Symmetry properties of positive solutions of parabolic equations: a survey

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Abstract

This survey is concerned with positive solutions of nonlinear parabolic equations. Assuming that the underlying domain and the equation have certain reflectional symmetries, the presented results show how positive solutions reflect the symmetries. Depending on the class of solutions considered, the symmetries for all times or asymptotic symmetries are established. Several classes of problems, including fully nonlinear equations on bounded domains, quasilinear equations on $\mathbb{R}^N$, asymptotically symmetric equations, and cooperative parabolic systems, are examined from this point of view. Applications of the symmetry results in the study of asymptotic temporal behavior of solutions are also shown.

Contents

1 Introduction: basic problems, results and some history

A conventional wisdom says that "parabolic flows reduce complexity." Although it should not be taken too seriously and universally, there are good examples where it manifests itself. Asymptotic symmetry of solutions, which is one of the main topics of this survey, is such an example. It shows a tendency of positive solutions of certain parabolic equations to "improve their symmetry" as time increases, becoming "symmetric in the limit" as $t \to \infty$.

Historically, first studies of symmetry properties of positive solutions of parabolic equations were carried out after similar properties for elliptic equations had been long understood. These studies brought about interesting qualitative results for parabolic equations, but at the same time they opened new perspectives for looking at the earlier results for elliptic equations. Viewing solutions of elliptic equations as equilibria (steady states) of the corresponding parabolic equations, one naturally tries to understand how their symmetry fits in the broader picture of the parabolic semiflow. For example, examining heteroclinic orbits between symmetric equilibria, one naturally asks if they are symmetric as well. Generalizing, one is subsequently lead to the problem of symmetry of entire solutions, that is, solutions defined for all times, positive and negative. Another symmetry problem is concerned with general solutions of parabolic equations, which are typically defined for positive times only. If the parabolic semiflow admits a Lyapunov functional, which forces bounded solutions to converge to steady states, the symmetry of the latter translates to the asymptotic symmetry of the solutions of the parabolic equation. This immediately raises the question whether the asymptotic symmetry can be established regardless of the presence of any Lyapunov functional, even if the solution does not converge to a steady state. This question is even more interesting for time-dependent parabolic problems, whose solutions typically do not converge to equilibria and their temporal behavior can be very complicated. In this case, the asymptotic symmetrization in space is to be studied independently of the temporal structure. Once it is understood, however, it often proves very useful for studying the temporal behavior of the solutions.

The previous paragraph indicates the sort of problems this survey is devoted to. Considering parabolic equations with certain symmetry properties, we want to understand how their solutions reflect the symmetry. The key issues to be discussed are the asymptotic symmetry properties for the Cauchy problem, symmetry of the entire solutions (and related to this, symmetry of unstable spaces of entire solutions), and applications of these results. We also mention key ideas of the proofs and discuss differences of their use for different type of problems.

To give a more specific overview of the results to be presented in this paper and to put them in context with similar results on elliptic equations, we first consider the following semilinear reaction-diffusion equation

$$u_t = \Delta u + f(t, u), \quad x \in \Omega, t \in J. \tag{1.1}$$

Here Ω is a domain in $\mathbb{R}^N$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, which is Lipschitz continuous with respect to u. We take either $J = (0, \infty)$ and consider a suitable initial-value problem, or $J = (-\infty, \infty)$ in case we want to consider entire solutions.

Although much simpler an equation than fully nonlinear ones examined is the forthcoming sections, for the purposes of the introduction (1.1) is sufficiently representative and it allows us to discuss some key issues in a more rudimentary way. The simplicity of (1.1) consists mainly in the fact that, without any additional assumptions, the equation is invariant under any Euclidean symmetries the domain may have. This is true regardless of the temporal dependence of f on time, which is not restricted by any assumption like periodicity or almost periodicity.

We assume that either $\Omega = \mathbb{R}^N$, or Ω is a bounded domain in $\mathbb{R}^N$ which is symmetric with respect to the hyperplane

$$H_0 := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$$

and convex in x_1 (which means that with any two points (x_1, x'), $(\tilde{x}_1, x')$, differing only at the first component, Ω contains the line segment connecting them). The specific choice of the direction e_1 for the reflectional symmetry is arbitrary, domains symmetric in other directions can be considered equally well. In case Ω is bounded, we complement the equation with Dirichlet boundary condition

$$u(x, t) = 0, \quad x \in \partial\Omega, t \in J. \tag{1.2}$$

If f is independent of t, steady states of (1.1), (1.2) are solutions of the elliptic equation

$$\Delta u + f(u) = 0, \quad x \in \Omega, \tag{1.3}$$

complemented (in case Ω is bounded) with the Dirichlet condition

$$u(x) = 0, \quad x \in \partial\Omega. \tag{1.4}$$

Symmetry theorems for (1.1) have somewhat different flavors and hypotheses for bounded domains Ω and for $\Omega = \mathbb{R}^N$, so we distinguish these case separately.

1.1 Equations on bounded domains

Reflectional symmetry of positive solutions of (1.3), (1.4) was first established by Gidas, Ni and Nirenberg [37]. They proved that if Ω is as above (convex and symmetric in x_1) and smooth (of class C^2), then each positive solution u of (1.3), (1.4) has the following symmetry and monotonicity properties:

$$\begin{aligned} &u(-x_1, x_2, \dots, x_N) = u(x_1, x_2, \dots, x_N) \quad (x \in \Omega), \\ &u_{x_1}(x_1, x_2, \dots, x_N) < 0 \quad (x \in \Omega,\, x_1 > 0). \end{aligned} \tag{1.5}$$

The method of moving hyperplanes, which is the basic geometric technique in their paper, was introduced earlier by Alexandrov [2] and further developed by Serrin [70] ([70] also contains a related result on radial symmetry). Generalizations and extensions of the symmetry result have been made by many authors. In particular, Li [50] extended it to fully nonlinear equations on smooth domains. Later Berestycki and Nirenberg [11] found a way of dealing with fully nonlinear equations on nonsmooth domains employing the elliptic maximum principle for domains with small measure (see also Dancer's contribution [29], where semilinear equations on nonsmooth domains are treated using a different method). There are many other related results, including further developments regarding symmetry of elliptic overdetermined problems, as considered in the original paper of Serrin [70], see for example [1, 73]. Additional references and more detailed overviews can be found in the surveys [7, 46, 58]. Let us also mention a more recent work by Da Lio and Sirakov [27], where the symmetry results are extended to viscosity solutions of general elliptic equations.

The proofs of the above results are based on the method of moving hyperplanes and various forms of the maximum principle. Below we shall indicate how these techniques are typically used. A different approach employing a continuous Steiner symmetrization was used in [12] (see also the survey [46]). It applies to positive solutions of (1.3), (1.4) and, as it relies on the variational structure of the problem and not so much the maximum principle, it allows for extensions of the results of [37] in different directions.

We remark that it is not always possible to generalize the above symmetry result to solutions which are merely nonnegative, rather than strictly positive, (counterexamples are easily constructed if $N = 1$). However, if $N > 1$ and some regularity assumptions are made on the domain, it can be proved that nonnegative solutions are necessarily strictly positive, hence symmetric (see [19, 33] for results of this sort). The issue whether some strict positivity assumption is needed or not will arise again in our discussion of parabolic equations below. Also the convexity of Ω in x_1 is an important assumption without which the result is not valid in general (however, see [47] for a symmetry result involving some nonconvex domains).

If Ω is a ball, say $\Omega = B(0, r_0)$ (0 is the center, r_0 the radius), then the reflectional symmetry theorem can be applied in any direction which leads to the following radial symmetry result. Any positive solution $u(x)$ of (1.3), (1.4) is radially symmetric (it only depends on $r = |x|$) and radially decreasing ($u_r(x) < 0$ for $r \in (0, r_0)$).

There are numerous application of the above symmetry results in further studies of positive solutions of (1.3), (1.4). For example, the radial symmetry property implies that positive solutions can be viewed as solutions of the ordinary differential equation (ODE)

$$u_{rr} + \frac{N-1}{r} u_r + f(u) = 0, \quad r \in (0, r_0), \tag{1.6}$$

and that $u_r(0) = 0$. Thus one immediately gains ODE tools, like the shooting method, for the study of positive solutions. Problems on multiplicities and/or bifurcations of positive solutions become then a lot more elementary. The reflectional symmetry results do not lead to such dramatic simplifications of the problem, but they are still very useful, especially if there are several directions in which the domain is symmetric.

For parabolic problems, such as (1.1), (1.2), first symmetry results of similar nature started to emerge much later. After a prelude [30] devoted to time-periodic solutions, symmetry of general positive solutions of parabolic equations on bounded domains was considered in [4, 5, 44] and later in [6, 63]. With Ω as above (convex and symmetric in x_1) and with suitable symmetry assumptions on the nonlinearity, symmetries of two classes of solutions were examined in these papers. Closer in spirit to the results for elliptic equations are symmetry theorems concerning entire solutions. A typical theorem in this category states that if u is a bounded positive solution of (1.1), (1.2) with $J = \mathbb{R}$ satisfying

$$\inf_{t \in \mathbb{R}} u(x,t) > 0 \quad (x \in \Omega,\, t \in J), \tag{1.7}$$

then u has the symmetry and monotonicity properties (1.5) for each $t \in \mathbb{R}$:

$$\begin{aligned} u(-x_1, x', t) = u(x_1, x', t) \quad (x = (x_1, x') \in \Omega,\, t \in \mathbb{R}),\\ u_{x_1}(x,t) < 0 \quad (x \in \Omega,\, x_1 > 0,\, t \in \mathbb{R}). \end{aligned} \tag{1.8}$$

This result follows from more general theorems of [4, 6], although to be precise we would need to include additional compactness assumptions on u (in the context of the semilinear problem (1.1), (1.2), the boundedness of u alone is sufficient if, for example, Ω has Lipschitz boundary and $t \to f(0,t)$ is a bounded function).

In a different type of symmetry results, nonnegative solutions of the Cauchy-Dirichlet problem for (1.1) are considered. These of course cannot be symmetric, unless they start from a symmetric initial function. However, it can be shown that they "achieve" the symmetry in the limit as $t \to \infty$. To formulate this more precisely assume that u is a bounded positive solution of (1.1), (1.2) with $J = (0, \infty)$ such that for some sequence $t_n \to \infty$

$$\liminf_{n \to \infty} u(x, t_n) > 0 \quad (x \in \Omega). \tag{1.9}$$

Then u is asymptotically symmetric in the sense that

$$\begin{aligned} \lim_{t \to \infty} (u(-x_1, x', t) - u(x_1, x', t)) = 0 \quad (x \in \Omega),\\ \limsup_{t \to \infty} u_{x_1}(x,t) \le 0 \quad (x \in \Omega,\, x_1 > 0). \end{aligned} \tag{1.10}$$

If $\{u(\cdot,t) : t \geq 1\}$ is relatively compact in $C(\bar{\Omega})$, then the asymptotic symmetry of u can be expressed in terms of its limit profiles, that is, elements of its ω-limit set,

$$\omega(u) := \{\phi : \phi = \lim u(\cdot,t_n) \text{ for some } t_n \to \infty\},$$

where the limit is in $C(\bar{\Omega})$ (with the supremum norm). It can be easily shown that condition (1.9) is equivalent to the requirement that there exist at least one element of $\omega(u)$ which is strictly positive in Ω and (1.10) translates to all elements of $\omega(u)$ being symmetric (even) in x_1 and monotone nonincreasing in $x_1 > 0$. This result is proved in a more general setting in [63]. Condition (1.9) is a relatively minor strict positivity condition (note that it is *not* assumed to be valid for all sequences $t_n \to \infty$), which cannot be omitted in general (a counterexample can be found in [63]). It is not needed, however, if the domain is sufficiently regular [44].

