Existence of quasiperiodic solutions of elliptic equations on the entire space with a quadratic nonlinearity

Peter Poláčik*
School of Mathematics
University of Minnesota
Minneapolis, MN 55455, United States

and

Darío A. Valdebenito†
Department of Mathematics and Statistics
McMaster University
Hamilton, ON L8S 4K1, Canada

Abstract

We consider the equation

$$\Delta u + u_{yy} + f(x, u) = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R} \tag{1}$$

where f is sufficiently regular, radially symmetric in x, and $f(\cdot, 0) \equiv 0$. We give sufficient conditions for the existence of solutions of (1) which are quasiperiodic in y and decaying as $|x| \to \infty$ uniformly in y. Such solutions are found using a center manifold reduction and results from the KAM theory. A required nondegeneracy condition is stated in terms of $f_u(x, 0)$ and $f_{uu}(x, 0)$, and is independent of higher-order terms in the Taylor expansion of $f(x, \cdot)$. In particular, our results apply to some quadratic nonlinearities.

Key words: Elliptic equations on the entire space, quasiperiodic solutions, center manifold, Birkhoff normal form, KAM theorem

AMS Classification: 35B08, 35B15, 35J61, 37J40, 37K55

Contents

1 Introduction **2**

2 Main results **3**

3 The reduced Hamiltonian **7**

*Supported in part by the NSF Grant DMS-1565388

†Supported in part by CONICYT-Chile Becas Chile, Convocatoria 2010

4 Application of a KAM-type theorem using Arnold's condition 12

5 Verification of Arnold's condition 18
5.1 Birkhoff normal form algorithm 19
5.2 The asymptotic behavior of $\mathcal{M}(s)$ 22

1 Introduction

In this paper, we continue our study, initiated in [11], of semilinear elliptic equations on the entire space:

$$\Delta u + u_{yy} + f_1(x, u) = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.1}$$

Here N is a positive integer, Δ is the Laplacian in x, and $f_1 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function satisfying $f_1(\cdot, 0) \equiv 0$. Our main goal in this research has been to give conditions on f_1 which guarantee that (1.1) possesses solutions which decay to 0 as $|x| \to \infty$, uniformly in y, and are quasiperiodic (and not periodic) in y. Our techniques are based on a spatial-dynamics approach to elliptic equations as proposed by Kirchgässner [9] and further developed by Mielke (see, for example, [10]) and many other authors, and results from the Kolmogorov-Arnold-Moser (KAM) theory. Previously, related ideas for finding quasiperiodic solutions of elliptic equations on an unbounded strip have been used by Scheurle [14] and Valls [17] (see [11] for a more detailed discussion and further related references).

The main contribution of our previous work [11] was twofold. First, we built a general framework for studying solutions of (1.1) using tools from dynamical systems, such as the center manifold reduction, Birkhoff normal form, and the KAM theory. Second, we showed how these techniques yield quasiperiodic solutions in some specific classes of equations. Considering (1.1) in the more specific form

$$\Delta u + u_{yy} + a_1(x)u + b\left(sa_2(x)u^2 + a_3(x)u^3\right) + u^4 f_2(x, u; s, b) = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.2}$$

with sufficiently regular functions f_2, a_1, a_2, a_3, and parameters $b \neq 0$ and $s \in \mathbb{R}$, we proved the existence of y-quasiperiodic solutions under various conditions involving eigenvalues and eigenfunctions of the Schrödinger operator $-\Delta - a_1(x)$ and the function a_3. We assumed that either $b \in \mathbb{R} \setminus \{0\}$ is fixed and $|s| \geq 0$ is sufficiently small (thus, the quadratic term is small relative to the cubic term), or that $s \in \mathbb{R}$ is fixed and $|b| > 0$ is sufficiently small (the quadratic and cubic terms become small at the same rate, as $b \to 0$). In both cases, it is crucial that $b \neq 0$, and $a_3 \neq 0$ satisfies a certain robust condition. While the quadratic term in (1.2) is always small in some sense, because of the smallness requirement on one of the parameters, no conditions, other than some smoothness, are imposed on a_2 itself. The coefficient a_2 may well vanish, which would actually make the whole theory a lot simpler. In [11], we made a point of including the quadratic term, although it complicates matters at several levels (for another perspective on this issue and a KAM-result for the Boussinesq equation with a quadratic nonlinearity see [15]). As we argued in [11], when considering applications of our results in specific classes of elliptic problems, say, when the functions a_1, a_2, a_3 come as the coefficients of the Taylor expansion of a nonlinearity along a y-independent solution, keeping any restrictive assumptions on these functions to a minimum is of paramount importance.

With this in mind, we now consider a case which is in some sense complementary to the cases considered in [11]: the quadratic term is the dominant one. In our present theorem, the existence of quasiperiodic solutions is proved not despite $a_2 \not\equiv 0$, but thanks to a_2 being present and satisfying a certain robust condition. This time, the cubic term is of no importance and it can well be identical to 0. We remark, however, that unlike in [11], where quasiperiodic solutions with any number of frequencies have been proved to exist, here we are able to handle only two (rationally independent) frequencies.

The remainder of the paper is organized as follows. Section 2 contains the formulation of our main result, an informal outline of the proof, and related remarks. As briefly explained there, while we mainly focus on equations and solutions which are radially symmetric in x, other settings are admissible as well. In Section 3, we apply a center manifold reduction to an abstract form of (1.1) and recall some results from [11] concerning the Hamiltonian structure of the reduced equation. Section 4 is devoted to an application of a KAM-type theorem, yielding quasiperiodic solutions with any number of frequencies under an additional assumption. In Section 5, we then verify the additional assumption in the case of two frequencies.

2 Main results

In this section, we introduce some terminology and provide the statement of our main result. Afterwards, we give an outline of the proof.

Throughout the paper, $\mathscr{C}_{\mathrm{b}}(\mathbb{R}^N)$ stands for the space of all continuous bounded (real-valued) functions on $\mathbb{R}^N$ and $\mathscr{C}^k_{\mathrm{b}}(\mathbb{R}^N)$ for the space of functions on $\mathbb{R}^N$ with continuous bounded derivatives up to order k, $k \in \mathbb{N} := \{0, 1, 2, \dots\}$. The spaces $\mathscr{C}_{\mathrm{rad}}(\mathbb{R}^N)$ and $\mathscr{C}^k_{\mathrm{rad}}(\mathbb{R}^N)$ are the subspaces of $\mathscr{C}_{\mathrm{b}}(\mathbb{R}^N)$ and $\mathscr{C}^k_{\mathrm{b}}(\mathbb{R}^N)$, respectively, consisting of all radially symmetric functions (that is, functions depending on $x \in \mathbb{R}^N$ only via $r = |x|$). When needed, we assume that these spaces are equipped with the usual norms. For $k \in \mathbb{N}$, the spaces $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ and $H^k_{\mathrm{rad}}(\mathbb{R}^N)$ are the subspaces of $L^2(\mathbb{R}^N)$ and $H^k(\mathbb{R}^N)$, respectively, consisting of radially symmetric functions. Throughout this paper we will abuse the notation slightly by viewing radially symmetric functions as functions of $x \in \mathbb{R}^N$ or functions of $r \geq 0$, depending on the context. This should cause no confusion.

Given integers $n \geq 2$, $k \geq 1$, a vector $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ is said to be *nonresonant up to order* k if

$$\omega \cdot \alpha \neq 0 \text{ for all } \alpha \in \mathbb{Z}^n \setminus \{0\} \text{ such that } \ |\alpha| \leq k. \tag{2.1}$$

(Here $|\alpha| = |\alpha_1| + \dots + |\alpha_n|$, and $\omega \cdot \alpha$ is the usual dot product.) If (2.1) holds for all $k = 1, 2, \dots$, we say that ω is *nonresonant*, or, equivalently, that the numbers $\omega_1, \dots, \omega_n$ are *rationally independent*. A special class of nonresonant vectors which will play a role later on is the class of *Diophantine* vectors, see Section 4.

A function $u : (x, y) \mapsto u(x, y) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is said to be *quasiperiodic* in y if there exist an integer $n \geq 2$, a nonresonant vector $\omega^* = (\omega^*_1, \dots, \omega^*_n) \in \mathbb{R}^n$, and an injective function U defined on $\mathbb{T}^n$ (the n-dimensional torus) with values in the space of real-valued functions on $\mathbb{R}^N$ such that

$$u(x, y) = U(\omega^*_1 y, \dots, \omega^*_n y)(x) \quad (x \in \mathbb{R}^N,\ y \in \mathbb{R}). \tag{2.2}$$

The vector ω^* is called a *frequency vector* of u.

We emphasize that the nonresonance of the frequency vector is a part of our definition. In particular, a quasiperiodic function is not periodic and, if it has some regularity properties, its image is dense in an n-dimensional manifold diffeomorphic to $\mathbb{T}^n$.

Consider now an elliptic equation

$$\Delta u + u_{yy} + a_1(x;s)u + f(x,u;s) = 0 \quad (x \in \mathbb{R}^N,\ y \in \mathbb{R}), \tag{2.3}$$

where $s \approx 0$ is a parameter and f is a nonlinearity of the form

$$f(x,u;s) = a_2(x;s)u^2 + u^3 g(x,u;s). \tag{2.4}$$

We assume that for some $\delta > 0$ the functions a_1, a_2, and g satisfy the following hypotheses, where K, m are integers with

$$K \geq 18, \quad m > \frac{N}{2}. \tag{2.5}$$

(S1) $a_1(\cdot;s) \in \mathscr{C}^{m+1}_{\mathrm{rad}}(\mathbb{R}^N)$ for each $s \in (-\delta,\delta)$, and the map $s \in (-\delta,\delta) \mapsto a_1(\cdot;s) \in \mathscr{C}^{m+1}_{\mathrm{rad}}(\mathbb{R}^N)$ is of class $\mathscr{C}^{K+1}$.

(S2) $a_2(\cdot;s) \in \mathscr{C}^{m+1}_{\mathrm{rad}}(\mathbb{R}^N)$ for each $s \in (-\delta,\delta)$, the map $s \in (-\delta,\delta) \mapsto a_2(\cdot;s) \in \mathscr{C}^{m+1}_{\mathrm{rad}}(\mathbb{R}^N)$ is of class $\mathscr{C}^{K+1}$; $g \in \mathscr{C}^{K+m+4}(\mathbb{R}^N \times \mathbb{R} \times (-\delta,\delta))$, and for all $\vartheta > 0$ the function g is bounded on $\mathbb{R}^N \times [-\vartheta,\vartheta] \times [0,\delta)$ together with all its partial derivatives up to order $K+m+4$. Also, $g = g(x,u;s)$ is radially symmetric in $x \in \mathbb{R}^N$.

The next hypotheses concern the Schrödinger operator $A_1(s) := -\Delta - a_1(r;s)$ acting on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ with domain $H^2_{\mathrm{rad}}(\mathbb{R}^N)$.

(A1)(a) There exists $L < 0$ such that $\limsup_{r\to\infty} a_1(r;s) \leq L$ for all $s \in (-\delta,\delta)$.

(A1)(b) For all $s \in [0,\delta)$, $A_1(s)$ has exactly 2 nonpositive eigenvalues $\mu_1(s) < \mu_2(s)$, and one has $\mu_2(s) < 0$ for all $s \in (0,\delta)$ and $\mu_2(0) = 0$.

(NR) Denoting $\omega_j(s) := \sqrt{|\mu_j(s)|}$, $j = 1,2$, the vector $\omega(s) = (\omega_1(s),\omega_2(s))$ is nonresonant up to order K for all $s \in (0,\delta)$.

Sometimes we will collectively refer to assumptions (A1)(a) and (A1)(b) as (A1). Hypotheses (A1)(b) and (NR) are assumed in our main theorem, but in some of our results we consider more general versions of (A1)(b) and (NR), namely:

(A1)(b') There is an integer $n \geq 2$ such that for all $s \in (0,\delta)$, $A_1(s)$ has exactly n nonpositive eigenvalues $\mu_1(s) < \mu_2(s) < \cdots < \mu_n(s)$, and one has $\mu_n(s) < 0$ for all $s \in (0,\delta)$. ($\mu_n(0) = 0$ is not required here.)

(NR') Denoting $\omega_j(s) := \sqrt{|\mu_j(s)|}$, $j = 1,\ldots,n$, the vector $\omega(s) = (\omega_1(s),\ldots,\omega_n(s))$ is nonresonant up to order K for all $s \in (0,\delta)$, where K satisfies

$$K \geq 6(n+1). \tag{2.6}$$

When hypotheses (A1)(b') and (NR') are assumed in lieu of (A1)(b) and (NR), the constant K in (S1), (S2) is also assumed to satisfy (2.6).

