felmer, A Liouville-type theorem for elliptic systems, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 21 (1994), 387–397.

[14] D. G. de Figueiredo and B. Sirakov, Liouville theorems, monotonicity results and a priori bounds for positive solutions of semilinear elliptic systems, *Math. Ann.*, to appear.

[15] D. G. de Figueiredo and J. Yang, A priori bounds for positive solutions of a non-variational elliptic system, *Commun. Partial. Differ. Equations.* **26** (2001), 2305–2321.

[16] B. Gidas and J. Spruck, Global and local behavior of positive solutions of non-linear elliptic equations, *Commun. Pure Appl. Math.* **34** (1981), 525–598

[17] B. Gidas and J. Spruck, A priori bounds for positive solutions of a nonlinear elliptic equations, *Commun. Partial. Differ. Equations.* **6** (1981), 883–901.

[18] B. Hu, Remarks on the blowup estimate for solutions of the heat equation with a nonlinear boundary condition, *Differ. Integral Equations* **9** (1996), 891–901.

[19] P.-L. Lions, Isolated singularities in semilinear problems, *J. Differ. Equations* **38** (1980), 441–550.

[20] E. Mitidieri, A Rellich type identity and applications, *Comm. Partial Differential Equations* **18** (1993), 125–151.

[21] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$, *Differ. Integral Equations* **9** (1996), 465–479.

[22] E. Mitidieri and S. Pohozhaev, A priori estimates and blow-up of solutions to nonlinear partial differential equations, *Proc. Steklov Inst. Math.* **234** (2001).

[23] P. Poláčik and P. Quittner, A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation, *Nonlinear Anal.*, to appear.

[24] P. Poláčik, P. Quittner and Ph. Souplet, Singularity and decay estimates in superlinear prooblems via Liouville-type theorems. Part II: parabolic equations, preprint (2005).

[25] W. Reichel and H. Zou, Non-existence results for semilinear cooperative elliptic systems via moving spheres, *J. Differ. Equations* **161** (2000), 219–243.

[26] J. Serrin, Local behavior of solutions of quasilinear equations, *Acta Math.* **111** (1964), 247–302.

[27] J. Serrin and H. Zou, Non-existence of positive solutions of Lane-Emden systems, *Differ. Integral Equations* **9** (1996), 635–653.

[28] J. Serrin and H. Zou, Existence of positive solutions of the Lane-Emden system, *Atti Sem. Mat. Fis. Univ. Modena* **46** (1998), suppl., 369–380.

[29] J. Serrin and H. Zou, Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, *Acta Math.* **189** (2002), 79–142.

[30] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differ. Equations* **51** (1984), 126–150.

[31] H. Zou, A priori estimates for a semilinear elliptic systems without variational structure and their applications, *Math. Ann.* **323** (2002), 713–735.

[32] H. Zou, A priori estimates and existence for strongly coupled semilinear cooperative elliptic systems, preprint.