There are connections between the two types of symmetry results, the asymptotic symmetry for the Cauchy-Dirichlet problem and the symmetry for entire solutions. Oftentimes, if the nonlinearity is sufficiently regular, the limit profiles of a solution of the Cauchy-Dirichlet problem can be shown to be given by entire solutions of suitable limit parabolic problems. Thus if the limit profiles are all positive (i.e., (1.9) holds for any sequence $t_n \to \infty$), the symmetry of entire solutions for the limit problems can be used to establish the asymptotic symmetry of positive solutions of the original Cauchy-Dirichlet problem. This is how the asymptotic symmetry is proved in [4, 6]. A different approach to asymptotic symmetry is used in [63]. It is based on direct estimates, not relying on any limit equation, thus the regularity and positivity requirements on the nonlinearity and the solutions are significantly relaxed compared to the earlier results. A yet different approach was used in the original paper [44]. While it requires more regularity of the nonlinearity and the domain, it does not assume any strict positivity condition. Also it has an interesting feature in that it shows that the symmetry of a positive solution u improves with time in the sense that a quantity which can be thought of as a measure of symmetry increases strictly along any positive solution which is not symmetric from the start.

See Section 3 for precise formulations of the above results in the context of fully nonlinear parabolic equations. Results on entire solutions given there also include a statement on the symmetry of unstable spaces of positive solutions. In the special case when the positive solution is a steady state of a time-autonomous equation, the statement says that the eigenfunctions of the linearization around the steady state corresponding to negative eigenvalues are all symmetric.

When considering the asymptotic symmetry of solutions, several natural questions come to mind. For example, can the asymptotic symmetry be proved if the equations itself is not symmetric, but rather is merely asymptotically symmetric as $t \to \infty$? One could for example think of equations (1.1) with an extra term added, say

$$u_t = \Delta u + f(t,u) + g(t,u), \quad x \in \Omega, t > 0,$$

where $g(t,u) \to 0$ as $t \to \infty$ for each u. Then one can also consider relaxing other conditions, like the assumption of positivity of the solutions, and only require them to be satisfied asymptotically. Sometimes such problems can be addressed in a relatively simple manner. Indeed, as in the discussion above, if the limit profiles of a solution considered can be shown to be given by positive entire solutions of a limit equation, then, the limit equations being symmetric by assumption, one can apply to these entire solution the symmetry results discussed above. This gives the asymptotic symmetry of the original solution. However, in a general setting such a simple argument may not be applicable, a simple possible reason being that the original solution does not have a strictly positive inferior limit as $t \to \infty$ at each point $x \in \Omega$. In that case, not only is the treatment of asymptotically symmetric problems more complicated, the result may not be true in the form one could expect. Asymptotically symmetric problems are considered in the recent work [35]. We include statements of the main theorems and some discussion in Section 3.3.

1.2 Equations on $\mathbb{R}^N$

Let us now take $\Omega = \mathbb{R}^N$. In their second symmetry paper [38], a sequel to [37], Gidas, Ni and Nirenberg considered elliptic equations on $\mathbb{R}^N$ including the following one

$$\Delta u + f(u) = 0, \quad x \in \mathbb{R}^N. \tag{1.11}$$

They assumed that $f(0) = 0$ and made other hypotheses on the behavior of $f(u)$ near $u = 0$. They proved that each positive solution $u(x)$ of (1.11) which decays to 0 as $|x| \to \infty$ at a suitable rate has to be radially symmetric around some $\xi \in \mathbb{R}^N$ and radially decreasing away from ξ. Later it was proved by Li and Ni [52] that a mere decay (with no specific rate) is sufficient for the symmetry of u if $f(0) = 0$ and f' is nonpositive near zero (under the stronger condition $f'(0) < 0$, this result was also proved in [51]). In both [51] and [52], general fully nonlinear equations satisfying suitable symmetry assumptions are treated. Many other extensions of the symmetry results are available. For example, one can consider some degenerate equations [71] on $\mathbb{R}^N$ or different types of unbounded domains [8, 9, 10, 69]. Again we refer the reader to the surveys [7, 58] for more details and references.

Contrary to elliptic equations, symmetry results for parabolic equations on $\mathbb{R}^N$ did not appear so soon after the first results on bounded domains. The fact that the possible center or hyperplane of symmetry in $\mathbb{R}^N$ is not fixed a priori (unlike on bounded domains) adds an interesting flavor to the symmetry problem and is the cause of major difficulties. Already in the simple autonomous case, the problem is by no means trivial. Consider for example the Cauchy problem

$$\begin{aligned} u_t &= \Delta u + f(u), && x \in \mathbb{R}^N,\, t > 0, \\ u &= u_0, && x \in \mathbb{R}^N,\, t = 0, \end{aligned} \tag{1.12}$$

where f is of class C^1, $f(0) = 0$, and u_0 is a positive continuous function on $\mathbb{R}^N$ decaying to 0 at $|x| = \infty$. Assume the solution u of (1.12) is global, bounded, and localized in the sense that

$$\sup_{t\geq 0} u(x,t) \to 0 \text{ as } |x| \to \infty. \tag{1.13}$$

It is not clear whether u is asymptotically radially symmetric around some center, even if it is known that its ω-limit set $\omega(u)$ consists of steady states, each of them being radially symmetric about *some* center. It is not obvious whether all the functions in $\omega(u)$ share the *same* center of symmetry and, in fact, that is not true in general. A counterexample can be found in [68] where equations (1.12) with $N \geq 11$, $f(u) = u^p$, and p sufficiently large are considered. The proof of the existence of a solution with no asymptotic center of symmetry, as given there, depends on the fact that the steady states, in particular the trivial steady state, are stable in some weighted norms but are unstable in $L^\infty(\mathbb{R}^N)$ (see [40, 41, 67]). If, on the other hand, one makes the assumption $f'(0) < 0$, which in particular implies that $u \equiv 0$ is asymptotically stable in $L^\infty(\mathbb{R}^N)$, then bounded solutions satisfying (1.13) do symmetrize as $t \to \infty$: they actually converge to a symmetric steady state. This convergence result is proved in [15], under slightly stronger hypotheses (exponential decay of the solution at spatial infinity); for more specific nonlinearities proofs can also be found in [26, 34]. The proofs of these convergence theorems depend heavily on energy estimates and are thus closely tied to the autonomous equations.

The symmetry problem for nonautonomous parabolic equations on $\Omega = \mathbb{R}^N$, such as (1.1), was addressed in [61, 62]. The asymptotic symmetry for solutions of the Cauchy problems as well as symmetry for all times for entire solutions is established in these papers, see Section 4 for the statements. It is worthwhile to mention that these result are available for quasilinear equations only, not for fully nonlinear as in the case of bounded domains. The technical reasons for this will be briefly explained in Section 4.

1.3 Cooperative systems

We shall now discuss extensions of the symmetry results to a class of parabolic systems. A model problem is the following cooperative system of reaction-diffusion equations

$$u_t = D(t)\Delta u + f(t,u), \quad (x,t) \in \Omega \times (0,\infty). \tag{1.14}$$

Here $D(t) = \operatorname{diag}(d_1(t), \dots, d_n(t))$ is a diagonal matrix whose diagonal entries are continuous functions bounded above and below by positive constants, and $f = (f_1, \dots, f_n) : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function which is Lipschitz continuous in $u \in \mathbb{R}^n$ and which satisfies the cooperativity condition $\partial f_i(t,u)/\partial u_j \geq 0$ whenever $i \neq j$ and the derivative exists (which is almost everywhere by the Lipschitz continuity). We couple (1.14) with the Dirichlet boundary conditions

$$u_i(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty),\ i = 1, \dots, n. \tag{1.15}$$

In case the diffusion coefficients D and the nonlinearity are time-independent and only steady state solutions are considered, we are lead to the elliptic system

$$D\Delta u + f(u) = 0, \quad x \in \Omega, \tag{1.16}$$

with Dirichlet boundary conditions.

Symmetry properties of positive solutions for such elliptic cooperative systems were established by Troy [75], then by Shaker [72] (see also [24]) who considered equations on smooth bounded domains. In [31], de Figueiredo removed the smoothness assumption on the domain in a similar way as Berestycki and Nirenberg [11] did for the scalar equation. For cooperative systems on the whole space, a general symmetry result was proved by Busca and Sirakov [16] (an earlier more restrictive result can be found in [32]). The cooperativity hypotheses which is assumed in all these references cannot be removed. Without it, neither is the maximum principle applicable nor do the symmetry result hold in general (see [17] and [72] for counterexamples).

Parabolic cooperative systems, such as (1.14), (1.15), were considered in [36], where the asymptotic symmetry of positive solutions is proved (see Section 5 for the results). A new difficulty that arises when dealing with parabolic systems, as opposed to scalar parabolic equations or elliptic systems, is that different components of the positive solution may be very small at different times. This situation has to be handled carefully using Harnack type estimates which were developed in [36] for this purpose.

Similar symmetry results for parabolic systems on $\mathbb{R}^N$ can be proved using ideas form [36] and [61], but they are not documented in literature.

1.4 Applications

When it comes to applications of the symmetry results in further qualitative studies of parabolic equations, the matters are more complicated than in the case of elliptic equations. Even when dealing with radially symmetric solutions of (1.1), the analogue of (1.6) is

$$u_t = u_{rr} + \frac{N-1}{r} u_r + f(t, u) = 0, \tag{1.17}$$

which is still a PDE. Even worse, studying positive solutions of the Cauchy problem, we only have the asymptotic symmetry results, hence (1.17) can only be valid asymptotically, if one can make a sense of that. Nonetheless, the symmetry theorems have proved very useful for further studies of positive solutions of parabolic problems. For example, they have been used in the proofs of convergence results for some autonomous and time-periodic equations We will sketch the proof of such a convergence theorem in Section 7. In that section we also discuss some open problems related to symmetry properties of positive solutions and indicate possible directions of further research.

We would like to emphasize that we have devoted this survey exclusively to symmetry properties related to the positivity of solutions. There are many other results in literature where symmetry is shown to be a consequence of other properties of solutions, like stability (see, for example, [57, 59] and references therein) or being a minimizer for some variational problems (see [55] and reference therein). Different types of parabolic symmetry results can also be found in [25, 45, 56, 69].