For $s \in [0, \delta)$ and $j = 1, \dots, n$ (with $n = 2$ in (A1)(b)), we denote by $\varphi_j(\cdot; s)$ the eigenfunction of $A_1(s)$ associated with $\mu_j(s)$, normalized in the L^2-norm and satisfying $\varphi_j(0; s) > 0$ ($\varphi_j(\cdot; s)$ is thus determined uniquely, cp. Remark 2.1(iii) below).

Our last hypothesis concerns the coefficient a_2 and the eigenfunction φ_2 when $s = 0$:

(A2) One has
$$\int_{\mathbb{R}^N} a_2(x; 0)\varphi_2^3(x; 0)dx \neq 0.$$

Hypotheses (S1), (S2), (A1)(a), (A1)(b'), and (NR') with $m > N/2$ and $K \geq 6(n+1)$ are assumed throughout the paper. In our main theorem and its proof (Section 5), we take $n = 2$ and assume also that (A1)(b) and (A2) hold.

Remark 2.1. (i) We shall mainly deal with nonnegative values of the parameter s (as in (A1)(b'), (NR'), (A1)(b), (NR), and (A2)), but it will be convenient to extend the parameter range to $(-\delta, \delta)$ (as in (S1), (S2), (A1)(a)).

(ii) Hypothesis (A1)(a) guarantees that for all s the essential spectrum $\sigma_{ess}(A_1(s))$ is contained in $[-L, \infty)$ [13]. The eigenfunctions corresponding to the eigenvalues in $(-\infty, -L)$ have exponential decay as $r = |x| \to \infty$ [1, 13]; in particular, the integral in (A2) exists. Since the eigenvalues in $(-\infty, -L)$ are isolated in $\sigma(A_1(s))$, hypotheses (A1)(a) and (A1)(b') imply that there is $\gamma > 0$ such that $(0, \gamma) \cap \sigma(A_1(s)) = \emptyset$ for all $s \in [0, \delta)$.

(iii) Since the operator $A_1(s)$ is considered on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$, its eigenfunctions φ can be viewed as solutions of a second-order ordinary differential equation (in the radial variable r) satisfying $\varphi_r(0) = 0$, $\varphi(0) \neq 0$; in particular, the eigenvalues of $A_1(s)$ in $(-\infty, -L)$ are automatically simple [13]. Moreover, (S1) implies that the eigenvalues $\mu_1(s)$, $\mu_2(s)$ in (A1)(b) (or $\mu_1(s), \dots, \mu_n(s)$ in (A1)(b')) are functions of s of class $\mathscr{C}^{K+1}$ (see [8]).

(iv) As long as the nonlinearity f in (2.3) is sufficiently smooth, (2.4) is simply a Taylor expansion of f around $u = 0$. Note that, other than the regularity assumptions in (S2), the only hypothesis concerning the nonlinear part of equation (2.3) is (A2). In particular, the case that the function g vanishes identically is allowed. This is very different from [11], where the presence of a nonzero cubic term in the nonlinearity f was essential.

(v) The importance of the parameter s lies mainly in hypothesis (A1)(b). The fact that $\mu_2(s) \to 0-$ (so $\omega_2(s) \to 0+$), together with (A2), will be crucial in the verification of a nondegeneracy condition (see the outline at the end of this section for more details). The dependence of the nonlinearity on the parameter is of little relevance in the proof of our results, but we include it for the sake of generality.

(vi) Since (NR) is a *finite-order* nonresonance condition, it is easy to find examples of families $a_1(\cdot; s)$, $s \approx 0$, satisfying conditions (S1), (A1), and (NR) (for example, one can use arguments from [11, Section 2.3]). For any such family, condition (A2) is obviously satisfied for "most" functions $a_2(\cdot; 0)$.

(vii) Our hypotheses (S1), (S2), (A1)(a), (A2)(b'), (NR') are analogous to some hypotheses in our previous paper [11]. This will allow us to use certain technical results from [11].

We can now state our main theorem.

Theorem 2.2. *Suppose that the hypotheses* (S1), (S2), (A1), (NR) *(with K, m as in* (2.5)*) and* (A2) *are satisfied. Then the following statements are valid, possibly after making $\delta > 0$ smaller, for each $s \in (0, \delta)$. There exists a solution $u(x, y)$ of equation* (2.3) *such that $u(x, y)$ is radially symmetric in x, $u(x, y) \to 0$ as $|x| \to \infty$, uniformly in y, and $u(x, y)$ is quasiperiodic in y. In fact, there is an uncountable family of such quasiperiodic solutions (disregarding translations), their frequency vectors forming an uncountable subset of $\mathbb{R}^2$.*

Remark 2.3. (i) Theorem 2.2 gives sufficient conditions in terms of the functions a_1, a_2, and g for the existence of quasiperiodic solutions of (2.3) with 2 frequencies. As already mentioned in Remark 2.1(vi), these conditions are easily verified for a large class of radial functions a_1, a_2 (the function g just needs to be radial and sufficiently smooth, cp. (S2)). For technical reasons (the verification of a certain nondegeneracy relation), in this theorem we need the parameter $s > 0$ to be sufficiently small and the number of frequencies to be restricted to $n = 2$. Below, we do include a theorem—see Theorem 4.4—where, assuming (S1), (S2), (A1)(a), (A1)(b'), and (NR'), we give a different sufficient condition for the existence of quasiperiodic solutions of (2.3) with any given number of frequencies and for a fixed value of s. However, that condition is rather implicit, and, we are unable to formulate it as a specific condition on a_1, a_2, unless $n = 2$ and s is allowed to vary.

(ii) As in [11], while we are primarily interested in radially symmetric solutions, our techniques are not limited to radially symmetric problems. One can lift the requirement of the radial symmetry on the functions a_1, a_2, g, and assume instead that the eigenvalues in (A1)(b) (or (A1)(b')) are simple (of course, $-\Delta - a_1(x)$ has to be considered as an operator on the full space $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$). Theorem 4.4 (minus the radial symmetry in x) then remains valid. See [11, Remark 2.1] for more on how the hypotheses can be adjusted to other settings.

In the proof of Theorems 2.2, 4.4, we first follow a general scheme from [11]. Applying a center manifold theorem, we obtain a system of ordinary differential equations (the "reduced equation") whose solutions are in one-to-one correspondence with a class of solutions of (2.3). The reduced equation is a Hamiltonian system with respect to a suitable symplectic form. One can choose local coordinates, using the Darboux theorem, so that the system is Hamiltonian with respect to the standard symplectic structure. The resulting Hamiltonian is then brought to its Birkhoff normal form up to a sufficiently high order, so that, when restricted to a neighborhood of the origin, it can be written as the sum of an integrable Hamiltonian and a small perturbation. If the integrable part satisfies a certain nondegeneracy condition, then KAM-type theorems can be used to prove the existence of quasiperiodic solutions for the reduced system, which in turn translate to quasiperiodic solutions of the original equation.

The most significant differences between [11] and the present paper lie in the verification of the nondegeneracy conditions. In [11], we verified the nondegeneracy (Kolmogorov's type) when a specific condition involving the cubic term in the nonlinearity f is satisfied and the quadratic term is small or suitably controlled. Here, the nondegeneracy (Arnold's type) comes from the condition on the quadratic term in f and the structure of the Birkhoff normal form. The assumption $\mu_2(s) \to 0$ as $s \to 0$ means that the nonresonance property of $(\mu_1(s), \mu_2(s))$ is lost at $s = 0$. This introduces certain singularities, with respect to s, in the fourth-order coefficients of the normal form, which can be effectively used to verify Arnold's condition when $n = 2$ and $s > 0$ is sufficiently small.

3 The reduced Hamiltonian

To a large extent, this section is a summary of results from Sections 3 and 4 of [11], with minor changes to account for the setting of the present article. We introduce the Hamiltonian of the equation obtained from a center manifold reduction of (2.3) and transform it to a form suitable for an application of a KAM-type theorem. Throughout the section we assume that hypotheses (A1)(a), (A1)(b'), (S1), (S2), and (NR') hold with $m > N/2$ and $K \geq 6(n+1)$.

We begin with the center manifold reduction. For that we first write equation (2.3) in an abstract form, using the spaces $X = H^{m+1}_{\text{rad}}(\mathbb{R}^N) \times H^m_{\text{rad}}(\mathbb{R}^N)$, and $Z = H^{m+2}_{\text{rad}}(\mathbb{R}^N) \times H^{m+1}_{\text{rad}}(\mathbb{R}^N)$. Let f be as in (2.4). Its Nemytskii operator $\tilde{f} : H^{m+2}_{\text{rad}}(\mathbb{R}^N) \times (-\delta, \delta) \to H^{m+1}_{\text{rad}}(\mathbb{R}^N)$ is given by

$$\tilde{f}(u; s)(r) = f(r, u(r); s).$$

This is a well defined map and of class $\mathscr{C}^{K+1}$ (see [11, Theorem A.1(b)]). The abstract form of (2.3) is given by

$$\begin{aligned} \frac{du_1}{dy} &= u_2, \\ \frac{du_2}{dy} &= A_1(s)u_1 - \tilde{f}(u_1; s). \end{aligned} \tag{3.1}$$

We rewrite this further as

$$\frac{du}{dy} = A(s)u + R(u; s), \tag{3.2}$$

where $u = (u_1, u_2)$,

$$\begin{aligned} A(s)(u_1, u_2) &= (u_2, A_1(s)u_1)^T, \\ R(u_1, u_2; s) &= (0, \tilde{f}(u_1; s))^T. \end{aligned} \tag{3.3}$$

Here, for each $s \in (-\delta, \delta)$, $A(s)$ is considered as an operator on X with domain $D(A(s)) = Z$, and R as a $\mathscr{C}^{K+1}$-map from $Z \times (-\delta, \delta)$ to Z. The concept of a solution of (3.2) on an interval $\mathcal{I}$ is as in [6, 18]: it is a function in $\mathscr{C}^1(\mathcal{I}, X) \cap \mathscr{C}(\mathcal{I}, Z)$ satisfying (3.2).

For $s \in [0, \delta)$ and $n \geq 2$ integer, let $\varphi_j(\cdot; s)$, $j = 1, \dots, n$, be the eigenfunctions of $A_1(s) := -\Delta - a_1(r; s)$ as introduced in Section 2. By elliptic regularity, (S1) implies that $\varphi_j(\cdot; s) \in H^{m+2}_{\text{rad}}(\mathbb{R}^N)$, for $j = 1, \dots, n$ and $s \in [0, \delta)$. Moreover, by [8], as $H^{m+2}(\mathbb{R}^N)$-valued functions of s, the $\varphi_j(\cdot; s)$ are of class $\mathscr{C}^{K+1}$ (cp. Remark 2.1(iii)). Define the space

$$X_c(s) := \{(h, \tilde{h})^T : h, \tilde{h} \in \text{span}\{\varphi_1(\cdot; s), \dots, \varphi_n(\cdot; s)\}\} \subset Z,$$

the orthogonal projection operator $\Pi(s) : L^2_{\mathrm{rad}} \to \operatorname{span}\{\varphi_1(\cdot;s), \ldots, \varphi_n(\cdot;s)\}$, and let $P_c(s) : X \to X_c(s)$ be given by $P_c(s)(v_1, v_2) = (\Pi(s)v_1, \Pi(s)v_2)$. As shown in [11, Section 3.2], $P_c(s)$ is the spectral projection for the operator $A(s)$ associated with the spectral set $\{\pm i\omega_j(s) : j = 1, \ldots, n\}$ (with $\omega_j(s)$ as in (NR'))—the spectrum of $A(s)$ is the union of this set and a set which is at a positive distance from the imaginary axis. Due to (S1), $P_c(s) \in \mathscr{L}(X)$ is of class $\mathscr{C}^{K+1}$ in $s \in [0, \delta)$; in fact, the smoothness of the maps $s \mapsto \varphi_j(\cdot;s)$ implies that $s \mapsto P_c(s)$ is of class $\mathscr{C}^{K+1}$ as an $\mathscr{L}(X, Z)$-valued map on $[0, \delta)$.