2 General notation

The following general notation is used throughout the paper. For $x_0 \in \mathbb{R}^N$ and $r > 0$, $B(x_0, r)$ stands for the ball centered at x_0 with radius r. For a set $\Omega \subset \mathbb{R}^N$ and functions v and w on Ω, the inequalities $v \geq 0$ and $w > 0$ are always understood in the pointwise sense: $v(x) \geq 0$, $w(x) > 0$ $(x \in \Omega)$. For a function z, z^+, z^- stand for the positive and negative parts of z, respectively:

$$\begin{aligned} z^+(x) &= (|z(x)| + z(x))/2 \geq 0, \\ z^-(x) &= (|z(x)| - z(x))/2 \geq 0. \end{aligned}$$

If D_0, D are subsets of $\mathbb{R}^m$ with D_0 bounded, the notation $D_0 \subset\subset D$ means $\bar{D}_0 \subset D$; $\operatorname{diam}(D)$ stands for the diameter of D; and $|D|$ for the (Lebesgue) measure of D (if D is measurable). In each section of the paper, Ω is a fixed domain in $\mathbb{R}^N$ and we denote

$$\begin{aligned} \ell &:= \sup\{x_1 : (x_1, x') \in \Omega \text{ for some } x' \in \mathbb{R}^{N-1}\} \leq \infty, \\ \Omega_\lambda &:= \{x \in \Omega : x_1 > \lambda\}, \\ H_\lambda &:= \{x \in \mathbb{R}^N : x_1 = \lambda\}, \\ \Gamma_\lambda &:= H_\lambda \cap \bar{\Omega}. \end{aligned} \tag{2.1}$$

By P_λ we denote the reflection in the hyperplane H_λ. Note that if Ω is convex in x_1 and symmetric in the hyperplane H_0, then $P_\lambda(\Omega_\lambda) \subset \Omega$ for each $\lambda \in [0, \ell)$. For a function $z(x) = z(x_1, x')$ defined on Ω, let z^λ and $V_\lambda z$ be defined by

$$\begin{aligned} z^\lambda(x) &= z(P_\lambda x) = z(2\lambda - x_1, x'), \\ V_\lambda z(x) &= z^\lambda(x) - z(x) \quad (x \in \Omega_\lambda). \end{aligned} \tag{2.2}$$

3 Fully nonlinear equations on bounded domains

In this section we consider fully nonlinear parabolic problems of the form

$$u_t = F(t, x, u, Du, D^2u), \quad x \in \Omega, t \in J, \tag{3.1}$$

$$u = 0, \qquad x \in \partial\Omega, t \in J. \tag{3.2}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain and J is either $(0, \infty)$ or $(-\infty, T)$ for some $T \leq \infty$. Taking $J = (0, \infty)$ we have the Cauchy-Dirichlet problem in mind, although usually we do not write down the initial condition explicitly (it does not play a role in our analysis). Included in the case $J = (-\infty, T)$ are entire solutions $(T = \infty)$, but the results we state apply also to $T < \infty$.

We make the following assumptions

(D1) $\Omega \subset \mathbb{R}^N$ is a bounded domain which is convex in x_1 and symmetric about the hyperplane $H_0 = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = 0\}$:

$$\{(-x_1, x') : (x_1, x') \in \Omega\} = \Omega.$$

(D2) For each $\lambda > 0$, the set

$$\Omega_\lambda := \{x \in \Omega : x_1 > \lambda\}$$

has only finitely many connected components.

Via a canonical isomorphism, we identify the space of $N \times N$-matrices with $\mathbb{R}^{N^2}$. The nonlinearity $F : (t, x, u, p, q) \mapsto F(t, x, u, p, q)$ is defined on $J \times \Omega \times \mathcal{B}$, where $\mathcal{B}$ is an open convex set in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2}$ which is invariant under the transformation Q defined by

$$\begin{aligned} &Q(u, p, q) = (u, -p_1, p_2, \dots, p_N, \bar{q}), \\ &\bar{q}_{ij} = \begin{cases} -q_{ij} & \text{if exactly one of } i,\ j \text{ equals } 1, \\ \ \ q_{ij} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.3}$$

We assume that F satisfies the following conditions:

(F1) (Regularity) F is continuous on $J \times \bar{\Omega} \times \mathcal{B}$ and Lipschitz in (u, p, q), uniformly with respect to (x, t): there is $\beta > 0$ such that

$$\sup_{x \in \Omega, t \geq 0} |F(t, x, u, p, q) - F(t, x, \tilde{u}, \tilde{p}, \tilde{q})| \leq \beta |(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q})| \qquad ((u, p, q), (\tilde{u}, \tilde{p}, \tilde{q}) \in \mathcal{B}). \tag{3.4}$$

Moreover, F is differentiable with respect to q on $J \times \Omega \times \mathcal{B}$.

(F2) (Ellipticity) There is a constant $\alpha_0 > 0$ such that

$$F_{q_{ij}}(t, x, u, p, q)\xi_i\xi_j \geq \alpha_0 |\xi|^2 \quad ((t, x, u, p, q) \in J \times \Omega \times \mathcal{B},\ \xi \in \mathbb{R}^N). \tag{3.5}$$

Here and below we use the summation convention (summation over repeated indices). For example, in the above formula the left hand side represents the sum over $i, j = 1, \dots, N$.

(F3) (Symmetry and Monotonicity) For any $(t, u, p, q) \in J \times \mathcal{B}$ and any $(x_1, x'), (\tilde{x}_1, x') \in \Omega$ with $\tilde{x}_1 > x_1 \geq 0$ one has

$$F(t, \pm x_1, x', Q(u, p, q)) = F(t, x_1, x', u, p, q) \geq F(t, \tilde{x}_1, x', u, p, q).$$

We consider global classical solutions u of (3.1), (3.2). By this we mean functions $u \in C^{2,1}(\Omega \times J) \cap C(\bar{\Omega} \times J)$ such that

$$(u(x, t), Du(x, t), D^2u(x, t)) \in \mathcal{B} \quad (x \in \Omega, t \in J)$$

and (3.1), (3.2) are satisfied everywhere. We assume the following conditions on u.

(U1) $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded uniformly in $t \in J$.

(U2) The family of functions $u(\cdot, \cdot+s+1)$, $s \in J$, is equicontinuous on $\Omega \times [0,1]$:

$$\lim_{h\to 0} \sup_{\substack{x,\bar{x}\in\bar{\Omega},\, t,\bar{t}\in[0,1],\\ |x-\bar{x}|,|t-\bar{t}|<h,\\ s\in J}} |u(x,t+s+1) - u(\bar{x},\bar{t}+s+1)| = 0.$$

Let us make some comments on our hypotheses and overall set up of the problem. Hypothesis (D2), which rules our some very irregular domains Ω, is assumed for technical reasons and we do not know if it can be removed from the hypotheses or not (it is not needed in some theorems, as explicitly indicated). The differentiability requirement on F with respect to q can be relaxed somewhat (see Remark 2.1 in [63]). Also it should be noted that although we assume the global Lipschitz continuity of F (and similarly for the ellipticity), it is clear that we only need this assumption on the range of (u, Du, D^2u) for a solution u considered. If the range is bounded, then local Lipschitz continuity, uniform in x and t, is a sufficient assumption. However, we would like to emphasize that only the boundedness of u, and not of its derivatives, is required in (U1). While it is often very easy to obtain a bound on a solution u of a Cauchy problem using a comparison argument, bounds on its derivative are usually not so easy to find, in particular if the equation is fully nonlinear. In this connection, we would also like to stress that in this paper we are not concerned with problems of global existence of solutions. Rather, given a global solution, we want to understand if there are consequences of its positivity on its symmetry properties. It is a different question whether a local solution u of an initial value problem which is only assumed bounded in $L^\infty(\Omega)$ is actually global (after the maximal extension of the existence time interval). This is always true in semilinear equations under suitable growth conditions (see [3] for example), but not in fully nonlinear equations. Still, it is of interest to derive a priori information on the behavior of a bounded solution u which is *assumed* global.

The equicontinuity condition (U2) presents no extra restriction if $(0,0,0) \in \mathcal{B}$, $F(x,t,0,0,0)$ is bounded, and Ω satisfies minor regularity conditions (Lipschitz continuity is sufficient, a weaker sufficient condition is Condition (A) of [49]). As shown in [63, Proposition 2.7], (U2) then follows from (U1) and parabolic boundary Hölder estimates. Without any regularity of Ω, (U2) can be shown to hold if (U1) holds together with the following stronger form of the boundary condition:

(U2)' $u(x,t) \to 0$ as $\operatorname{dist}(x,\partial\Omega) \to 0$, uniformly with respect to $t \in J$.

We focus on reflectional symmetries of solutions, but, as noted in the introduction, if the domain and the equation are invariant under all rotations of $\mathbb{R}^N$ (or only rotations of a subspace of $\mathbb{R}^N$), then reflectional symmetries in all admissible variables can be used to establish a rotational symmetry of positive solutions. We formulate, as an illustration, one result on radial symmetry in the next subsection, but refrain from giving such standard corollaries in the other subsections.

3.1 Solutions on $(0, \infty)$

In this subsection we take $J = (0, \infty)$. Hypotheses (U1), (U2) imply in particular that $\{u(\cdot, t) : t > 1\}$ is relatively compact in $C(\bar{\Omega})$. We introduce the ω-limit set of u in this space:

$$\omega(u) = \{z \in C(\bar{\Omega}) : \|u(\cdot, t_k) - z\|_{L^\infty(\Omega)} \to 0 \text{ for some } t_k \to \infty\}.$$

Observe that $\{u(\cdot, t) : t > 1\}$ is also compact in $C_0(\bar{\Omega})$, the closed subspace of $C(\bar{\Omega})$ consisting of all continuous functions on $\bar{\Omega}$ vanishing on $\partial\Omega$. Hence $\omega(u) \subset C_0(\bar{\Omega})$. Also,

$$\operatorname{dist}_{C(\bar{\Omega})}(u(\cdot, t), \omega(u)) \to 0 \text{ as } t \to \infty.$$

We can now state the first asymptotic symmetry result.

Theorem 3.1. *Let* (D1), (D2), (F1)–(F3) *hold and let u be a nonnegative solution of* (3.1), (3.2) *satisfying* (U1), (U2). *Assume that there is $\phi \in \omega(u)$ such that $\phi > 0$ on Ω. Then u is asymptotically symmetric and monotone in x_1. More specifically, for each $z \in \omega(u)$ one has*

$$z(-x_1, x') = z(x_1, x') \quad ((x_1, x') \in \Omega), \tag{3.6}$$

and either $z \equiv 0$ or else z is strictly decreasing in x_1 on Ω_0. The latter holds in the form $z_{x_1} < 0$ on Ω_0, provided $z_{x_1} \in C(\Omega_0)$.

This is the same as Theorem 2.2 of [63]. Note that without extra conditions, like boundedness of spatial derivatives of u, we cannot in general assume that elements of $\omega(u)$ are differentiable. We have $z_{x_1} \in C(\Omega_0)$ for each $z \in \omega(u)$, provided $\{u_{x_1}(\cdot, t) : t > 1\}$ is relatively compact in $C(\bar{B})$ for each closed ball $\bar{B} \subset \Omega_0$.

We remark that the assumption that $\phi > 0$ for some $\phi \in \omega(u)$ cannot be removed even if u is strictly positive. A counterexample is given in [63]. It consists in finding a solution u of the semilinear problem

$$u_t = \Delta u + f(x_2, u) \quad x \in \Omega, t > 0, \tag{3.7}$$

$$u = 0, \qquad x \in \partial\Omega, t > 0, \tag{3.8}$$

where $\Omega := (-1, 1) \times (-1, 1)$ and f is a suitable Lipschitz nonlinearity, with the following properties: u is (strictly) positive in $\Omega \times (0, \infty)$ and $u(\cdot, t) \to z$ in $C_0(\bar{\Omega})$ where $z \in C_0(\bar{\Omega})$ is a nonnegative function such that $z > 0$ in Ω_0 and $z(0, x_2) = 0$ for each $x_2 \in [-1, 1]$. In particular, z is not monotone in $x_1 > 0$.

Observe that Theorem 3.1 in particular implies that if some $\phi \in \omega(u)$ is positive in Ω then each $\phi \in \omega(u) \setminus \{0\}$ is positive in Ω.

One can give several sufficient conditions for the existence of a positive element of $\omega(u)$. For example, Theorem 2.5 of [63] says that if $(0, 0, 0) \in \mathcal{B}$ and

$$\liminf_{x \in \Omega,\, t \to \infty} F(t, x, 0, 0, 0) \geq 0,$$

then either $\omega(u) = \{0\}$ or else there exists $\phi \in \omega(u)$ with $\phi > 0$ in Ω. Also notice that if Ω is a ball, as in Corollary 3.3 below, then the assumption that $\phi > 0$ for some $\phi \in \omega(u)$ is not needed. On more general domains, the following partial symmetry can be proved without that assumption.

Theorem 3.2. *Let* (D1), (F1)–(F3) *hold and assume that Ω_λ is connected for each $\lambda \in (0, \ell)$. Let u be a nonnegative solution of* (3.1), (3.2) *satisfying* (U1), (U2). *Then there exists $\lambda \geq 0$ such that for each $z \in \omega(u)$ the following is true: z is monotone nonincreasing in x_1 on Ω_λ and*

$$z(2\lambda - x_1, x') = z(x_1, x') \quad ((x_1, x') \in \Omega_\lambda). \tag{3.9}$$

A slightly more general version of this theorem is stated as Theorem 2.4 of [63].

Let us now state one result, taken from [63], on radial symmetry, assuming Ω is a ball. Other problems in which Ω is rotationally symmetric and convex in some variables only can be formulated analogously. The proof of such results are rather standard: one applies the reflectional symmetry results in all admissible directions.

Thus let Ω be the unit ball centered at the origin and consider the problem

$$u_t = f(t, |x|, u, |\nabla u|, \Delta u) \quad x \in \Omega, t > 0, \tag{3.10}$$

$$u = 0, \quad x \in \partial\Omega, t > 0, \tag{3.11}$$

where $f(t, r, u, \eta, \xi)$ is defined on $[0, \infty) \times [0, 1] \times \mathbb{R}^3$. We assume that f satisfies the following conditions:

(F1)$_{rad}$ $f : [0, \infty) \times [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous in all variables, Lipschitz in (u, η, ξ), uniformly with respect to (x, t) and differentiable with respect to ξ.

(F2)$_{rad}$ $f_\xi \geq \alpha_0$ on $[0, \infty) \times [0, 1] \times \mathbb{R}^3$ for some positive constant α_0.

(F3)$_{rad}$ f is monotone nonincreasing in r.

Corollary 3.3. *Let Ω be the unit ball,* (F1)$_{rad}$–(F3)$_{rad}$ *hold, and let u be a nonnegative solution of* (3.10), (3.11) *satisfying* (U1), (U2). *Then u is asymptotically radially symmetric and radially nonincreasing. More specifically, each $z \in \omega(u)$ is a radial function and either $z \equiv 0$ or else z is strictly decreasing in $r = |x|$. If $z \in C^1(\Omega)$ and $z \not\equiv 0$ then $z_r < 0$ for $r \in (0, 1)$.*

3.2 Solutions: $(-\infty, T)$ and linearized problems

In this subsection we assume $J = (-\infty, T)$ for some $T \leq \infty$. The first theorem gives the symmetry of positive solutions of (3.1), (3.2) for all times $t \in J$.

Theorem 3.4. *Let* (D1), (D2), (F1)–(F3) *hold and let u be a nonnegative solution of* (3.1), (3.2) *satisfying* (U1), (U2). *Assume that there is a sequence $t_n \to -\infty$ such that*

$$\liminf_{n\to\infty} u(x, t_n) > 0 \quad (x \in \Omega). \tag{3.12}$$

Then u is symmetric and monotone in x_1:

$$\begin{aligned} &u(-x_1,x',t)=u(x_1,x',t) \quad (x=(x_1,x')\in\Omega,\, t\in J),\\ &u_{x_1}(x_1,x',t)<0 \quad (x\in\Omega,\, x_1>0,\, t\in J). \end{aligned} \tag{3.13}$$

The strict positivity condition (3.12) is equivalent to the existence of a positive element of the α-limit set of u,

$$\alpha(u)=\{z\in C(\bar\Omega):\|u(\cdot,t_k)-z\|_{L^\infty(\Omega)}\to 0 \text{ for some } t_k\to-\infty\}.$$

The symmetry of u for all times, as given in (3.13), in particular implies that all elements of $\alpha(u)$ are symmetric and monotone in x_1. Theorem 3.4 is proved in [4, 6] under stronger assumptions, requiring in particular that (3.12) holds for any sequence $t_n\to-\infty$ (in other words, all element of $\alpha(u)$ are positive). As stated, Theorem 3.4 can be proved using arguments similar to those in the proof of Theorem 3.1, as given in [63], although the general scheme of the proof has to be adjusted to solutions on $(-\infty,0]$ (see Section 6 below).

We next want to consider the linearization along a positive solution u of (3.1), (3.2). For that purpose we need the following additional hypothesis on F.

(F4) $F(t,x_1,x',u,p,q)$ is differentiable in x_1, u, p, q.

Assume u is a solution of (3.1), (3.2) and consider the following linearized problem

$$v_t=L^u(x,t)v, \quad x\in\Omega, t\in(-\infty,T), \tag{3.14}$$

$$v=0, \quad x\in\partial\Omega, t\in(-\infty,T), \tag{3.15}$$

where

$$L^u(x,t)v:=\frac{\partial F}{\partial q_{ij}}v_{x_ix_j}+\frac{\partial F}{\partial p_i}v_{x_i}+\frac{\partial F}{\partial u}v \tag{3.16}$$

and the derivatives of F are evaluated at $(t,x,u(x,t),Du(x,t),D^2u(x,t))$.

Theorem 3.5. *Let* (D1), (F1)–(F4) *hold and let u be a nonnegative solution of* (3.1), (3.2) *satisfying* (U1), (U2) *and such that $u(\cdot,t)\in C^3(\Omega)$ for all $t<T$, the derivatives u_{x_1t}, u_{tx_1} exist and are continuous, and* (3.12) *holds for any sequence $t_n\to-\infty$. Let v be a solution of* (3.14), (3.15) *such that*

$$\sup_{t<T}\|v(\cdot,t)\|_{L^\infty(\Omega)}<\infty. \tag{3.17}$$

Assume further that one of the following conditions (i)-(iii) is satisfied.

(i) $\|v(\cdot,t)\|_{L^\infty(\Omega)}\to 0$ *as $t\to-\infty$.*

(ii) There exist $y\in\partial\Omega$ and positive constants ϵ, m such that

$$u_{x_1}\le -m \quad (x\in\Omega\cap B(y,\epsilon), t<T).$$

(iii) There exist $y \in \Omega$ and positive constants ϵ, m such that

$$F_{x_1}(t, x, u(x,t), Du(x,t), D^2u(x,t)) \leq -m \quad (x \in B(y, \epsilon), t < T).$$

Then v is even in x_1: $v(-x_1, x', t) = v(x_1, x', t)$ $(x \in \Omega, t < T)$.

This is essentially a result of [5, 6], although our assumptions are a little different (the arguments of [5, 6] can be easily adapted). A special case of linearizations around periodic solutions was considered in [23].

Let us define the *center-unstable space* of the solution u as the space of all bounded solutions of the linearized equation (3.14), (3.15) and the *unstable space* as the space of all solutions v of (3.14), (3.15) such that condition (i) holds. The previous result says that the unstable space and, under additional conditions on u or F, also the center-unstable space, consists of functions that are even in x_1.

As a corollary to this theorem one can establish the symmetry of generalized eigenfunctions of the linearization at a solutions of an elliptic equation. Specifically, assuming that u is a positive steady state of a time independent problem (3.1), (3.2), consider the eigenvalue problem problem

$$L^u(x)v + \lambda v = 0, \quad x \in \Omega, \tag{3.18}$$

$$v = 0, \quad x \in \partial\Omega, \tag{3.19}$$

where $L^u(x)$ is as in (3.16), only now u and F are independent of t.

Corollary 3.6. *Let* (D1), (F1)–(F4) *hold and let F be independent of t. Let u be a positive steady state of* (3.1), (3.2) *such that $u \in C^3(\Omega)$. Let v be a generalized eigenfunction of* (3.18), (3.19) *corresponding to an eigenvalue λ with* $\operatorname{Re}\lambda \leq 0$. *Further assume that either* $\operatorname{Re}\lambda < 0$ *or (*$\operatorname{Re}\lambda = 0$*,) v is an eigenfunction, and one of the conditions (ii), (iii) of Theorem 3.5 holds. Then v is even in x_1: $v(-x_1, x') = v(x_1, x')$ $(x \in \Omega)$.*

One proves this corollary by applying Theorem 3.5 to a suitable solution of the parabolic problem (3.14), (3.15). For example, if v is an eigenfunction of (3.18), (3.19) and $\operatorname{Re}\lambda \leq 0$, then $\operatorname{Re} e^{\lambda t}v(x)$, $\operatorname{Im} e^{\lambda t}v(x)$ are bounded solutions of (3.14), (3.15). For generalized eigenfunctions slightly more complicated formulas, involving the exponentials and polynomials of t, are used.

Similar results on symmetry of real eigenfunctions of linearized elliptic problems can be derived directly by elliptic comparison arguments (see [20, 28, 54, 43, 74, 76]). The advantage of the approach relying on the parabolic linearized equations is that one can simultaneously treat complex eigenfunctions and generalized eigenfunctions.