Also define $P_h(s) = I_X - P_c(s)$, I_X being the identity map on X, and, for $j = 1, \ldots, n$,

$$\psi_j(\cdot;s) = (\varphi_j(\cdot;s), 0)^T, \quad \zeta_j(\cdot;s) = (0, \varphi_j(\cdot;s))^T. \tag{3.4}$$

A basis of $X_c(s)$ is given by

$$\mathscr{B}(s) := \{\psi_1(\cdot;s), \ldots, \psi_n(\cdot;s), \zeta_1(\cdot;s), \ldots, \zeta_n(\cdot;s)\}.$$

For $z \in X_c(s)$, we denote by $\{z\}_{\mathscr{B}}$ the coordinates of z with respect to the basis $\mathscr{B}(s)$. Denote further

$$\begin{aligned} \psi(s) &:= (\psi_1(\cdot;s), \ldots, \psi_n(\cdot;s)), \\ \zeta(s) &:= (\zeta_1(\cdot;s), \ldots, \zeta_n(\cdot;s)). \end{aligned} \tag{3.5}$$

Proposition 3.1. *Using the above notation the following statement is valid, possibly after making $\delta > 0$ smaller. There exist a map $\sigma : (\xi, \eta; s) \in \mathbb{R}^{2n} \times [0, \delta) \mapsto \sigma(\xi, \eta; s) \in Z$ of class $\mathscr{C}^{K+1}$ and a neighborhood $\mathscr{N}$ of 0 in Z such that for each $s \in [0, \delta)$ one has*

$$\sigma(\xi, \eta; s) \in P_h(s)Z \quad ((\xi, \eta) \in \mathbb{R}^{2n}), \tag{3.6}$$

$$\sigma(0, 0; s) = 0, \quad D_{(\xi,\eta)}\sigma(0, 0; s) = 0, \tag{3.7}$$

and the manifold

$$W_c(s) = \{\xi \cdot \psi(s) + \eta \cdot \zeta(s) + \sigma(\xi, \eta; s) : (\xi, \eta) = (\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{2n}\} \subset Z$$

has the following properties:

(a) If $u(y)$ is a solution of (3.1) on $\mathcal{I} = \mathbb{R}$ and $u(y) \in \mathscr{N}$ for all $y \in \mathbb{R}$, then $u(y) \in W_c(s)$ for all $y \in \mathbb{R}$; that is, $W_c(s)$ contains the orbit of each solution of (3.1) which stays in $\mathscr{N}$ for all $y \in \mathbb{R}$.

(b) If $z : \mathbb{R} \to X_c(s)$ is a solution of the equation

$$\frac{dz}{dy} = A(s)\big|_{X_c(s)} z + P_c(s)R(z + \sigma(\{z\}_{\mathscr{B}}; s); s) \tag{3.8}$$

on some interval $\mathcal{I}$, and $u(y) := z(y) + \sigma(\{z(y)\}_{\mathscr{B}}; s) \in \mathscr{N}$ for all $y \in \mathcal{I}$, then $u : \mathcal{I} \to Z$ is a solution of (3.1) on $\mathcal{I}$.

Moreover, σ satisfies the following relation:

(c) If $2 \leq \ell \leq K$ is an integer, then $\sigma(\{u\}_{\mathscr{B}}; s) = \mathcal{O}(\|u\|^{\ell+1})$ as $u \to 0$ whenever $s \in [0, \delta)$ is such that $R(u; s) = \mathcal{O}(\|u\|^{\ell+1})$ as $u \to 0$.

In the sequel, the function σ is called the *reduction function*, $X_c(s)$ is the *center space*, $W_c(s)$ is the *center manifold*, and equation (3.8) is the *reduced equation*.

Proof of Proposition 3.1. Using the results of [11, Section 3], one can show (without even shrinking the parameter domain) that there is map $\sigma(\xi, \eta; s)$ of class $\mathscr{C}^{K+1}$ in (ξ, η) satisfying (3.6), (3.7), and statements (a)–(c). However, because of the s-dependence in the linear operator $A(s)$, we cannot refer to [11] for the $\mathscr{C}^{K+1}$-regularity in s—in [11], parameters appear only in the nonlinearity R—and we need to prove the existence of σ differently. We derive it from standard center manifold theorems using the fact that $A(s)$ depends on s in its bounded part only.

Write equation (3.2) in the form

$$\frac{du}{dy} = A_0 u + \bar{R}(u; s), \tag{3.9}$$

where $A_0 := A(0)$ and $\bar{R}(u; s) = (A(s) - A_0)u + R(u; s)$. Due to (S1), (S2), $\bar{R} : Z \times (-\delta, \delta) \to Z$ is of class $\mathscr{C}^{K+1}$, just like R. Multiplying $\bar{R}$ by a suitable cutoff function on the Hilbert space $Z \times \mathbb{R}$, one finds a $\mathscr{C}_{\mathrm{b}}^{K+1}$-map $\tilde{R} : Z \times \mathbb{R} \to Z$ having a sufficiently small (global) Lipschitz constant and satisfying $\tilde{R} \equiv \bar{R}$ on a small neighborhood of $(0, 0) \in Z \times \mathbb{R}$, say, on $\mathscr{N} \times (-\delta_0, \delta_0)$ ($\mathscr{N}$ is a neighborhood of $0 \in Z$ and $\delta_0 \in (0, \delta)$). One then applies the global center manifold theorem to equation (3.9) with $\bar{R}$ replaced by $\tilde{R}$, augmented by the "stationary-parameter equation" $ds/dy = 0$ (cp. [6, 18]). This yields a $\mathscr{C}_{\mathrm{b}}^{K+1}$-map $\tilde{\sigma} : X_c(0) \times \mathbb{R} \mapsto Z$ taking values in $P_h(0)Z$, such that for each $s \in \mathbb{R}$

$$W^c(s) := \{w + \tilde{\sigma}(w; s) : w \in X_c(0)\} \tag{3.10}$$

is the *global center manifold* for (3.9). This means, by definition, that $W^c(s)$ is the set of all points $u_0 \in Z$ with the following property: there is a solution $u(y)$ of (3.9) defined for all $y \in \mathbb{R}$ such that $u(0) = u_0$ and

$$\sup_{y \in \mathbb{R}} \|u(y)\|_Z e^{-\epsilon |y|} < \infty \quad (\epsilon > 0).$$

In particular, since $u \equiv 0$ is a solution of (3.9) due to the relation $\tilde{R}(0, s) = \bar{R}(0, s) = 0$, one has $\tilde{\sigma}(0, s) = 0$ for all $s \in (-\delta_0, \delta_0)$. The applicability of [6, 18] to (3.9) is verified in [11]: in addition to the $\mathscr{C}^{K+1}$-regularity of $\bar{R}$ already mentioned above, this verification amounts to showing a certain resolvent bound on the operator $A(0)$. The bound determines how small the Lipschitz constant of $\tilde{R}$ needs to be, and the cutoff function is selected accordingly.

If $s \geq 0$ and it is small enough, $W^c(s)$ can be written as the graph of a map $\bar{\sigma}(\cdot; s) : X_c(s) \to P_h(s)Z$. To find $\bar{\sigma}$, for $w \in X_c(0)$ we write $w + \tilde{\sigma}(w; s)$ as

$$w + \tilde{\sigma}(w; s) = P_c(s)(w + \tilde{\sigma}(w; s)) + P_h(s)(w + \tilde{\sigma}(w; s)). \tag{3.11}$$

Given any $v \in X_c(s) = P_c(s)Z$, we want to solve the equation

$$P_c(s)w + P_c(s)\tilde{\sigma}(w; s) = v \tag{3.12}$$

for $w \in X_c(0)$. To that goal, define, for any $s \in [0, \delta_0)$,

$$Q(s) := P_c(s)P_c(0) + P_h(s)P_h(0) \in \mathscr{L}(X) \tag{3.13}$$

and note that $Q(0) = I_X$—the identity on X, and $Q(s)w = P_c(s)w$ for $w \in X_c(0)$ (in particular, $Q(s)$ takes $X_c(0)$ to $X_c(s)$). As mentioned above, $P_c(s) \in \mathscr{L}(X)$ is of class $\mathscr{C}^{K+1}$ in $s \in (-\delta_0, \delta_0)$, hence $Q(s) \in \mathscr{L}(X)$ is such as well. It follows that for sufficiently small $s \geq 0$ (say, for $s \in [0, \delta_1)$, with some $\delta_1 \in (0, \delta_0]$), the inverse $Q^{-1}(s) \in \mathscr{L}(X)$ exists and is of class $\mathscr{C}^{K+1}$ in s. For such s and for any $v \in X_c(s)$, equation (3.12) can be equivalently written as

$$w = Q^{-1}(s)P_c(s)v - Q^{-1}(s)P_c(s)P_h(0)\tilde{\sigma}(w; s), \tag{3.14}$$

where we have used the relations $Q(s)w = P_c(s)w$, $P_c(s)v = v$, and $\tilde{\sigma}(w; s) = P_h(0)\tilde{\sigma}(w; s)$. Since $\tilde{\sigma}$ is of class $\mathscr{C}^{K+1}_{\mathrm{b}}$ and $P_c(0)P_h(0) = 0$, we observe that if $\delta_2 \in (0, \delta_1)$ is small enough, then the map on the right-hand side of (3.14) is a 1/2-contraction in $w \in X_c(0)$—assuming the norm from X on $X_c(0)$—for all $s \in [0, \delta_2)$ and $v \in X$ (not just $v \in X_c(s)$). The uniform contraction principle implies that equation (3.14) has a unique solution $w \in X_c(0)$ given by

$$w = \Upsilon(v, s), \tag{3.15}$$

where $\Upsilon : X \times (-\delta_2, \delta_2) \to X_c(0)$ is a $\mathscr{C}^{K+1}$ map. We now define $\bar{\sigma}$ by

$$\bar{\sigma}(v; s) := P_h(s)(\Upsilon(v, s) + \tilde{\sigma}(\Upsilon(v, s); s)). \tag{3.16}$$

Clearly, $\bar{\sigma} : X \times [0, \delta_2) \to Z$ is of class $\mathscr{C}^{K+1}$ and, by (3.11),

$$W^c(s) = \{w + \tilde{\sigma}(w; s) : w \in X_c(0)\} = \{v + \bar{\sigma}(v; s) : v \in X_c(s)\}. \tag{3.17}$$

To conclude, define $\sigma : \mathbb{R}^{2n} \times [0, \delta_2) \to Z$ by

$$\sigma(\xi, \eta; s) := \bar{\sigma}(\xi \cdot \psi(s) + \eta \cdot \zeta(s); s) \quad ((\xi, \eta) \in \mathbb{R}^{2n},\ s \in [0, \delta_2)), \tag{3.18}$$

with $\psi(s)$, $\zeta(s)$ as in (3.5), (3.4). Since the functions $\varphi_j(\cdot; s) \in H^{m+2}(\mathbb{R}^N)$ are of class $\mathscr{C}^{K+1}$ in s, σ is of class $\mathscr{C}^{K+1}$. It is straightforward to verify, using properties of $\tilde{\sigma}$, that $\sigma(0, 0; s) = 0$ for all $s \in [0, \delta_2)$, and statements (a), (b) hold. Relation (3.6) is a direct consequence of the definition of $\bar{\sigma}$, (3.16). The remaining statements of the proposition can be verified, as in [11, Section 2.1], using the following "invariance identity" for $\bar{\sigma}(\cdot; s)$:

$$D_u\bar{\sigma}(u; s)[A(s)u + P_c(s)R(u + \bar{\sigma}(u; s); s)] = A(s)\bar{\sigma}(u; s) + P_h(s)R(u + \bar{\sigma}(u; s); s) \\ (s \in [0, \delta_2),\ u \in X_c(s), u \approx 0). \tag{3.19}$$

This identity follows from the definition of $W^c(s)$, relation (3.17), and the fact that for $s \in [0, \delta_2)$ and $u \approx 0$ one has $A_0 + \tilde{R}(u, s) = A(s) + R(u, s)$. From (3.19) and $D_uR(0; s) = 0$ it follows that $D_u\bar{\sigma}(0; s) = 0$, which gives the second relation in (3.7). For a detailed verification of the relations in (c), we refer the reader to [11, Section 2.1] (or, see [10, Section 2]). □

Next, we examine the Hamiltonian structure of the reduced equation. For $(u, v) \in Z$ and any fixed $s \in [0, \delta)$, let

$$H(u, v) = \int_{\mathbb{R}^N} \frac{-1}{2}|\nabla u(x)|^2 + \frac{1}{2}a_1(x; s)u^2(x) + F(x, u(x); s) + \frac{1}{2}v^2(x)\ dx, \tag{3.20}$$

where

$$F(x, u; s) = \int_0^u f(x, \vartheta; s) d\vartheta.$$

Equation (3.1) has a formal Hamiltonian structure with respect to the functional H and this structure is inherited in a certain way by the reduced equation. More specifically, denoting by Φ the composition of the maps $(\xi, \eta) \to \sigma(\xi, \eta; s) : \mathbb{R}^{2n} \to Z$ and $H : Z \to \mathbb{R}$, (3.8) is the Hamiltonian system with respect to the Hamiltonian Φ and a certain symplectic structure defined in a neighborhood of $(0, 0) \in \mathbb{R}^{2n}$. This can be proved using general results of [10], but in [11] we instead gave a proof, with some additional useful specifics, using direct explicit computations. We have then transformed the system successively performing three coordinate changes:

(T) a Darboux transformation, normal form transformation, and action-angle variables.

By the first change of coordinates, we achieve that the transformed system is Hamiltonian with respect to the standard symplectic form on $\mathbb{R}^{2n}$ (and the transformed Hamiltonian). The existence of such a local transformation is guaranteed by the Darboux theorem, but in [11] we took some care to keep track of how the symplectic structure and the Darboux transformation depend on the parameters. In particular, the computations in [11] show that the Darboux transformation can be chosen as the sum of the identity map (on $\mathbb{R}^{2n}$) and terms of order $\mathcal{O}(|(\xi, \eta)|^3)$, with the cubic terms having coefficients of class $\mathscr{C}^K$ in s. In the new coordinates (ξ', η') resulting from such a transformation, the Hamiltonian takes the following form for $(\xi', \eta') \approx (0, 0)$:

$$\Phi(\xi', \eta'; s) = \frac{1}{2} \sum_{j=1}^{n} (-\mu_j(s)(\xi_j')^2 + (\eta_j')^2) + \frac{1}{3} \int_{\mathbb{R}^N} a_2(x; s)(\xi' \cdot \varphi(x; s))^3 \, dx \\ + \Phi_4(\xi', \eta'; s) + \Phi'(\xi', \eta'; s). \quad (3.21)$$

Here, $\mu_j(s)$, $\varphi_j(x; s)$ are the eigenvalues and eigenfunctions as above,

$$\varphi(x; s) = (\varphi_1(x; s), \ldots, \varphi_n(x; s)),$$

Φ_4 is a homogeneous polynomial in (ξ', η') of degree 4 whose coefficients are of class $\mathscr{C}^K$ in s, and Φ' is a function of class $\mathscr{C}^K$ in all its arguments and of order $\mathcal{O}(|(\xi', \eta')|^5)$ as $(\xi', \eta') \to (0, 0)$.

Remark 3.2. In the sequel, we will only consider the Hamiltonian Φ for $s > 0$. Note, however, that the coefficients of the polynomial on the right hand side of (3.21) with Φ' deleted, depend continuously on s up to $s = 0$. In particular, they stay bounded as $s \to 0$. This fact will be used in the asymptotic analysis in Section 5.

The second transformation in (T) puts the Hamiltonian $\Phi(\cdot, \cdot; s)$, for $s > 0$, to the normal form up to order $2k_B + 1$, where $k_B = [K/2] - 1$, $[K/2]$ being the integer part of $K/2$. More precisely, we showed in [11] that near $(0, 0)$ there is a canonical coordinate transformation—recall that a canonical transformation does not change the symplectic structure—such that in the new coordinates $(\bar{\xi}, \bar{\eta})$ the Hamiltonian can be written as follows. Let $(\bar{\xi}, \bar{\eta}) = (\bar{\xi}_1, \dots, \bar{\xi}_n, \bar{\eta}_1, \dots, \bar{\eta}_n)$,

$$I_j = \frac{1}{2}(\bar{\xi}_j^2 + \bar{\eta}_j^2), \tag{3.22}$$

and $I = (I_1, \dots, I_n)$. Then

$$\Phi(\bar{\xi}, \bar{\eta}; s) = \omega(s) \cdot I + \Phi_0(I; s) + \Phi_1(\bar{\xi}, \bar{\eta}; s), \tag{3.23}$$

where $\omega(s) = (\omega_1(s), \dots, \omega_n(s))$ (cp. (NR')), Φ_0 is a polynomial in I of degree at most k_B, and Φ_1 a $\mathscr{C}^K$ function of order $\mathcal{O}(|(\bar{\xi}, \bar{\eta})|^{2k_B+2})$ as $(\bar{\xi}, \bar{\eta}) \to (0, 0)$. The polynomial Φ_0 is of the form

$$\Phi_0(I; s) = \frac{1}{2} I \cdot M(s) I + \hat{P}(I; s), \tag{3.24}$$

where, for $s \in (0, \delta)$, $M(s)$ is an $n \times n$ matrix and $\hat{P}(I; s)$ a polynomial in I (of degree at most k_B) with no constant, linear, or quadratic terms. The entries of $M(s)$ and the coefficients of $\hat{P}(\cdot; s)$ are of class $\mathscr{C}^K$ in s.

Finally, we introduce the *action-angle* variables $I = (I_1, \dots, I_n) \in \mathbb{R}^n$, $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{T}^n$ by

$$(\bar{\xi}_j, \bar{\eta}_j) = \sqrt{2I_j}(\cos\theta_j, \sin\theta_j).$$

The change of coordinates from $(\bar{\xi}_j, \bar{\eta}_j)$ to (θ, I) is defined in regions where $I_j = (\bar{\xi}_j^2 + \bar{\eta}_j^2)/2 > 0$ for all $j \in \{1, \dots, n\}$, and it is well known that this transformation is canonical. In these coordinates, Φ looks as follows:

$$\Phi(\theta, I; s) = \omega(s) \cdot I + \Phi_0(I; s) + \Phi_1(\theta, I; s). \tag{3.25}$$

($\Phi_1(\theta, I; s)$ actually stands for the function $\Phi(\bar{\xi}(\theta, I), \bar{\eta}(\theta, I); s)$.) Thus, the Hamiltonian Φ is the sum of an integrable Hamiltonian (the first two terms on the right hand side of (3.25)) and a "perturbation" (the last term in (3.25)). This is a form suitable for an application of a KAM theorem.

We remark that for the proof of Theorem 2.2, we will need more precise information on the lower order terms resulting from the Birkhoff normal form transformation. This issue will be dealt with in Section 5.

4 Application of a KAM-type theorem using Arnold's condition

In the previous section, we have summarized how the reduced Hamiltonian Φ can be written in a form suitable for an application of a KAM theorem: it is the sum of an integrable Hamiltonian (a function of the action variables only) and a perturbation of a high order (see (3.25)). We have to work in a finite-differentiability KAM setting here, for in the center manifold reduction we would in general lose the analytic structure—or, for that matter, the $\mathscr{C}^\infty$ structure—even if the original nonlinearity had one. Similarly as in [11], a theorem of Pöschel is suitable for our purposes. Although it has Kolmogorov's nondegeneracy condition as an assumption, a trick from [4], which we recall below, shows that Arnold's isoenergetic condition can be assumed instead. The validity of Arnold's condition for the reduced Hamiltonian will be addressed in Section 5.

To recall Pöschel's theorem, let $n \geq 2$ be an integer and consider a Hamiltonian $H : \mathbb{T}^n \times \Omega \to \mathbb{R}$ given by

$$H(\theta, I) = H^0(I) + H^1(\theta, I), \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ is a domain, and $\mathbb{T}^n$ is the n-dimensional torus (in other words, $H^1(\theta, I)$ is periodic in $\theta_1, \ldots, \theta_n$ with a common period, 2π, say). The Hamiltonian system corresponding to H is

$$\begin{aligned} \dot{\theta} &= \nabla_I H(\theta, I), \\ \dot{I} &= -\nabla_\theta H(\theta, I). \end{aligned} \tag{4.2}$$

We make the assumption that H^0 is analytic on Ω and its *frequency map* $\omega^*(I) := \nabla H^0(I) : \Omega \to \mathbb{R}^n$ is a diffeomorphism onto its image $V := \{\omega^*(I) : I \in \Omega\}$; in particular, the Hessian matrix

$$\frac{\partial^2 H^0}{\partial I^2}(I) \tag{4.3}$$

is nonsingular on Ω (this is usually referred to as *Kolmogorov's condition*). Moreover, we assume that there is a complex neighborhood Ω^ρ of Ω,

$$\Omega^\rho = \bigcup_{I \in \Omega} \{\zeta \in \mathbb{C}^n : |\zeta - I| < \rho\} \tag{4.4}$$

with $\rho > 0$, such that H^0 has an analytic extension to Ω^ρ whose Hessian is nonsingular on Ω^ρ and $\omega^*(I)$ is a one-to-one map of Ω^ρ onto its image in $\mathbb{C}^n$.

The perturbation term H^1 is assumed sufficiently small (as specified in the theorem, see equation (4.9)) in a Hölder norm: if $\vartheta > 0$ is a noninteger, $\|H\|_{\mathscr{C}^\vartheta(\mathbb{T}^n \times \Omega)}$ is the infimum of all constants c satisfying the following inequalities:

$$\|D^J H(\theta, I)\|_{L^\infty(\mathbb{T}^n \times \Omega)} \leq c \text{ for all } J \in \mathbb{N}^{2n}, |J| \leq [\vartheta],$$

and

$$\sup_{\substack{I \in \Omega \\ \theta, \theta' \in \mathbb{T}^n \\ \theta \neq \theta'}} \frac{|D^J H(\theta, I) - D^J H(\theta', I)|}{|\theta - \theta'|^{\vartheta - [\vartheta]}} \leq c, \qquad \sup_{\substack{\theta \in \mathbb{T}^n \\ I, I' \in \Omega \\ I \neq I'}} \frac{|D^J H(\theta, I) - D^J H(\theta, I')|}{|I - I'|^{\vartheta - [\vartheta]}} \leq c$$

for all $J \in \mathbb{N}^{2n}$ such that $|J| = [\vartheta]$. Here $[\vartheta]$ is the integer part of ϑ and, for $J = (j_1, \ldots, j_n, \ell_1, \ldots, \ell_n)$,

$$D^J = \frac{\partial^{|J|}}{\partial \theta_1^{j_1} \ldots \partial \theta_n^{j_n} \partial I_1^{\ell_1} \ldots \partial I_n^{\ell_n}}, \qquad |J| = j_1 + \cdots + j_n + \ell_1 + \cdots + \ell_n.$$

A vector $\omega \in \mathbb{R}^n$ is said to be *κ, ν-Diophantine*, for some $\kappa > 0$ and $\nu > n - 1$, if

$$|\omega \cdot \alpha| \geq \kappa |\alpha|^{-\nu} \quad (\alpha \in \mathbb{Z}^n \setminus \{0\}). \tag{4.5}$$

For $\kappa > 0$ and $\nu > n - 1$, we define

$$V_\kappa := \{\omega \in V : \operatorname{dist}(\omega, \partial V) \geq \kappa \text{ and } \omega \text{ is } \kappa, \nu\text{-Diophantine}\}. \tag{4.6}$$

(We only emphasize the dependence on κ of the set V_κ in our notation, since ν will be fixed.)

The following statement is contained (in a stronger form) in [12, Theorem A].

Theorem 4.1. *Let Ω, H^0, ρ, and V be as above. Suppose additionally that for some $R > 0$ one has*

$$\left|\frac{\partial^2 H^0}{\partial I^2}(I)\right|, \left|\left(\frac{\partial^2 H^0}{\partial I^2}\right)^{-1}(I)\right| \leq R \quad (I \in \Omega^\rho). \tag{4.7}$$

Fix constants λ, ν and α satisfying

$$\lambda > \nu + 1 > n, \quad \alpha > 1, \quad \alpha \notin \{\ell/\lambda + j : j, \ell \in \mathbb{N}\}. \tag{4.8}$$

Then there exists a positive constant δ_{KAM}, depending on n, ν, λ, ρ, R (but independent of Ω and κ), such that for any $\kappa \in (0, \rho/R)$ and $H^1 \in \mathscr{C}^{\alpha\lambda+\lambda+\nu}(\mathbb{T}^n \times \Omega)$ satisfying

$$\|H^1\|_{\mathscr{C}^{\alpha\lambda+\lambda+\nu}(\mathbb{T}^n\times\Omega)} \leq \kappa^2 \delta_{\mathrm{KAM}} \tag{4.9}$$

the Hamiltonian $H = H^0 + H^1$ has the following property. There exists a diffeomorphism $T : \mathbb{T}^n \times V \to \mathbb{T}^n \times \Omega$ of class $\mathscr{C}^\alpha$ such that for each $I \in \Omega$ with $\omega^*(I) \in V_\kappa$ the manifold $\tilde{\mathbb{T}}_I := T(\mathbb{T}^n \times \omega^*(I))$ is invariant under the flow of (4.2) and the solution of (4.2) with the initial condition $T(\theta_0, \omega^*(I))$, $\theta_0 \in \mathbb{T}^n$, is given by $T(\theta_0 + \omega^*(I)t, \omega^*(I))$, $t \in \mathbb{R}$.