We remark that if none of the conditions (i)-(iii) is satisfied, the symmetry conclusion of the Theorem 3.6 may fail. Counterexamples are found in equations (1.3) with positive solutions satisfying simultaneously Dirichlet and Neumann boundary conditions.

3.3 Asymptotically symmetric equations

In this subsection $J = (0, \infty)$. We consider a nonsymmetric perturbation of (3.1), (3.2), which decays to zero as $t \to \infty$, making the problem asymptotically symmetric. Specifically, we consider the following problem

$$\begin{aligned} \partial_t u &= F(t, x, u, Du, D^2u) + G_1(x, t), && (x, t) \in \Omega \times (0, \infty), && (3.20)\\ u(x, t) &= G_2(x, t), && (x, t) \in \partial\Omega \times (0, \infty), && (3.21) \end{aligned}$$

where Ω and F satisfy conditions (D1), (D2), (F1)-(F3). The perturbation terms G_1 and G_2 are assumed to be continuous functions defined on $\Omega \times [0, \infty)$ and $\partial\Omega \times [0, \infty)$, respectively, such that the following decay condition holds.

(G1) $G_1 \in L^{N+1}(\Omega \times (0, T))$ for each $T \in (0, \infty)$, and

$$\lim_{t\to\infty} \max \left\{ \|G_1\|_{L^{N+1}(\Omega\times(t,t+1))}, \|G_2(\cdot, t)\|_{L^\infty(\partial\Omega)} \right\} = 0. \qquad (3.22)$$

For some results we shall require the convergence in (3.22) to be exponential, that is, we shall assume the existence of some $\gamma \in (0, \infty)$ such that the following condition holds.

$(\mathrm{G1})_\gamma$ G_1, G_2, are as in (G1) and there exists a positive constant C such that for each $t \in (0, \infty)$

$$\max \left\{ \|G_1\|_{L^{N+1}(\Omega\times(t,t+1))}, \|G_2(\cdot, t)\|_{L^\infty(\partial\Omega)} \right\} \le Ce^{-\gamma t}.$$

Changing the constant C, the condition on G_1 as stated in $(\mathrm{G1})_\gamma$ can be equivalently formulated as

$$\|G_1\|_{L^{N+1}(\Omega\times(t,\infty))} \le Ce^{-\gamma t} \quad (t > 0).$$

We remark that the form of problem (3.20), (3.21) is general enough to cover equations with nonlinear nonsymmetric perturbation terms. For example, if $G_1(x, t)$ in the first equation is replaced with $\tilde{G}_1(x, t, u, Du, D^2u)$, where $\tilde{G}_1(x, t, u, p, q)$ is a function defined on $\bar{\Omega} \times [0, \infty) \times \mathbb{R}^{1+N+N^2}$ then, given a solution u of the modified equation, we set

$$G_1(x, t) = \tilde{G}_1(x, t, u, Du, D^2u)$$

to bring the equation to the form (3.20). The results formulated below are then applicable, provided assumptions on $\tilde{G}_1$ are such that the resulting function G_1 satisfies (G1) (or $(\mathrm{G1})_\gamma$) for any global solution u one wishes to consider. A condition like

$$\sup_{(x,u,p,q)\in\bar{\Omega}\times\mathbb{R}^{1+N+N^2}} |\tilde{G}_1(x, t, u, p, q)| \le Ce^{-\gamma t} \quad (t > 0)$$

would suffice.

Another way in which the symmetric problem (3.1), (3.2) can be perturbed is allowing the domain Ω to depend on time and be asymptotically symmetric in a suitable sense. This can be covered by problems considered here to some extent. In case the variable domain was sufficiently smooth, by a time dependent change of coordinates one could transform the problem to a problem on a fixed symmetric domain and the equation in the transformed problem would be asymptotically symmetric.

We consider classical solutions of (3.20), (3.21) satisfying (U1), (U2). We do not assume that they are nonnegative, rather we assume that they are asymptotically nonnegative in the sense that all elements of their limit sets are nonnegative.

We first state an asymptotic symmetry theorem assuming that G_1, G_2 decay as in (G1), not necessarily exponentially, but the solutions are assumed asymptotically (strictly) positive. Note that hypotheses (D2) is not needed in this theorem.

Theorem 3.7. *Assume* (D1), (F1)-(F3), (G1) *and let u be a global solution of* (3.20)*,* (3.21) *satisfying* (U1), (U2). *Further assume that for each $z \in \omega(u)$ one has $z > 0$ on Ω. Then for each $z \in \omega(u)$*

$$z(-x_1, x') = z(x_1, x') \quad ((x_1, x') \in \Omega),$$

and z strictly decreasing in x_1 on $\Omega_0 = \{x \in \Omega : x_1 > 0\}$. The latter holds in the form $z_{x_1} < 0$ provided $z_{x_1} \in C(\Omega_0)$.

The is theorem is proved in [35]. Because of the assumption that *all* elements of $\omega(u)$ are positive, this is not a generalization of Theorem 3.1. It is quite natural to ask whether the positivity assumption can be relaxed so as to only require that *at least one* element of $\omega(u)$ be positive, as in Theorem 3.1. The answer is negative if G_1, G_2 are merely assumed to decay as in (G1), or even if they decay exponentially as in $(\mathrm{G1})_\gamma$ with a small exponent $\gamma > 0$. See [35] for a counterexample. As the next theorem of [35] shows, the weaker asymptotic positivity condition is sufficient, provided $(\mathrm{G1})_\gamma$ holds with a sufficiently large $\gamma > 0$.

Theorem 3.8. *Assume* (D1), (D2), (F1) - (F3). *Then there exists $\gamma > 0$ depending on α_0, β, N, Ω such that if* $(\mathrm{G1})_\gamma$ *holds, the following is true. If u is a solution of* (3.20), (3.21) *satisfying* (U1), (U2), *such that $z \geq 0$ for each $z \in \omega(u)$ and there exists $\phi \in \omega(u)$ with $\phi > 0$ on Ω, then for each $z \in \omega(u)$*

$$z(-x_1, x') = z(x_1, x') \quad ((x_1, x') \in \Omega),$$

and either $z \equiv 0$ on Ω or z is strictly decreasing in x_1 on Ω_0. The latter holds in the form $z_{x_1} < 0$ if $z_{x_1} \in C(\Omega_0)$.

Problems with rotational asymptotic symmetry, such as asymptotically symmetric perturbations of problem (3.10), (3.11), can be treated in a usual way, applying the reflectional symmetry results after any rotation, see [35]. An extension of the partial symmetry result of Theorem 3.2 to asymptotically symmetric problems can also be found in [35].

4 Quasilinear equations on $\mathbb{R}^N$

In this section we consider quasilinear parabolic equations on $\mathbb{R}^N$ of the following form (using the summation convention as usual)

$$u_t = A_{ij}(t, u, \nabla u) u_{x_i x_j} + f(t, u, \nabla u), \quad x \in \mathbb{R}^N,\ t \in J. \tag{4.1}$$

Similarly as for problems on bounded domains, we take either $J = (0, \infty)$ (and think of solutions of a Cauchy problem, not always indicating the initial condition explicitly) or $J = (-\infty, T)$ for some $T \le \infty$.

We assume the following conditions on the functions $A_{ij}, f : J \times [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$.

(Q1) $A_{ij}(t, u, p)$, $f(t, u, p)$ are of class C^1 in u and $p = (p_1, \dots, p_N)$ uniformly with respect to t. This means that A_{ij}, f are continuous on $J \times [0, \infty) \times \mathbb{R}^N$ together with their partial derivatives $\partial_u A_{ij}$, $\partial_u f$, $\partial_{p_1} A_{ij}, \dots, \partial_{p_N} A_{ij}$, $\partial_{p_1} f, \dots, \partial_{p_N} f$; and if h stands for any of these partial derivatives, then for each $M > 0$ one has

$$\lim_{\substack{0 \le u, v, |p|, |q| \le M,\ t \in J \\ |u-v| + |p-q| \to 0}} |h(t, u, p) - h(t, v, q)| = 0. \tag{4.2}$$

(Q2) $(A_{ij})_{i,j}$ is locally uniformly elliptic in the following sense: for each $M > 0$ there is $\alpha_0^M > 0$ such that

$$\begin{aligned} &A_{ij}(t, u, p)\xi_i \xi_j \ge \alpha_0^M |\xi|^2 \\ &\qquad\qquad (\xi = (\xi_1, \dots \xi_N) \in \mathbb{R}^N, t \in J, u \in [0, M], |p| \le M). \end{aligned} \tag{4.3}$$

(Q3) $f(t, 0, 0) = 0$ $(t \in J)$ and there is a constant $\gamma > 0$ such that

$$\partial_u f(t, 0, 0) < -\gamma \quad (t \in J). \tag{4.4}$$

(Q4) For each $(t, u, p) \in J \times [0, \infty) \times \mathbb{R}^N$ and $i, j = 1, \dots, N$ one has

$$\begin{gathered} A_{ij}(t, u, P_0 p) = A_{ij}(t, u, p), \quad f(t, u, P_0 p) = f(t, u, p), \\ A_{1j} \equiv A_{j1} \equiv 0 \ \text{ if } j \ne 1. \end{gathered}$$

We consider global positive solutions of (4.1) satisfying the following boundedness and decay conditions:

$$u(x, t), |u_{x_i}(x, t)|, |u_{x_i x_j}(x, t)| < d_0 \quad (x \in \mathbb{R}^N, t \in J), \tag{4.5}$$

where d_0 is a positive constant, and

$$\lim_{|x| \to \infty} \sup\{u(x, t), |u_{x_i}(x, t)|, |u_{x_i x_j}(x, t)| : t \in J, i, j = 1, \dots, N\} = 0 \tag{4.6}$$

The above assumptions on A_{ij} and f are suited for reflectional symmetry results (with respect to a hyperplane perpendicular to $e_1 = (1, 0, \dots, 0)$). We will also formulate rotational symmetry results, under additional assumptions. The assumptions in this section are the same as in [61, 62]. Note in particular, that the equation is quasilinear, rather than fully nonlinear as in the previous section. This is an essential requirement for the method used in [61, 62]. The fact that the nonlinearities are independent of x is not so important. A generalization to x-dependent problems under suitable monotonicity assumptions is not difficult, but the proofs would become a little cumbersome (see [61, Section 4] for a discussion of the hypotheses and a comparison to elliptic problems on $\mathbb{R}^N$).

We further remark that if the functions A_{ij} and f are slightly more regular (Hölder continuous in t) then it is sufficient to assume the boundedness and decay of u and u_{x_i}. For semilinear equations considered in the introduction it is sufficient to assume the boundedness and decay of u alone.