The stated property of the diffeomorphism T means that T *conjugates* the flow of (4.2) to a flow for which each torus $\mathbb{T}^N \times \{\bar{\omega}\}$, $\bar{\omega} \in V_\kappa$, is invariant and whose restriction to this torus is a linear flow with frequencies $\bar{\omega}$. The theorem thus provides a class of quasiperiodic solutions of (4.2) whose frequencies cover V_κ. Of course, to use this conclusion, we want $V_\kappa \neq \emptyset$, or, better, $|V_\kappa| > 0$, where $|\cdot|$ stands for the Lebesgue measure.

In [11], choosing the domain Ω and a constant κ with $|V_\kappa| > 0$ suitably, we showed that Theorem 4.1 applies directly to the reduced Hamiltonian (3.25). In particular, a hypothesis of [11] was tailored so as to imply Kolmogorov's condition. In the present setting, we make use of Arnold's nondegeneracy condition instead and verify that Theorem 4.1 applies to the Hamiltonian H given by

$$H(\theta, I) = G(\theta, I) + (G(\theta, I))^2, \tag{4.10}$$

where $G(\theta, I) := G^0(I) + G^1(\theta, I)$, with

$$G^0(I) := \omega \cdot I + \Phi_0(I) = \omega \cdot I + \frac{1}{2} I \cdot MI + \hat{P}(I), \qquad G^1(\theta, I) := \Phi_1(\theta, I), \tag{4.11}$$

and Φ_0, Φ_1, M, and $\hat{P}$ are as in (3.24), (3.25). Here, and throughout this section, the argument s in these functions and in ω is omitted; the dependence on s is not employed in the section and the value of s can be considered fixed. Thus $G(\theta, I)$ is equal to the reduced Hamiltonian $\Phi(\theta, I; s)$ for a fixed value of $s \in (0, \delta)$. Recall that $\hat{P}(I)$ is a polynomial in I with no constant, linear or quadratic terms.

The Hamiltonian defined in (4.10) can be written in the form (4.1) by setting

$$\begin{aligned} H^0(I) &= G^0(I) + (G^0(I))^2, \\ H^1(\theta, I) &= G^1(\theta, I) + 2G^0(I)G^1(\theta, I) + (G^1(\theta, I))^2. \end{aligned} \tag{4.12}$$

The Hamiltonian H is relevant for two reasons, both observed in [4]. First, on any energy level, the Hamiltonian vector fields of H and G differ only by a multiplicative constant. This will be elaborated on and used later on. The second reason is that H^0 satisfies Kolmogorov's condition if G^0 satisfies *Arnold's nondegeneracy condition* requiring the matrix

$$\begin{bmatrix} \dfrac{\partial^2 G^0}{\partial I^2}(I) & \dfrac{\partial G^0}{\partial I}(I) \\ \\ \dfrac{\partial G^0}{\partial I}(I)^T & 0 \end{bmatrix} \tag{4.13}$$

to be nonsingular in a selected domain. To be more specific, consider the $(n+1) \times (n+1)$ matrix

$$\mathcal{M} := \begin{bmatrix} M & \omega \\ \omega^T & 0 \end{bmatrix}. \tag{4.14}$$

Note that this is the matrix (4.13) evaluated at $I = 0$ (cp. (4.11)).

Lemma 4.2. *Suppose that the matrix $\mathcal{M}$ is nonsingular, while the matrix M is singular. If H^0 is as in* (4.12)*, then the Hessian matrix* (4.3) *is nonsingular for $I = 0$ (and hence for any I sufficiently close to* 0*).*

Proof. Note first that

$$\frac{\partial^2 H^0}{\partial I^2}(0) = \left((1 + 2G^0(I)) \frac{\partial^2 G^0}{\partial I^2}(I) + 2 \frac{\partial G^0}{\partial I}(I) \otimes \frac{\partial G^0}{\partial I}(I) \right) \Bigg|_{I=0} = M + 2\omega \otimes \omega, \tag{4.15}$$

where, for vectors $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, $v \otimes w$ is the exterior product of v and w, i.e., $(v \otimes w)_{ij} = v_i w_j$. Also recall the following determinant identity for block matrices: if A is an $n \times n$ matrix and v, $w \in \mathbb{R}^n$ are (column) vectors, then

$$\begin{vmatrix} A & v \\ w^T & 0 \end{vmatrix} = \det A - \det(A + v \otimes w). \tag{4.16}$$

Applying this identity with $A = M$, $v = \omega$, $w = 2\omega$, and using that M is singular, we obtain

$$\det(M + 2\omega \otimes \omega) = -2 \det \mathcal{M} \neq 0.$$

The conclusion of the lemma follows immediately from this and (4.15). $\square$

Lemma 4.2 facilitates an application of Theorem 4.1 to the Hamiltonian H defined in (4.10). We consider G^0 and G^1 (cp. (4.11)) as real-valued maps on Ω and $\mathbb{T}^n \times \Omega$, respectively, where

$$\Omega = \Omega_q := \{I \in \mathbb{R}^n : \ q \leq I_j \leq 2q \ \ (j = 1, \ldots, n)\} \tag{4.17}$$

with a sufficiently small $q > 0$. Note that these maps are defined, as are Φ_0, Φ_1, if $q > 0$ is small enough.

Lemma 4.3. *Suppose the hypotheses* (S1), (S2), (A1)(a), (A1)(b')*, and* (NR') *are satisfied and set $k_B = [K/2] - 1$. Let (for some fixed $s \in (0, \delta)$) M be as in* (4.11) *and $\mathcal{M}$ as in* (4.14)*, and assume that* $\det M = 0$ *and* $\det \mathcal{M} \neq 0$.

Fix constants α, λ, ν such that

$$3n > \alpha\lambda + \lambda + \nu \text{ and relations (4.8) hold.}^1 \tag{4.18}$$

Then there exists $q^* > 0$ such that for each $q \in (0, q^*)$ the maps H^0, H^1 defined in (4.12) *have the following properties.*

(a) H^0 is a polynomial in I, and there are R, $\rho > 0$ such that (4.7) *holds (with Ω^ρ as in* (4.4)*) and the map*

$$I \mapsto \omega^*(I) = \nabla H^0(I) \tag{4.19}$$

is one-to-one on Ω^ρ. We denote by V the image of Ω under this map ω^*.

(b) $H^1 \in \mathscr{C}^{\alpha\lambda+\lambda+\nu}(\mathbb{T}^n \times \Omega)$ and, with R, ρ the constants in statement (a) and $\delta_{\mathrm{KAM}} = \delta_{\mathrm{KAM}}(n, \nu, \alpha, \rho, R)$ as in Theorem 4.1, there is $\kappa \in (0, \rho/R)$ such that (4.9) *holds and $|V_\kappa| > 0$ (V_κ is defined in* (4.6)*).*

Proof. This lemma is analogous to Lemma 5.2 of [11]. In that lemma, we showed that—using the present notation—if $\det M \neq 0$, then (a), (b) hold when one takes $H^0 = G^0$, $H^1 = G^1$, that is, when $H^0 + H^1$ is the reduced Hamiltonian itself.

In the present case, assuming that $\det M = 0$ and $\det \mathcal{M} \neq 0$, and using Lemma 4.2, we discover that with H^0 and H^1 as in (4.12) the Hamiltonian $H = H^0 + H^1$ has the same structure as in the proof of [11, Lemma 5.2]. More specifically, one has

$$H^0(I) = \omega \cdot I + \frac{1}{2} I \cdot \tilde{M} I + \tilde{P}(I),$$

where $\tilde{M}$ is the Hessian matrix of H^0 as in (4.15) (hence $\det \tilde{M} \neq 0$) and $\tilde{P}(I)$ is a polynomial in I with no constant, linear or quadratic terms. Further, H^1 is a $\mathscr{C}^K$ map on $\mathbb{T}^n \times \Omega_q$ of order $\mathcal{O}(|I|^{k_B+1})$ uniformly in $\theta \in \mathbb{T}^n$. This is all the structure one needs (regardless of the origin of the Hamiltonian H) to show, following [11, Proof of Lemma 5.2], that (a), (b) hold if q^* is sufficiently small. □

We can now state the main result of this section. We make the following assumption on the matrix $\mathcal{M}$ defined in (4.14) (with M and ω as in (4.11)).

(AN) The $(n+1) \times (n+1)$ matrix $\mathcal{M}$ is nonsingular.

We remark that although $\mathcal{M}$ is defined in terms of the reduced Hamiltonian, it is not easy to formulate explicit sufficient conditions for (AN) in terms of the functions in the original equation (2.3). Indeed, M is the Hessian matrix $\frac{\partial^2 \Phi_0}{\partial I^2}(0)$ of the reduced Hamiltonian in the action variables, thus, it depends on fourth-order terms of the Hamiltonian in the variables (ξ', η') (cp. (3.21)), and these in turn depend on terms up to order 3 of the reduction function and the Darboux transformation. These are hardly ever computable explicitly. Nonetheless, we will show in the next section that (AN) holds if $n = 2$, hypothesis (A2) is satisfied, and $s > 0$ is sufficiently small.

[1] for example, take α, λ, ν as in (4.8) with $\lambda \approx \nu + 1 \approx n$, $\alpha \approx 1$.

Theorem 4.4. *Assume the hypotheses* (A1)(a)*,* (A1)(b')*,* (S1)*,* (S2)*,* (NR') *(with $m > N/2$ and $K \geq 6(n+1)$) are satisfied, and also that (for some fixed $s \in (0, \delta)$)* (AN) *holds. Then there exists a solution $u(x, y)$ of equation* (2.3) *such that $u(x, y) \to 0$ as $|x| \to \infty$ uniformly in y, and $u(x, y)$ is radially symmetric in x and quasiperiodic in y with n frequencies (n is as in* (A1)(b'), (AN)*). In fact, there is a uncountable family of such quasiperiodic solutions, their frequency vectors forming an uncountable subset of $\mathbb{R}^n$.*

Proof. If $M = \frac{\partial^2 \Phi_0}{\partial I^2}(0)$ is nonsingular, then the result is a direct consequence of Theorem 2.2 and Remark 5.5(b) in [11].

We proceed assuming that M is singular. As above, we consider the maps G^0 and G^1 (cp. (4.11)) with domains Ω and $\mathbb{T}^n \times \Omega$, respectively, where $\Omega = \Omega_q$ is as in (4.17). Denote $G = G^0 + G^1$ and let H^0, H^1 be as in (4.12). According to Lemma 4.3, for all sufficiently small $q > 0$ the conclusion of Theorem 4.1 applies to the Hamiltonian $H = H^0 + H^1$, with $|V_\kappa| > 0$. We choose a small $q > 0$ such that in addition, $|G(\theta, I)| < 1/4$ and $|H(\theta, I)| < 1/8$ for all $(\theta, I) \in \mathbb{T}^n \times \Omega$.

Let T be the diffeomorphism from Theorem 4.1, and $I^* \in \Omega$ be such that $\omega^*(I^*) \in V_\kappa$, V_κ being the set defined in (4.6). Since the manifold $T(\mathbb{T}^n \times \{\omega^*(I^*)\})$ is invariant under the Hamiltonian vector field of H, it is contained in the level set (relative to H) $M_\epsilon := \{(\theta, I) \in \mathbb{T}^n \times \Omega : H(\theta, I) = \epsilon\}$, for some $\epsilon = \epsilon(I^*) \in (-1/8, 1/8)$. This set coincides with the c-level set of G for $c = c(I^*) := (1/2)(-1 + \sqrt{1 + 4\epsilon})$, as found by taking the inverse of the map

$$t \in \left(\frac{\sqrt{2} - 2}{4}, \frac{\sqrt{6} - 2}{4} \right) \mapsto t^2 + t \in (-1/8, 1/8).$$

The gradients of H and G are related as follows:

$$\nabla H(\theta, I) = \nabla \big(G(\theta, I) + (G(\theta, I))^2 \big) = (1 + 2G(\theta, I)) \nabla G(\theta, I); \tag{4.20}$$

in particular, when ∇H and ∇G are restricted to M_ϵ, one has

$$\nabla H(\theta, I) \Big|_{M_\epsilon} = (1 + 2c) \nabla G(\theta, I) \Big|_{M_\epsilon}. \tag{4.21}$$

It follows that, up to a multiplicative constant, the Hamiltonian vector fields of G and H coincide when restricted to M_ϵ (this is the first of the two observations from [4] mentioned above). By Theorem 4.1, the solution of

$$\begin{aligned} \dot{\theta} &= \nabla_I H(\theta, I), \\ \dot{I} &= -\nabla_\theta H(\theta, I), \end{aligned} \tag{4.22}$$

with the initial condition $T(\theta_0, \omega^*(I^*))$, is given by $T(\theta_0 + \omega^*(I^*)t, \omega^*(I))$. Using (4.21), it is easy to see that the manifold $T\left(\mathbb{T}^n \times \left\{ \frac{1}{1+2c} \omega^*(I^*) \right\}\right)$ is invariant under the flow of

$$\begin{aligned} \dot{\theta} &= \nabla_I G(\theta, I), \\ \dot{I} &= -\nabla_\theta G(\theta, I), \end{aligned} \tag{4.23}$$

and the solution of (4.23) with the initial condition $T\left(\theta_0, \frac{1}{1+2c}\omega^*(I^*)\right)$, $\theta_0 \in \mathbb{T}^n$, is given by

$$T\left(\theta_0 + \frac{1}{1+2c}\omega^*(I^*)t, \frac{1}{1+2c}\omega^*(I^*)\right) \quad (t \in \mathbb{R}). \tag{4.24}$$

Note that since $\omega^*(I^*) \in V_\kappa$, the solution in (4.24) is quasiperiodic, with frequency vector $\omega^*(I^*)/(1+2c)$. Note also that the trajectory of this solution is contained in $\mathbb{T}^n \times \Omega_q$. Adjusting $q > 0$, we can assume that the solutions obtained this way are as close to $\mathbb{T}^n \times \{0\}$ as we like.

We pause at this point to observe that the frequencies $\omega^*(I^*)/(1+2c)$, $c = c(I^*)$, found above form an uncountable set in $\mathbb{R}^n$. More specifically, since $|V_\kappa| > 0$, there exists an uncountable set $W_\kappa \subset V_\kappa$ such that no two elements of W_κ are multiples of each other. Let Ω^W be the preimage of W_κ via the frequency map $\omega^*(I) = \nabla H^0(I)$. This is an uncountable set, as $\omega^*(I)$ is a diffeomorphism. The frequencies

$$\frac{\omega^*(I^*)}{1+2c(I^*)} \quad (I^* \in \Omega^W), \tag{4.25}$$

are mutually distinct, due to the defining property of the set W_κ, thus they form an uncountable set in $\mathbb{R}^n$.

Remembering that G is equal to the reduced Hamiltonian $\Phi(\theta, I; s)$ for a fixed value of $s \in (0, \delta)$, we now reverse the transformations identified in Section 3 (see the list (T)), to get back to the reduced equation (3.8). This yields quasiperiodic solutions z_{I^*}, $I^* \in \Omega^W$, of (3.8), whose frequency vectors form the uncountable set described by (4.25). Moreover, we can assume that all the trajectories of these solutions are contained in a small neighborhood of $(0,0) \in \mathbb{R}^{2n}$ (this is guaranteed by choosing q small enough); in particular, $z_{I^*}(y) \in \mathscr{N}$ for all $y \in \mathbb{R}$, $\mathscr{N}$ being the neighborhood of $0 \in Z$ from Proposition 3.1. By Proposition 3.1(b), for each $I^* \in \Omega^W$,

$$U(y) = (U_1(y), U_2(y))^T = z_{I^*}(y) + \sigma(\{z_{I^*}(y)\}_{\mathscr{B}}) \in Z$$

is a solution of system (3.1). Letting

$$u(x, y) = U_1(y)(x), \tag{4.26}$$

we obtain a solution of (2.3). This solution is quasiperiodic in y, in the sense of the definition given in Section 2, with frequency vector $\omega^*(I^*)/(1+2c(I^*))$.

It remains to show that the solution $u(x, y)$ in (4.26) decays to 0 as $|x| \to \infty$, uniformly in y. This follows immediately from the fact that the set $\{u(\cdot, y) : y \in \mathbb{R}\}$ is contained in a compact set—continuous image of a torus—in $H^{m+2}_{\mathrm{rad}}(\mathbb{R}^N)$, with $m > N/2$. $\square$

5 Verification of Arnold's condition

Throughout this section, we take $n = 2$ and assume that hypotheses (S1), (S2), (A1), (NR) (with K and m as in (2.5)), and (A2) hold. We prove that condition (AN), stated in Section 4, holds for all sufficiently small $s > 0$. Once this fact has been established, our main theorem will become a direct consequence of Theorem 4.4.

The verification of hypothesis (AN) will be carried out by studying the terms of degree 4 arising from the computation of the Birkhoff normal form of the Hamiltonian Φ (cp. (3.21)). Hypothesis (A2) will be instrumental in describing the asymptotic behavior of these terms as $s \to 0$ (which actually means $s \to 0+$, as we only take $s \geq 0$ in this section). Since $\omega_2(s) = \sqrt{|\mu_2(s)|} \to 0$, at $s = 0$ we arrive at the resonant case: a double 0 eigenvalue and a pair of purely imaginary eigenvalues $\pm i\omega_1(0)$. Therefore, an analysis of this resonance, as carried out in [3, 16], for example, could probably be put to some good use here. We proceed differently, however. Keeping track of the parameter dependence in the Birkhoff normal form procedure (see Section 5.1), we single out the 4th-order term in the normal form with the highest singularity as $s \to 0$, and relate it to the functions a_2 and φ_2 via the integral in (A2).

5.1 Birkhoff normal form algorithm

To compute the terms of order 4 (order 2 in I), we use some steps from the normal form algorithm, which we now summarize (the details can be found in [2, 5, 7], for example). Although we need the following computations for $n = 2$ only, there is no need for such restriction in this subsection.

Recall that if $h_1(\xi, \eta)$ and $h_2(\xi, \eta)$ are $\mathscr{C}^2$ functions on a domain in $\mathbb{R}^{2n}$, their Poisson bracket $\{h_1, h_2\}$ is defined by

$$\{h_1, h_2\} := \sum_{j=1}^{n} \left(\frac{\partial h_1}{\partial \xi_j} \frac{\partial h_2}{\partial \eta_j} - \frac{\partial h_1}{\partial \eta_j} \frac{\partial h_2}{\partial \xi_j} \right). \tag{5.1}$$

Consider a Hamiltonian on $\mathbb{R}^{2n}$ in the form $H = H_2 + H_3 + \cdots + H_\ell + \text{h.o.t.}$, where

$$H_2(\xi, \eta) = \sum_{j=1}^{n} \omega_j \frac{\xi_j^2 + \eta_j^2}{2},$$

H_j is a homogeneous polynomial of degree j, $j = 2, \ldots, \ell$, and "h.o.t." stands for terms of order $\mathcal{O}(|(\xi, \eta)|^{\ell+1})$. We assume that the vector $\omega = (\omega_1, \ldots, \omega_n)$ is nonresonant up to order $\ell \geq 4$. In the normal form procedure, one successively eliminates the nonresonant terms (as defined below) in $H_3, H_4, \ldots$. The cubic terms are all nonresonant and they are eliminated by a first transformation. This transformation alters terms of degree 4 and higher, but does not change the quadratic terms. The next transformation eliminates the nonresonant terms from the (altered) 4th-order terms, keeping the quadratic and cubic terms intact and altering the terms of degree 5 and higher; and so on. The transformations in this procedure are always found as the Lie transforms corresponding to a polynomial Hamiltonian, which guarantees that they are canonical (in the new coordinates, the symplectic form is still the standard one and the Poisson brackets are computed in the same way as in (5.1)). The key observation here is that if χ_ℓ is a homogeneous polynomial on $\mathbb{R}^{2n}$ of degree $\ell \geq 3$ and ν_ℓ is the time-1 map of the Hamiltonian flow with the Hamiltonian χ_ℓ (ν_ℓ is defined in a neighborhood of the origin and it is a near identity transformation), then

$$H \circ \nu_\ell = H_2 + H_3 + \cdots + H_\ell + \{H_2, \chi_\ell\} + \text{h.o.t.} \tag{5.2}$$

Thus, if χ_ℓ can be chosen such that

$$\{H_2, \chi_\ell\} = -H_\ell, \tag{5.3}$$

then the terms of degree ℓ can be completely eliminated; otherwise only certain terms of degree ℓ can be eliminated by a suitable choice of χ_ℓ.

The *homological equation* (5.3) is easiest to consider in the complex coordinates $(\alpha, \beta) = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$ given by

$$(\alpha_j, \beta_j) = \frac{1}{\sqrt{2}}\big(\xi_j + i\eta_j, i(\xi_j - i\eta_j)\big). \tag{5.4}$$

We note that these coordinates are canonical in the sense that for any two polynomials h_1, h_2 expressed in these coordinates, their Poisson bracket is still computed as in (5.1) with ξ, η replaced by α, β, respectively.

The new coordinates (α, β) "diagonalize" equation (5.3) in the following sense. Substituting the inverse relations $\xi_j = \frac{1}{\sqrt{2}}(\alpha_j - i\beta_j)$, $\eta_j = \frac{1}{\sqrt{2}}(\beta_j - i\alpha_j)$ in a homogeneous polynomial, one obtains a homogeneous polynomials in (α, β) of the same degree. Assuming that $\chi_\ell(\alpha, \beta)$ is a homogeneous polynomials of degree ℓ makes (5.3) a linear nonhomogeneous equation on the space of such polynomials. Consider the basis of this space consisting of the monomials

$$\alpha^J \beta^L, \quad J,\ L \in \mathbb{N}^n, \quad |J| + |L| = \ell, \tag{5.5}$$

where, for any multiindex $J = (j_1, \dots, j_n) \in \mathbb{N}^n$, we denote $|J| = j_1 + \dots + j_n$, $\alpha^J = \alpha_1^{j_1} \dots \alpha_n^{j_n}$; and similarly for L, β^L. Equation (5.3) can be written, with respect to the basis (5.5), as a system of linear equations with the diagonal coefficient matrix $\mathrm{diag}(i\omega \cdot (L - J))_{|J|+|L|=\ell}$. If $\omega \cdot (J - L) = 0$, the monomial $h^{JL}\alpha^J\beta^L$ (for any $h^{JL} \in \mathbb{C}$) is said to be *resonant*; otherwise, it is *nonresonant*. Due to the nonresonance assumption on ω, there are no resonant terms of degree ℓ if ℓ is odd. Therefore, (5.3) has a unique solution χ_ℓ and the terms of degree ℓ can be completely eliminated in (5.2). If ℓ is even, only nonresonant terms of degree ℓ can be eliminated by a suitable (nonunique) choice of χ_ℓ.

In the first step of the normal form procedure, one takes the (unique) solution χ_3 of

$$\{H_2, \chi_3\} = -H_3. \tag{5.6}$$

The corresponding Lie transform ν_3 eliminates the cubic terms and alters the quartic terms as follows (see [2, 5, 7] for details):

$$\begin{aligned} H \circ \nu_3 &= H_2 + H_4 + \frac{1}{2}\{\{H_2, \chi_3\}, \chi_3\} + \{H_3, \chi_3\} + \text{h.o.t.} \\ &= H_2 + H_4 + \frac{1}{2}\{H_3, \chi_3\} + \text{h.o.t.} \end{aligned} \tag{5.7}$$

where "h.o.t." now stands for terms of order 5 or higher and (5.6) was used to get the second equality in (5.7). Thus, the new degree-four homogeneous polynomial is $H_4 + \frac{1}{2}\{H_3, \chi_3\}$. The second step is to determine which terms in this polynomial can be eliminated by the next transformation ν_4.