4.1 Solutions on $(0, \infty)$

Let $J = (0, \infty)$. As in Section 3.1, the asymptotic symmetry of solutions u of (4.1) will be described in terms of the functions in the ω-limit set of u. Assuming (4.5), (4.6), the orbit $\{u(\cdot, t) : t \geq 1\}$ is relatively compact in $C_0(\mathbb{R}^N)$ the Banach space of all continuous functions on $\mathbb{R}^N$ decaying to 0 at $|x| = \infty$ (it is equipped with the supremum norm). Consequently, the ω-limit set

$$\omega(u) := \{\phi : \phi = \lim u(\cdot, t_n) \text{ for some } t_n \to \infty\}, \tag{4.7}$$

with the limit in $C_0(\mathbb{R}^N)$, is a nonempty compact subset of $C_0(\mathbb{R}^N)$ and one has

$$\lim_{t\to\infty} \operatorname{dist}_{C_0(\mathbb{R}^N)}(u(\cdot, t), \omega(u)) = 0.$$

We now state a theorem on asymptotic reflectional symmetry and its corollary on asymptotic radial symmetry of positive solutions of (4.5), (4.6) (the proofs are given in [61]).

Theorem 4.1. *Assume* (Q1)–(Q4). *Let u be a global positive solution of* (4.1) *satisfying* (4.5) *and* (4.6). *Then either $u(\cdot, t) \to 0$ in $L^\infty(\mathbb{R}^N)$ or else there exists $\lambda \in \mathbb{R}$ such that for each $\phi \in \omega(u)$ and each x in the halfspace $\mathbb{R}^N_\lambda = \{x : x_1 > \lambda\}$ one has*

$$\begin{aligned} &\phi(P_\lambda x) = \phi(x), \\ &\partial_{x_1}\phi(x) < 0. \end{aligned} \tag{4.8}$$

The next corollary concerns rotationally invariant equations of the form

$$u_t = A(t, u, |\nabla u|)\Delta u + f(t, u, |\nabla u|). \tag{4.9}$$

Corollary 4.2. *Let* (Q1)–(Q3) *hold, let $A_{ij} \equiv 0$ if $i \neq j$, $A_{ii} \equiv A_{jj}$ for any i, j, and*

$$A_{ii}(t, u, p) = A_{ii}(t, u, q), \quad f(t, u, p) = f(t, u, q),$$

whenever $|p| = |q|$. *Let* u *be a positive solution of* (4.1) *satisfying* (4.5) *and* (4.6). *Then either* $u(\cdot, t) \to 0$ *in* $L^\infty(\mathbb{R}^N)$ *or else there exists* $\xi \in \mathbb{R}^N$ *such that for each* $\phi \in \omega(u)$ *and each* $x, y \in \mathbb{R}^N$ *with* $|y - \xi| = |x - \xi| > 0$ *one has one has*

$$\begin{aligned} \phi(x-\xi) &= \phi(y-\xi), \\ \nabla\phi(x-\xi)\cdot(x-\xi) &< 0. \end{aligned}$$

We emphasize that the hyperplane (or center) of symmetry is the same for all elements of $\omega(u)$. In the case of radial symmetry, it in particular follows that, unless u decays to 0 as $t \to \infty$, all $\phi \in \omega(u)$ have a unique point of maximum which is independent of ϕ. This and the fact that $u(\cdot, t)$ approaches its ω-limit set in a C^1 sense (thanks to (4.5), (4.6)) imply that for large t the function $u(\cdot, t)$ has a unique point of maximum which stabilizes (converges as $t \to \infty$) to the point $\xi \in \mathbb{R}^N$. Of course, the solution itself may not stabilize.

4.2 Solutions on $(-\infty, T)$ and linearized problems

Now we take $J = (-\infty, T)$ for some $T \leq \infty$. The results in this subsection are taken from [62]. We consider solutions of (4.1) satisfying (4.5) and (4.6). It can be proved (see [62]) that with $J = (-\infty, T)$, the decay condition (4.6) on u is satisfied if u is sufficiently small near $|x| = \infty$ (uniformly in t) and then the decay is necessarily exponential.

We first formulate theorems on symmetry of positive solutions for all times.

Theorem 4.3. *Assume* (Q1)–(Q4) *and let* u *be a positive solution of* (4.1) *on* $(-\infty, T)$ *satisfying* (4.5) *and* (4.6). *Then there exist* $\lambda \in \mathbb{R}$ *such that for each* $t < T$ *and* x *in the halfspace* $\mathbb{R}^N_\lambda = \{x : x_1 > \lambda\}$ *one has*

$$\begin{aligned} u(P_\lambda x, t) &= u(x, t), \\ \partial_{x_1} u(x, t) &< 0. \end{aligned} \tag{4.10}$$

For equations in the rotationally invariant form (4.9) we have the following result.

Corollary 4.4. *Let* (Q1)–(Q3) *hold. Assume that* $A_{ij} \equiv 0$ *for* $i \neq j$, $A_{ii} \equiv A_{jj}$ *for any* i, j, *and*

$$A_{ii}(t, u, p) = A_{ii}(t, u, q), \quad f(t, u, p) = f(t, u, q)$$

whenever $|p| = |q|$. *Let* u *be a positive solution of* (4.1) *on* $(-\infty, T)$ *satisfying* (4.5) *and* (4.6). *Then there exists* $\xi \in \mathbb{R}^N$ *such that for each* $t < T$ *and each* $x, y \in \mathbb{R}^N$ *with* $|y - \xi| = |x - \xi| > 0$ *one has*

$$u(x, t) = u(y, t), \tag{4.11}$$

$$\nabla u(x - \xi, t) \cdot (x - \xi) < 0. \tag{4.12}$$

Now assume u is a positive solution of (3.1) on $(-\infty, T)$ as in Theorem 4.3. We examine symmetry properties of bounded solutions of the linearized equation

$$v_t = a_{ij}(x,t)v_{x_ix_j} + b_i(x,t)v_{x_i} + c(x,t)v, \quad x \in \mathbb{R}^N, t \in (-\infty, T) \tag{4.13}$$

where

$$\begin{aligned} a_{ij}(x,t) &= A_{ij}(t, u(x,t), \nabla u(x,t)), \\ b_i(x,t) &= f_{p_i}(t, u(x,t), \nabla u(x,t)) + u_{x_kx_\ell}(x,t)A_{k\ell\, p_i}(t, u(x,t), \nabla u(x,t)), \\ c(x,t) &= f_u(t, u(x,t), \nabla u(x,t)) + u_{x_kx_\ell}(x,t)A_{k\ell\, u}(t, u(x,t), \nabla u(x,t)). \end{aligned} \tag{4.14}$$

Theorem 4.5. *Assume* (Q1)–(Q4). *Let u and λ be as in Theorem 4.3 and let v be a bounded solution of* (4.13) *on $(-\infty, T)$. Then there exist a constant c_1 and a solution ψ of* (4.13) *on $(-\infty, T)$ such that*

$$\psi(P_\lambda x, t) = \psi(x,t) \quad (x \in \mathbb{R}^N, t < T), \tag{4.15}$$

and

$$v \equiv \psi + c_1 \partial_{x_1} u. \tag{4.16}$$

If, in addition,

$$\lim_{t \to -\infty} (v(P_\lambda x, t) - v(x,t)) = 0 \tag{4.17}$$

for some x in $\{x : x_1 > \lambda\}$, then $c_1 = 0$.

The corresponding result concerning the radial symmetry reads as follows.

Theorem 4.6. *Let the hypotheses of Corollary 3.3 be satisfied and let u and ξ be as in that corollary. Let v be a bounded solution of* (4.13) *on $(-\infty, T)$. Then there exist constants $c_1, \dots, c_N$ and a solution ψ of* (4.13) *on $(-\infty, T)$ such that $x \mapsto \psi(x - \xi, t)$ is radially symmetric for each $t < T$ and*

$$v \equiv \psi + c_1 \partial_{x_1} u + \dots + c_N \partial_{x_N} u. \tag{4.18}$$

If, in addition, there is a point x with $x_i > \xi_i$, $i = 1, \dots, N$, such that

$$\lim_{t \to -\infty} (v(P^i_{\xi_i} x, t) - v(x,t)) = 0 \quad (i = 1, \dots, N), \tag{4.19}$$

where $P^i_{\xi_i}$ is the reflection in the hyperplane $\{x : x_i = \xi_i\}$, then $v(x - \xi, t)$ is radially symmetric for each $t < T$.

Similarly as for problems on bounded domains, let us define the *center-unstable space* of the solution u as the space of all bounded solutions of the linearized equation (4.13) on $(-\infty, T)$ and the *unstable space* as the space of all solutions v of (4.13) such that $\|v(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to -\infty$. The previous result says that the unstable space is spanned by radial solutions, and the center-unstable space is spanned by radial solutions and the spatial derivatives of u (which, of course, are not radially symmetric). This is a natural extension of the results on the symmetry of the center-unstable space of positive entire solutions of parabolic equations on bounded domains, as formulated in the previous section. If equation (4.1) is semilinear and autonomous, $u_t = \Delta u + f(u)$, and the solution u is a positive steady state, Theorem 4.6 restates a well known result on the unstable space of the linearized Schrödinger operator (see [34, 60], for example).

5 Cooperative systems

We now give extensions of Theorems 3.1, 3.2 to cooperative systems. In this section Ω is a bounded domain in $\mathbb{R}^N$ satisfying conditions (D1), (D2) of Section 3.

We consider the following Dirichlet problem for a system of nonlinear parabolic equations

$$\partial_t u_i = F_i(t, x, u, Du_i, D^2 u_i), \quad (x,t) \in \Omega \times (0,\infty),\ i = 1, \dots, n, \tag{5.1}$$

$$u_i(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty),\ i = 1, \dots, n. \tag{5.2}$$

Here $n \geq 1$ is an integer, $u := (u_1, \cdots, u_n)$, and for each $i \in S := \{1, \dots, n\}$

$$F_i : (t, x, u, p, q) \mapsto F_i(t, x, u, p, q) \in \mathbb{R}^n$$

is a function defined on $[0,\infty) \times \bar{\Omega} \times \mathcal{O}_i$, where $\mathcal{O}_i$ is an open convex subset of $\mathbb{R}^{n+N+N^2}$ invariant under the transformation

$$Q : (u, p, q) \mapsto (u, -p_1, p_2, \dots, p_N, \tilde{q}),$$

$$\tilde{q}_{ij} = \begin{cases} -q_{ij} & \text{if exactly one of } i, j \text{ equals } 1, \\ q_{ij} & \text{otherwise.} \end{cases}$$

Note that while system (5.1) is fully nonlinear, it is only weakly coupled in the sense that the arguments of F_i do not involve the derivatives of u_j for $j \neq i$. We assume that $F = (F_1, \dots, F_n)$ satisfies the following hypotheses:

(N1) For each $i \in S$ the function $F_i : [0,\infty) \times \bar{\Omega} \times \mathcal{O}_i \to \mathbb{R}^n$ is continuous, differentiable with respect to q and Lipschitz continuous in (u, p, q) uniformly with respect to $(x,t) \in \bar{\Omega} \times \mathbb{R}^+$: there is $\beta > 0$ such that

$$|F_i(t, x, u, p, q) - F_i(t, x, \tilde{u}, \tilde{p}, \tilde{q})| \leq \beta |(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q})|$$
$$((x,t) \in \bar{\Omega} \times \mathbb{R}^+,\ (u, p, q), (\tilde{u}, \tilde{p}, \tilde{q}) \in \mathcal{O}_i).$$

(N2) There is a positive constant α_0 such that for all $i \in S$, $(t, x, u, p, q) \in [0,\infty) \times \bar{\Omega} \times \mathcal{O}_i$, and $\xi \in \mathbb{R}^N$ one has

$$\sum_{j,k=1}^{N} \frac{\partial F_i}{\partial q_{jk}}(t, x, u, p, q) \xi_j \xi_k \geq \alpha_0 |\xi|^2.$$

(N3) For each $i \in S$, $(t,u,p,q) \in [0,\infty) \times \mathcal{O}_i$, and any $(x_1,x'), (\tilde{x}_1,x') \in \Omega$ with $\tilde{x}_1 > x_1 \geq 0$ one has

$$F_i(t,\pm x_1,x',Q(u,p,q)) = F_i(t,x_1,x',u,p,q) \geq F_i(t,\tilde{x}_1,x',u,p,q)\,.$$

(N4) For all $i,j \in S$, $i \neq j$, $(t,x,u,p,q) \in [0,\infty) \times \bar{\Omega} \times \mathcal{O}_i$ one has

$$\frac{\partial F_i}{\partial u_j}(t,x,u,p,q) \geq 0,$$

whenever the derivative exists.