If, in the coordinates (α, β),

$$H_3(\alpha,\beta) = \sum_{\substack{J,L\in\mathbb{N}^n \\ |J|+|L|=3}} h_3^{JL} \alpha^J \beta^L,$$

for some coefficients h_3^{JL}, then the polynomial χ_3 is given by

$$\chi_3(\alpha,\beta) = \sum_{\substack{J,L\in\mathbb{N}^n \\ |J|+|L|=3}} \frac{h_3^{JL}}{i\omega\cdot(L-J)} \alpha^J \beta^L. \tag{5.8}$$

Computing the 4th-order term $\tilde{H}_4 := H_4 + \frac{1}{2}\{H_3, \chi_3\}$ in (5.7), one finds coefficients h_4^{JL} such that

$$\tilde{H}_4(\alpha,\beta) = \sum_{\substack{J,L\in\mathbb{N}^n \\ |J|+|L|=4}} h_4^{JL} \alpha^J \beta^L. \tag{5.9}$$

Due to the nonresonance assumption on ω, for any J, L with $|J| + |L| = 4$, the term $h^{JL}\alpha^J\beta^L$ is resonant if and only if $J = L$. The second step consists in choosing a homogeneous polynomial χ_4 which is real in the coordinates (ξ, η) and such that the corresponding transformation ν_4 eliminates all nonresonant terms in (5.9) while keeping the resonant terms intact. The final form of the quartic terms in $H \circ \nu_3 \circ \nu_4$ is then obtained by substituting (5.4) in the sum of all the resonant terms,

$$\sum_{|J|=2} h_4^{JJ} \alpha^J \beta^J, \tag{5.10}$$

and noting that for $|J| = 2$ one gets $h_4^{JJ}\alpha^J\beta^J = -h_4^{JJ} I^J$, with $I = (I_1, \dots, I_n)$ as in (3.22), (3.23).

To conclude these preparatory remarks, we rewrite (5.10) in a more convenient form. For any $J = (J_1, \dots, J_n)$ with $|J| = 2$, there exist two integers $1 \le j_2 \le j_1 \le n$ such that either $j_2 < j_1$ and

$$J_j = \begin{cases} 1 & \text{if } j = j_1 \text{ or } j = j_2 \\ 0 & \text{otherwise,} \end{cases}$$

or $j_1 = j_2$ and

$$J_j = \begin{cases} 2 & \text{if } j = j_1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, denoting $\bar{h}_4^{j_1,j_2} = h_4^{JJ}$, we have

$$\sum_{|J|=2} h_4^{JJ} \alpha^J \beta^J = \sum_{j_1=1}^{n} \sum_{j_2=1}^{j_1} \bar{h}_4^{j_1,j_2} \alpha_{j_1}\alpha_{j_2}\beta_{j_1}\beta_{j_2}. \tag{5.11}$$

5.2 The asymptotic behavior of $\mathcal{M}(s)$

In this section, we consider the matrix $\mathcal{M}$ defined in (4.14). Bringing the parameter $s \in (0,\delta)$ back into play, we have

$$\mathcal{M}(s) := \begin{bmatrix} M(s) & \omega(s) \\ \\ \omega^T(s) & 0 \end{bmatrix}, \tag{5.12}$$

where $M(s)$ is as in (3.24). We only consider the case of $n = 2$ frequencies here, thus, $\omega(s) = (\omega_1(s), \omega_2(s))$ (this is the vector defined in (NR)) and $M(s)$ is a 2×2 matrix. We are going to examine the behavior of $\mathcal{M}(s)$ as $s \to 0$. To simplify the notation, while keeping the dependence of ω and M on $s > 0$ in mind, when there is no danger of confusion, we will often omit the argument s in ω, M, and related quantities and functions.

Recall from (3.21) that after the Darboux transformation described in Section 3, the Hamiltonian of the reduced equation (3.8) takes the form

$$\Phi(\xi', \eta'; s) = \frac{1}{2}\sum_{j=1}^{2}(-\mu_j(s)(\xi'_j)^2 + (\eta'_j)^2) + \frac{1}{3}\int_{\mathbb{R}^N} a_2(x;s)(\xi' \cdot \varphi(x;s))^3\,dx$$
$$+ \Phi_4(\xi', \eta'; s) + \Phi'(\xi', \eta'; s),$$

where Φ_4 is a homogeneous polynomial in (ξ', η') of degree 4, while Φ' is a function of class $\mathscr{C}^K$ in all its arguments, and of order $\mathcal{O}(|(\xi', \eta')|^5)$ as $(\xi', \eta') \to (0, 0)$. Denote by $\Phi_2(\xi', \eta')$, $\Phi_3(\xi', \eta')$ and $\Phi_4(\xi', \eta')$ the homogeneous polynomials in Φ of degree 2, 3 and 4, respectively.

Remark 5.1. The coefficients of the polynomials Φ_2, Φ_3 and Φ_4 are bounded functions of $s \in (0, \delta)$, see Remark 3.2.

We now change the variables by

$$\xi'_j = \frac{1}{\sqrt{\omega_j}}\xi_j, \quad \eta'_j = \sqrt{\omega_j}\,\eta_j \quad (j \in \{1, 2\}),$$

so the quadratic part of Φ becomes

$$\Phi_2(\xi, \eta) := \frac{1}{2}\sum_{j=1}^{2}\omega_j(\xi_j^2 + \eta_j^2),$$

(similarly as above, in a slight abuse of notation we write $\Phi_2(\xi, \eta)$ for the the function $\Phi_2(\xi(\xi'), \eta(\eta')))$. We now write the cubic terms,

$$\Phi_3(\xi, \eta) = \int_{\mathbb{R}^N} \frac{a_2}{3}(\xi' \cdot \varphi)^3\,dx, \tag{5.13}$$

explicitly in terms of (ξ, η): first,

$$(\xi' \cdot \varphi)^3 = \sum_{j,k,\ell=1}^{2} \xi'_j\xi'_k\xi'_\ell\varphi_j\varphi_k\varphi_\ell = \sum_{j,k,\ell=1}^{2} \frac{\xi_j\xi_k\xi_\ell}{(\omega_j\omega_k\omega_\ell)^{1/2}}\varphi_j\varphi_k\varphi_\ell,$$

so

$$\Phi_3(\xi,\eta) = \frac{1}{3}\sum_{j,k,\ell=1}^{2} \frac{\xi_j\xi_k\xi_\ell}{(\omega_j\omega_k\omega_\ell)^{1/2}}\int_{\mathbb{R}^N} a_2\varphi_j\varphi_k\varphi_\ell dx = \sum_{j,k,\ell=1}^{2}\Theta(j,k,\ell;s)\xi_j\xi_k\xi_\ell, \tag{5.14}$$

where

$$\Theta(j,k,\ell) = \frac{1}{3(\omega_j\omega_k\omega_\ell)^{1/2}}\int_{\mathbb{R}^N} a_2\varphi_j\varphi_k\varphi_\ell dx, \quad (j,k,\ell\in\{1,2\}). \tag{5.15}$$

(Even though Φ_3 is independent of η, for consistency we write it as a function of (ξ,η) anyway.) Of course, the quantities $\Theta(j,k,\ell)$ depend on $s>0$. As functions of s, $\Theta(j,k,\ell;s)$ satisfy the following estimates.

Lemma 5.2. *As $s\to 0$, one has $\omega_2=\omega_2(s)\to 0$ and*

$$\Theta(j,k,\ell;s) = \mathcal{O}\left(\omega_2^{-(j+k+\ell-3)/2}\right) \quad (j,k,\ell\in\{1,2\}).$$

In particular,

$$\Theta(2,2,2;s) = \mathcal{O}\big(\omega_2^{-3/2}\big),$$

and if $(j,k,\ell)\neq(2,2,2)$, then

$$\Theta(j,k,\ell;s) = \mathcal{O}\big(\omega_2^{-1}\big).$$

Proof. By the continuity of the maps $s\in[0,\delta)\mapsto a_2(\cdot;s)\in\mathscr{C}_{\mathrm{b}}^{m+1}$ and $s\in[0,\delta)\mapsto\varphi_j(\cdot;s)\in L^p_{\mathrm{rad}}(\mathbb{R}^N)$ for $1\le p\le\infty$ and $j=1,2$, it follows that the integral on the right hand side of (5.15) is bounded by a constant independent of s. Also, $\omega_1(s)=\sqrt{|\mu_1(s)|}>\omega_2(s)$ for $s\in[0,\delta)$; in particular, $\omega_1(s)$ stays away from 0 as $s\to 0$. Our assertion follows immediately from these remarks. $\square$

Using a similar reasoning, combined with Remark 5.1, one proves the following result:

Corollary 5.3. *The coefficients of the polynomial $\Phi_4(\xi,\eta;s)$ are of order $\mathcal{O}(\omega_2^{-2})$ as $s\to 0$.*

We now carefully examine the fourth-order term resulting from the Birkhoff normal form procedure applied to Φ. Recall that the first transformation in the procedure eliminates all terms of degree 3 in (ξ,η). For any $s>0$, let $\chi_3=\chi_3(\xi,\eta;s)$ be the unique solution of

$$\{\Phi_2,\chi_3\} = -\Phi_3 \tag{5.16}$$

(cp. (5.6)). If ν_3 is the time-1 map generated by χ_3, then (cp. (5.7))

$$\Phi\circ\nu_3 = \Phi_2+\Phi_4+\frac{1}{2}\{\Phi_3,\chi_3\}+\text{h.o.t.}, \tag{5.17}$$

where "h.o.t." stands for terms of order $\mathcal{O}(|(\xi,\eta)|^5)$.

We now use the complex coordinates (5.4) (and, as customary, write $\Phi_2(\alpha,\beta)$ for $\Phi_2(\xi(\alpha,\beta),\eta(\alpha,\beta))$, and similarly for other functions). As before, although not always explicitly indicated in the notation, the functions involved in (5.17) depend on $s>0$.

The complex coordinates will help us to identify the resonant terms (see the definition in the paragraph following (5.5)) of degree 4 in $\Phi\circ\nu_3$. These are the only terms we need to care about; they remain intact after the second transformation in the normal from procedure, while all the nonresonant terms get eliminated.

Lemma 5.4. *For each $s \in (0,\delta)$,*

$$(\Phi \circ \nu_3)(\alpha,\beta) = \Phi_2(\alpha,\beta) + \frac{5}{12\omega_2^4}\left(\int_{\mathbb{R}^N} a_2(x;s)\varphi_2^3(x;s)dx\right)^2 \alpha_2^2\beta_2^2 + \tilde{\Phi}(\alpha,\beta) + \\ + \textit{nonresonant terms} + \textit{h.o.t.}, \quad (5.18)$$

where $\tilde{\Phi}(\alpha,\beta)$ is a homogeneous polynomial of degree 4 whose coefficients are of order $\mathcal{O}(\omega_2^{-7/2})$ as $s \to 0$, "nonresonant terms" contains all nonresonant terms of degree 4 not in $\tilde{\Phi}$, and "h.o.t." stands for terms of order $\mathcal{O}(|(\alpha,\beta)|^5)$.

Proof. Returning to (5.17), we recall that the coefficients of Φ_4 are of order $\mathcal{O}(\omega_2^{-2})$ as $s \to 0$ by Corollary 5.3. Looking for terms with the highest singularity in s, we focus our attention on the term $(1/2)\{\Phi_3,\chi_3\}$ in (5.17). Using Lemma 5.2 and (5.14), we can write

$$\Phi_3(\xi,\eta) = \Theta(2,2,2)\xi_2^3 + \Phi_3' =: \bar{\Phi}_3 + \Phi_3', \quad (5.19)$$

where $\bar{\Phi}_3(\xi,\eta) = \Theta(2,2,2)\xi_2^3$, and Φ_3' is a homogeneous polynomial in (ξ,η) of degree 3, whose coefficients are of order $\mathcal{O}(\omega_2^{-1})$. Equation (5.16) can be rewritten as

$$\{\Phi_2,\chi_3\} = -\bar{\Phi}_3 - \Phi_3'.$$

There are unique $\bar{\chi}_3$ and χ_3', both homogeneous polynomials in (ξ,η) of degree 3, such that

$$\{\Phi_2,\bar{\chi}_3\} = -\bar{\Phi}_3, \quad (5.20)$$

$$\{\Phi_2,\chi_3'\} = -\Phi_3'. \quad (5.21)$$

Thus,

$$\chi_3(\xi,\eta) = \bar{\chi}_3(\xi,\eta) + \chi_3'(\xi,\eta) \quad (5.22)$$

is the unique solution of (5.16). Using (5.19) and (5.22),

$$\frac{1}{2}\{\Phi_3,\chi_3\} = \frac{1}{2}\left(\{\bar{\Phi}_3,\bar{\chi}_3\} + \{\bar{\Phi}_3,\chi_3'\} + \{\Phi_3',\bar{\chi}_3\} + \{\Phi_3',\chi_3'\}\right), \quad (5.23)$$

and the rest of the proof consists of studying the asymptotic behavior (as $s \to 0$) of each of the brackets on the right hand side of (5.23).

Recall that if $J = (j_1,j_2) \in \mathbb{N}^2$ is a multiindex, we write $\alpha^J = \alpha_1^{j_1}\alpha_2^{j_2}$. As mentioned in Section 5.1, the homological equation $\{\Phi_2,\chi_3\} = \psi$ is diagonalized in the complex coordinates (5.4): with respect to the basis given by the monomials

$$\alpha^J\beta^L, \quad J,\ L \in \mathbb{N}^2, \quad |J|+|L| = 3, \quad (5.24)$$

$\{\Phi_2,\chi_3\} = \psi$ becomes a (consistent) linear nonhomogeneous system with the coefficient matrix $\mathrm{diag}(i\omega\cdot(L-J))_{|J|+|L|=3}$. Since the transformation (5.4) (and therefore also the transition matrix from the basis $\{\xi^J\eta^L : |J|+|L| = 3\}$ to the basis (5.24)), is independent of s, when solving the equation $\{\Phi_2,\chi_3\} = \psi$ (by inverting the diagonal matrix) one introduces a singularity of order at most $\mathcal{O}(\omega_2^{-1})$. Therefore, (5.21) in conjunction with the fact that the coefficients of Φ_3' are of order $\mathcal{O}(\omega_2^{-1})$ imply that the coefficients of the polynomial χ_3' are of order $\mathcal{O}(\omega_2^{-2})$ as $s \to 0$.

Next, in the coordinates (α, β),

$$
\begin{aligned}
\bar{\Phi}_3(\alpha,\beta) &= \frac{\Theta(2,2,2)}{(\sqrt{2})^3}(\alpha_2 - i\beta_2)^3 \\
&= \frac{\Theta(2,2,2)}{2\sqrt{2}}(\alpha_2^3 - 3i\alpha_2^2\beta_2 - 3\alpha_2\beta_2^2 + i\beta_2^3).
\end{aligned} \tag{5.25}
$$

By (5.8), to find $\bar{\chi}_3$ we need to divide each term $\alpha^J\beta^L$ in $\bar{\Phi}_3$ by $i\omega\cdot(L-J)$, with $J=(j_1,j_2)$, $L=(\ell_1,\ell_2)$ as in (5.24). Note that none of the terms on the right hand side of (5.25) involves α_1 or β_1, so $j_1=\ell_1=0$, $i\omega\cdot(L-J)=i\omega_2(\ell_2-j_2)$, and the condition $|J|+|L|=3$ becomes $j_2+\ell_2=3$. Thus,

$$
\bar{\chi}_3(\alpha,\beta) = \frac{\Theta(2,2,2)}{2\sqrt{2}i\omega_2}\left(\frac{-1}{3}\alpha_2^3 + 3i\alpha_2^2\beta_2 - 3\alpha_2\beta_2^2 + \frac{i}{3}\beta_2^3\right).
$$

In particular, by Lemma 5.4, $\bar{\chi}_3$ is a homogeneous polynomial in (α,β) of degree 3 whose coefficients are of order $\mathcal{O}(\omega_2^{-5/2})$ as $s\to 0$.

Using the formulas for $\bar{\Phi}_3$ and $\bar{\chi}_3$, we expand the first bracket on the right hand side of (5.23) as follows:

$$
\begin{aligned}
\{\bar{\Phi}_3,\bar{\chi}_3\} &= \frac{\partial\bar{\Phi}_3}{\partial\alpha_2}\frac{\partial\bar{\chi}_3}{\partial\beta_2} - \frac{\partial\bar{\chi}_3}{\partial\alpha_2}\frac{\partial\bar{\Phi}_3}{\partial\beta_2} \\
&= \frac{\Theta(2,2,2)^2}{8i\omega_2}\big[(3\alpha_2^2 - 6i\alpha_2\beta_2 - 3\beta_2^2)(3i\alpha_2^2 - 6\alpha_2\beta_2 + i\beta_2^2) - \\
&\qquad (-\alpha_2^2 + 6i\alpha_2\beta_2 - 3\beta_2^2)(-3i\alpha_2^2 - 6\alpha_2\beta_2 + 3i\beta_2^2)\big] \\
&= \frac{\Theta(2,2,2)^2}{8i\omega_2}[3i + 36i - 9i - (-3i - 36i + 9i)]\alpha_2^2\beta_2^2 + \text{nonresonant terms} \\
&= \frac{15\Theta(2,2,2)^2}{2\omega_2}\alpha_2^2\beta_2^2 + \text{nonresonant terms} \\
&= \frac{15}{18\omega_2^4}\left(\int_{\mathbb{R}^N} a_2(x)\varphi_2^3(x)dx\right)^2\alpha_2^2\beta_2^2 + \text{nonresonant terms},
\end{aligned}
$$

where we have used (5.15) and the fact that the fourth-order resonant terms are of the form (5.11).

From our previous observations, we can easily find the asymptotic behavior of the last three terms on the right hand side of (5.23): the coefficients of the polynomials $\{\bar{\Phi}_3,\chi_3'\}$ and $\{\Phi_3',\bar{\chi}_3\}$ are of order $\mathcal{O}(\omega_2^{-7/2})$ as $s\to 0$, while the coefficients of $\{\Phi_3',\chi_3'\}$ are of order $\mathcal{O}(\omega_2^{-3})$. Setting

$$
\tilde{\Phi} = \frac{1}{2}\left(\{\bar{\Phi}_3,\chi_3'\} + \{\Phi_3',\bar{\chi}_3\} + \{\Phi_3',\chi_3'\}\right) + \Phi_4,
$$

we see that all statements of the lemma are valid and the proof is complete. □

We are now ready for the final step which is to consider the asymptotics of the determinant of the matrix $\mathcal{M}(s)$ in (5.12).

Lemma 5.5. *For all sufficiently small $s>0$, one has $\det\mathcal{M}(s)\neq 0$.*

Proof. Recall that the matrix $M = M(s)$ in (5.12) is the same as the matrix in (3.24). This matrix is determined by the first two steps of the Birkhoff normal form algorithm, since the third and subsequent steps do not alter terms of degree lower than or equal to 4 in (ξ', η') (that is, degree 2 in $I = (I_1, I_2)$).

Denote by m_{ij}, $i, j \in \{1, 2\}$, the entries of M, with $m_{12} = m_{21}$. Expanding,

$$\frac{1}{2} I \cdot MI = \frac{1}{2} \left(m_{11} I_1^2 + (m_{12} + m_{21}) I_1 I_2 + m_{22} I_2^2 \right). \tag{5.26}$$

If (α_j, β_j), $j \in \{1, 2\}$, are as in (5.4), then

$$\alpha_j \beta_j = i(\xi_j^2 + \eta_j^2)/2 = i I_j.$$

Compare (5.26) with the expansion in Lemma 5.4. The resonant term $\alpha_2^2 \beta_2^2 = -I_2^2$ is present in (5.18) in the second term of the right hand side (and its coefficient is explicitly given) and, possibly, in $\tilde{\Phi}(\alpha, \beta)$, in which case its coefficient is of order $\mathcal{O}(\omega_2^{-7/2})$. Thus, as $s \to 0$, we have the following asymptotic formula for $m_{22} = m_{22}(s)$:

$$m_{22} = -\frac{5}{12\omega_2^4} \left(\int_{\mathbb{R}^N} a_2(x; s) \varphi_2^3(x; s) dx \right)^2 + \mathcal{O}(\omega_2^{-7/2}). \tag{5.27}$$

The integral in (5.27) depends continuously on s, thus, by hypothesis (A2), it is bounded from above and below by positive constants for all sufficiently small $s \geq 0$. The other nonresonant terms, namely, multiples of the monomials $\alpha_1^2 \beta_1^2 = -I_1^2$ and $\alpha_1 \beta_1 \alpha_2 \beta_2 = -I_1 I_2$, are all gathered in $\tilde{\Phi}(\alpha, \beta)$ on the right hand side of (5.18). Therefore, Lemma 5.4 implies that m_{11} and m_{12}, being the coefficients of these terms (cp. (5.26)), are of order $\mathcal{O}(\omega_2^{-7/2})$.

Expand the determinant $\det \mathcal{M}(s)$ along the last row:

$$\begin{aligned} \det \mathcal{M}(s) &= \omega_1(m_{12}\omega_2 - m_{22}\omega_1) - \omega_2(m_{11}\omega_2 - m_{21}\omega_1) \\ &= -m_{22}\omega_1^2 + \omega_1\omega_2(m_{12} + m_{21}) - m_{11}\omega_2^2. \end{aligned}$$

Since $m_{12} = m_{21}$ and, as $s \to 0$, $\omega_2(s) \to 0$, m_{11} and m_{12} are of order $\mathcal{O}(\omega_2^{-7/2})$, and $\omega_1(s)$ stays away from zero, using (5.27) we find

$$\det \mathcal{M}(s) = -\omega_1^2 m_{22} + \mathcal{O}(\omega_2^{-7/2}) = \psi(s)\omega_2^{-4} + \mathcal{O}(\omega_2^{-7/2}),$$

where $\psi(s)$ is bounded from below by a positive constant for all sufficiently small $s > 0$. It follows that $\det \mathcal{M}(s) \to \infty$ as $s \to 0$, proving in particular the conclusion of the lemma. □

Proof of Theorem 2.2. The theorem is a direct consequence of Theorem 4.4 and Lemma 5.5 □

References

[1] S. Agmon. *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, volume 29 of *Mathematical Notes*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982.

[2] D. Bambusi. An introduction to Birkhoff normal form. Università di Milano, 2014.

[3] H. W. Broer, S.-N. Chow, Y. Kim, and G. Vegter. A normally elliptic Hamiltonian bifurcation. *Z. Angew. Math. Phys.*, 44:389–432, 1993.

[4] H. W. Broer and G. B. Huitema. A proof of the isoenergetic KAM-theorem from the "ordinary" one. *J. Differential Equations*, 90:52–60, 1991.

[5] B. Grébert. Birkhoff Normal Form and Hamiltonian PDEs. *Séminaries & Congrès*, 15:1–46, 2007.

[6] M. Haragus and G. Iooss. *Local bifurcations, center manifolds, and normal forms in infinite-dimensional dynamical systems.* Springer, 2010.

[7] H. Hofer and E. Zehnder. *Symplectic invariants and Hamiltonian dynamics.* Birkhäuser, 1994.

[8] T. Kato. *Perturbation theory for linear operators.* Springer, 1980.

[9] K. Kirchgässner. Wave solutions of reversible systems and applications. *Journal of Differential Equations*, 45:113–127, 1982.

[10] A. Mielke. *Hamiltonian and Lagrangian flows on center manifolds.* Lecture Notes in Mathematics, 1489. Springer-Verlag, 1991.

[11] P. Poláčik and D. A. Valdebenito. Existence of quasiperiodic solutions of elliptic equations on $\mathbb{R}^{N+1}$ via center manifold and KAM theorems. *J. Differential Equations*, 262:6109–6164, 2017.

[12] J. Pöschel. Integrability of Hamiltonian systems on Cantor sets. *Comm. Pure Appl. Math.*, 35:653–695, 1982.

[13] M. Reed and B. Simon. *Methods of Mathematical Physics, volume IV.* Academic Press, New York, 1978.

[14] J. Scheurle. Bifurcation of quasiperiodic solutions from equilibrium points of reversible dynamical systems. *Arch. Rational Mech. Anal.*, 97:103–139, 1987.

[15] Y Shi, J. Xu, and X. Xu. On quasi-periodic solutions for generalized Boussinesq equation with quadratic nonlinearity. *J. Math. Phys.*, 56(2):022703, 15, 2015.

[16] J. M. Tuwankotta and F. Verhulst. Hamiltonian systems with widely separated frequencies. *Nonlinearity*, 16:689–706, 2003.

[17] C. Valls. Existence of quasi-periodic solutions for elliptic equations on a cylindrical domain. *Comentarii Mathematici Helvetici*, 81:783–800, 2006.

[18] A. Vanderbauwhede and G. Iooss. Center manifold theory in infinite dimensions. In *Dynamics Reported*, volume 1, pages 125–163. Springer-Verlag, 1992.