In some results we need to complement (N4) with the following condition.

(N5) There exists $\sigma > 0$ such that for any nonempty subsets $I, J \subset S$ with $I \cap J = \emptyset$, $I \cup J = S$ there exist $i \in I$, $j \in J$ such that

$$\frac{\partial F_i}{\partial u_j}(t,x,u,p,q) \geq \sigma$$

for all $(t,x,u,p,q) \in [0,\infty) \times \bar{\Omega} \times \mathcal{O}_i$ such the derivative exists.

Hypotheses (N1)-(N3) are analogous to those assumed in the scalar case in Section 3. Condition (N4) characterizes the cooperativity structure of the system and (N5) is a form of an irreducibility condition. Note that the derivatives in (N4) and (N5) exist almost everywhere by (N1). A model problem is the cooperative reaction-diffusion system (1.14) considered in the introduction.

As in the scalar case, we consider classical solutions of (5.1), (5.2) satisfying the following two conditions

$$\sup_{t\in[0,\infty)} \max_{i\in S} \|u_i(\cdot,t)\|_{L^\infty(\Omega)} < \infty, \tag{5.3}$$

$$\lim_{h\to 0} \sup_{\substack{x,\bar{x}\in\bar{\Omega},\, t,\bar{t}\geq 1,\\ |t-\bar{t}|,|x-\bar{x}|<h}} |u(x,t)-u(\bar{x},\bar{t})| = 0. \tag{5.4}$$

The orbit $\{u(\cdot,t) : t \geq 1\}$ of a solution satisfying (5.3), (5.4) is relatively compact in the space $E = (C(\bar{\Omega}))^n$ and then the ω-limit set

$$\omega(u) = \{z : z = \lim_{k\to\infty} u(\cdot,t_k) \text{ for some } t_k \to \infty\},$$

is nonempty, compact in E and it attracts $u(\cdot,t)$ as $t \to \infty$.

By a nonnegative solution, we mean a solution which is nonnegative in all its components. The following result on asymptotic symmetry of nonnegative solutions is proved in [36].

Theorem 5.1. *Assume* (D1), (D2), (N1) - (N4). *Let* u *be a nonnegative solution of* (5.1), (5.2) *satisfying* (5.3) *and* (5.4). *Assume in addition that one of the following conditions holds:*

(i) *there exists* $\varphi = (\varphi_1, \dots, \varphi_n) \in \omega(u)$ *such that* $\varphi_i > 0$ *in* Ω *for all* $i \in \{1, \dots, n\}$*;*

(ii) (N5) *holds and there is* $\varphi \in \omega(u)$ *such that* $\varphi_i > 0$ *in* Ω *for some* $i \in \{1, \dots, n\}$*.*

Then for each $z = (z_1, \dots, z_n) \in \omega(u)$ *and* $i \in S$*, the function* z_i *is even in* x_1*:*

$$z_i(x_1, x') = z_i(-x_1, x') \quad ((x_1, x') \in \Omega_0), \tag{5.5}$$

and either $z_i \equiv 0$ *on* Ω *or* z_i *is strictly decreasing in* x_1 *on* Ω_0*. The latter holds in the form* $(z_i)_{x_1} < 0$ *if* $(z_i)_{x_1} \in C(\Omega_0)$*.*

It can be proved (see [36]) that if (N5) holds, in addition to all the other hypotheses of Theorem 5.1, then for any $\varphi \in \omega(u)$ the relation $\varphi_i(x) > 0$ holds *for all* $x \in \Omega$ and $i \in S$ as soon as it holds *for some* $x \in \Omega$ and $i \in S$. Hence either $\varphi \equiv 0$ or all its components are strictly positive in Ω. This of course may not be true if (N5) does not hold as can be seen on examples of decoupled systems. Similarly to the scalar case, one can give sufficient conditions for the existence of positive elements of $\omega(u)$. Specifically, if (D1), (D2), (N1), (N2), (N4) hold, and for each $i \in S$ one has $(0, 0, 0) \in \mathcal{O}_i$ and

$$\liminf_{t \to \infty,\, x \in \Omega} F_i(t, x, 0, 0, 0) \geq 0, \tag{5.6}$$

then the following holds for any a nonnegative solution u of (5.1), (5.2) satisfying (5.3) and (5.4): for each $i \in S$ either $\|u_i(\cdot, t)\|_{L^\infty(\Omega)} \to 0$ or else there exists $\varphi \in \omega(u)$ with $\varphi_i > 0$ in Ω.

Without any strict positivity assumption, Theorem 5.1 is not true even in the scalar case (see Section 3.1), but we have the following partial symmetry result similar to Theorem 3.2.

Theorem 5.2. *Let* (D1), (N1) - (N5) *hold and assume that* Ω_λ *is connected for each* $\lambda \in (0, \ell)$*. Let* u *be a nonnegative solution of* (5.1), (5.2) *satisfying* (5.3) *and* (5.4)*. Then there exists* $\lambda \in [0, \ell)$ *such that for all* $z = (z_1, \dots, z_n) \in \omega(u) \setminus \{0\}$ *and* $i \in S$ *one has*

$$z_i(x_1, x') = z_i(2\lambda - x_1, x') \quad ((x_1, x') \in \Omega_\lambda)$$

and z_i *is strictly decreasing in* x_1 *on* Ω_λ*. The latter holds in the form* $(z_i)_{x_1} < 0$ *if* $(z_i)_{x_1} \in C(\Omega_\lambda)$*.*

A more general version of this theorem is proved in [36].

6 On the proofs: a comparison of bounded and unbounded domains

In this section we describe, on a rather general level, the methods used in the proofs of some of the above symmetry results. Doing so, we want to highlight key differences in the proofs of theorems dealing with bounded domains and those for $\mathbb{R}^N$. For simplicity, we consider spatially homogeneous equations

$$u_t = F(t, u, Du, D^2u), \quad x \in \Omega, t \in J, \tag{6.1}$$

where Ω is either a bounded domain as in Section 3 (it satisfies (D1), (D2)) or $\Omega = \mathbb{R}^N$, and $J = (0, \infty)$ or $J = (-\infty, T)$. If Ω is bounded, we couple (6.1) with Dirichlet boundary condition (3.1). In the case $\Omega = \mathbb{R}^N$, we restrict the class of admissible equations to quasilinear ones, as in Section 4, and we assume the hypotheses (Q1)-(Q4) of that section to be satisfied. Likewise, for equations on bounded domains we assume the hypotheses (F1)-(F3), of Section 3 to be satisfied.

Consider first the case $J = (0, \infty)$. Assume that a nonnegative solution u satisfies (U1), (U2) (if Ω is bounded) or (4.5), (4.6) (if $\Omega = \mathbb{R}^N$), and that the ω-limit set $\omega(u)$ of u contains a strictly positive element ϕ. The statement to be proved is that for some λ, $\lambda = 0$ if Ω is bounded, each $z \in \omega(u) \setminus \{0\}$ is symmetric about the hyperplane H_λ and decreasing in $x_1 > \lambda$.

Before discussing some technical points, let us outline a basic scheme of the moving hyperplane method as used in the proofs. For $\lambda \in \mathbb{R}$, taking only $\lambda \in [0, \ell)$ in case Ω is bounded, consider the statement

$(\mathrm{S})_\lambda \qquad \|(V_\lambda u)^-(\cdot, t)\|_{L^\infty(\Omega_\lambda)} \to 0$ as $t \to \infty$

(we are using the notation introduced in Section 2). Note that for each fixed λ, $(\mathrm{S})_\lambda$ is equivalent to the following statement

$$V_\lambda z \geq 0 \text{ in } \Omega_\lambda \text{ for all } z \in \omega(u). \tag{6.2}$$

As the first step of the proof one shows that $(\mathrm{S})_\lambda$ holds if $\lambda < \ell$, $\lambda \approx \ell$ (recall that $\ell = \infty$ if $\Omega = \mathbb{R}^N$). Having proved that, one sets

$$\lambda_0 := \inf\{\mu : (\mathrm{S})_\lambda \text{ holds for each } \lambda \in [\mu, \ell)\}.$$

In view of (6.2), the definition of λ_0 clearly implies that each $z \in \omega(u)$ is monotone nonincreasing in $x_1 > \lambda_0$ (by additional arguments, not to be recalled here, one can show that z is actually strictly decreasing unless it is identical to 0). Also, by continuity, $V_{\lambda_0} z \geq 0$, in Ω_{λ_0}. To prove the symmetry of each $z \in \omega(u)$ about H_{λ_0}, one needs the opposite inequality: $V_{\lambda_0} z \leq 0$ in Ω_{λ_0}. To prove it, set

$$\lambda_0^- := \sup\{\mu : (\mathrm{S}^-)_\lambda \text{ holds for each } \lambda \in (-\ell, \mu] \},$$

where $(\mathrm{S}^-)_\lambda$ is the statement

$(\mathrm{S}^-)_\lambda \qquad \|(V_\lambda u)^-(\cdot, t)\|_{L^\infty(\Omega_\lambda^-)} \to 0$ as $t \to \infty$,

with $\Omega_\lambda^- := \{x \in \Omega : x_1 < \lambda\}$ (we only take $\lambda \in (-\ell, 0]$ if Ω is bounded). The desired symmetry follows from the second key step which consists in showing that $\lambda_0^- = \lambda_0$ if $\Omega = \mathbb{R}^N$ and that $\lambda_0 = 0$ if Ω is bounded (analogous arguments then also give $\lambda_0^- = 0$).

The technical background of the above steps comprises several estimates of solutions of linear parabolic problems of the form

$$v_t = a_{ij}^\lambda(x,t) v_{x_i x_j} + b_i^\lambda(x,t) v_{x_i} + c^\lambda(x,t) v, \quad x \in \Omega_\lambda,\ t > 0, \tag{6.3}$$

$$v \geq 0, \qquad x \in \partial\Omega_\lambda,\ t > 0. \tag{6.4}$$

Using Hadamard formulas and the assumptions on (5.1), one shows that the function $v = V_\lambda u = u^\lambda - u$ satisfies such a linear problem with bounded measurable coefficients (they are bounded by β, the a Lipschitz constant of F with respect to (u, p, q)) and with uniformly elliptic leading part $(a_{ij})_{i,j=1}^N$ (the ellipticity constant coincides with that of (5.1)).

To carry out the first step of the above scheme, i.e., to show that $(S)_\lambda$ holds for $\lambda \approx \ell$, in case Ω is bounded, one applies a parabolic version of the maximum principle on small domains. It says, in essence, that if Ω_λ has sufficiently small measure (which is the case if $\lambda \approx \ell$), then any solution v of (6.3) satisfies

$$\|v^-(\cdot,t)\|_{L^\infty(\Omega_\lambda)} \leq 2e^{-k(t-\tau)} \|v^-(\cdot,\tau)\|_{L^\infty(\Omega_\lambda)} \quad (t > \tau > 0), \tag{6.5}$$

where the exponent $k > 0$ can be taken as large as desired, upon making $|\Omega_\lambda|$ smaller (see [63] for the proof; similar results appeared earlier in [4, 6] and the elliptic predecessor can be found in [11]).

If $\Omega = \mathbb{R}^N$ then Ω_λ is a halfspace and the small domain maximum principle does not help in the first step. Instead, assumption (4.4) is used which implies that the coefficient c^λ in (6.3) is less than $-\gamma/2$ if $\lambda \approx \infty$ and this gives (6.5) with $k = -\gamma/2$. Note that this time k cannot be taken arbitrarily large. This point is not so relevant here, but it is in the second step.

In the second step, first with Ω bounded, it is to be shown that $\lambda_0 = 0$. The proof is by contradiction. Assuming $\lambda_0 > 0$, one first utilizes the definition of λ_0 and the assumption that there is a positive element $\phi \in \omega(u)$ to conclude that $v := V_{\lambda_0} u$ has the following property. There is a sequence $t_n \to \infty$ such that for any $D \subset\subset \Omega_{\lambda_0}$

$$\liminf_{x \in D, n \to \infty} v(x, t_n) > 0. \tag{6.6}$$

Also, since $(S)_{\lambda_0}$ holds,

$$\|v^-(\cdot,t)\|_{L^\infty(\Omega_{\lambda_0})} \to 0 \text{ as } t \to \infty. \tag{6.7}$$

The contradiction consists in showing that (6.7) remains valid if v is replaced with $\tilde{v} := V_{\tilde{\lambda}} u$, where $\tilde{\lambda} < \lambda_0$ is sufficiently close to λ_0 (this means that $(S)_{\tilde{\lambda}}$ holds for all $\tilde{\lambda} \leq \lambda_0$ sufficiently close to λ_0, in contradiction to the definition of λ_0). The argument goes as follows. Take $D \subset\subset \Omega_{\lambda_0}$ such that $\Omega_{\lambda_0} \setminus D$ has small measure, hence also $|\Omega_{\tilde{\lambda}} \setminus D|$ is small for $\tilde{\lambda} \approx \lambda_0$. By the equicontinuity assumption on u, (6.6) remains valid if v is replaced with $\tilde{v} = V_{\tilde{\lambda}} u$, where $\tilde{\lambda} < \lambda_0$ is sufficiently close to λ_0. Also $\|\tilde{v}^-(\cdot, t_n)\|_{L^\infty(\Omega_{\tilde{\lambda}})}$ can be assumed small for such $\tilde{\lambda}$, by (6.7). Now, $\tilde{v}(\cdot,t)$ remains positive in D for t in some interval $[t_n, T)$. Take the maximal T with that property. Two estimates on $\tilde{v}$ are next derived. In the first one, one shows that $\|\tilde{v}^-(\cdot,t)\|_{L^\infty(\Omega_{\tilde{\lambda}}\setminus D)}$ decays exponentially in $t \in [t_n, T)$ with an *exponent that can be assumed as large as desired*, upon adjusting D and taking $\tilde{\lambda}$ closer to λ_0, if necessary. This follows by an application of the small domain maximum principle to (6.3), (6.4), with $\lambda = \tilde{\lambda}$ and with Ω_λ replaced with $\Omega_{\tilde{\lambda}} \setminus \bar{D}$. The second estimate concerns the function $\sup_{x\in D} \tilde{v}(\cdot,t)$ for $t \in [t_n, T)$. While it can decay exponentially as well, one can control its decay by an exponential function with a *fixed exponent* (the exponent is determined by ellipticity constant of (6.3) and a bound on its coefficient - quantities independent of λ). Hence, for $t \in [t_n, T)$ and $\tilde{\lambda} \approx \lambda_0$, $\|\tilde{v}^-(\cdot,t)\|_{L^\infty(\Omega_{\tilde{\lambda}}\setminus D)}$ decays much faster than $\sup_{x\in D} \tilde{v}(\cdot,t)$.

The two estimates are next combined, via an extension of the Harnack inequality which applies to sign-changing solutions. The resulting estimate shows that $\tilde{v}(\cdot,t) = \tilde{v}^+(\cdot,t) - \tilde{v}^-(\cdot,t)$ remains positive in D for all times, i.e., $T = \infty$. In particular, the exponential decay estimate on $\tilde{v}^-$ remains valid for all $t > t_n$ which gives the desired contradiction.

The details of the above proof are given in [63]. We remark that the arguments involving the two kinds of exponential estimates are not needed if $u(x,t)$ is assumed to stay away from zero for all $x \in \Omega$ (this is in particular true when u is a steady state). In that case the proof is considerably simpler (see Appendix A of [63]).

When carrying out the second step of the above scheme for $\Omega = \mathbb{R}^N$, i.e., when ruling out the possibility that $\lambda_0^- < \lambda_0$, one does not have the luxury of using the small domain maximum principle (although a maximum principle for narrow domains does play a role). Still, there are some similarities with the above arguments, in particular, two exponential estimates of $\tilde{v} = V_{\tilde{\lambda}}u$, with $\tilde{\lambda} < \lambda_0$, $\tilde{\lambda} \approx \lambda_0$, are derived in the proof, one in a large bounded domain $D \subset\subset \mathbb{R}^N_{\lambda_0}$ and another one in $\mathbb{R}^N_{\tilde{\lambda}} \setminus D$ (see [61]). The key difference is that in the latter exponential estimate one *cannot assume the exponent to be arbitrarily large*, at best one can assume it to be close to the constant γ from assumption (4.4). Therefore the first estimate has to be done much more carefully: one wants to control $\sup_{x\in D} \tilde{v}(\cdot,t)$ from below by an exponential function with a *sufficiently small* exponent. This can be done invoking a perturbation argument, relying on the behavior of $V_\lambda u$ for $\lambda \geq \lambda_0$. However, the perturbation argument gives the desired result only if the leading coefficients a_{ij} in (6.3) are independent of λ. This is where the quasilinear structure of the equation is used.

The scheme for proving the symmetry results for $J = (-\infty, T)$ is similar, except one considers the following statement in place of $(\mathrm{S})_\lambda$.

$$(\mathrm{T})_\lambda \qquad (V_\lambda u)(x,t) \geq 0 \quad (x \in \Omega_\lambda, t \in (-\infty, T)).$$

Although there are differences at several technical steps, key arguments of the proof for $J = (0, \infty)$ can be easily adapted.

When dealing with asymptotically symmetric equations, such as those considered in Section 3.3, additional technical difficulties stem from the fact that the equation corresponding to (6.3) has an additional term, say $f(x,t)$, which makes it a linear nonhomogeneous equation. One needs to quantify the exponential decay that has to be assumed on f for the exponential estimates on the solutions to go through. Alexandrov-Krylov estimate (see [48, Theorem III.3.9] or [53, Theorem VII.7.1]), which is the parabolic extension of Alexandrov-Bakelman-Pucci estimate (see [18] for example), plays an important role in the proof (see [35]).

7 Applications and some open problems

As mentioned in the introduction, once asymptotic symmetry of positive solutions of a parabolic problem is established, it can be very useful in the study of the temporal asymptotic behavior the solutions. To illustrate this, we consider the following nonautonomous reaction-diffusion problem on the unit ball in $\mathbb{R}^N$:

$$u_t = \Delta u + f(t,u), \quad x \in \Omega := B(0,1), t > 0, \tag{7.1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, t > 0. \tag{7.2}$$

We assume that $f : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is a C^1-function which is time-periodic with period $\tau > 0$: $f(t+\tau,u) \equiv f(t,u)$. We consider a nonnegative bounded solution u of (7.1), (7.2). By standard parabolic estimates, such a solution is also bounded in $C^1(\bar{\Omega})$ hence it satisfies the equicontinuity condition (U2) of Section 3. Therefore, by Corollary 3.3, u is asymptotically radially symmetric. The question is what can be said about the asymptotic behavior of the solution in relation to the time-periodic structure of the solution. The following theorem answers that question.

Theorem 7.1. *Under the above assumptions, let u be a nonnegative bounded solution of* (7.1), (7.2) *. Then there exists a solution $p(x,t)$ of* (7.1), (7.2) *which is τ-periodic in t: $p(\cdot,\cdot+\tau) \equiv p$, and such that*

$$\lim_{t\to\infty} \|u(\cdot,t) - p(\cdot,t)\|_{L^\infty(\Omega)} \to 0.$$

Thus u approaches, as $t \to \infty$, a τ-periodic solution of (7.1), (7.2). Such convergence results are true in one-space dimension in a more general setting and without any symmetry structure (see [14]). However, in higher space dimension the convergence does not take place in general and the symmetry is crucial (see for example [65, 66] and references therein). Theorem 7.1 is proved in [23] (a similar convergence result in the autonomous case can be found in [43]). Let us indicate how symmetry results are used in the proof. We already know, by the asymptotic symmetry, that all elements of $\omega(u)$ are radial functions. An invariance principle for time-periodic equations says that $\omega(u)$ consists of entire solutions of (7.1), (7.2). Since these are radial solutions, they can be viewed as solutions of the one-dimensional problem

$$w_t = w_{rr} + \frac{N-1}{r} w_r + f(t,w) = 0, \quad r \in (0,1), t \in \mathbb{R}, \tag{7.3}$$

$$w_r(0,t) = w(1,t) = 0, \quad t \in \mathbb{R}. \tag{7.4}$$

Moreover, it follows from other general properties of ω-limit sets, that the entire solutions forming $\omega(u)$ belong to what is called the chain recurrent set of the periodic-parabolic problem (7.3), (7.4). Analysis of this one-dimensional problem based on intersection comparison arguments (see [22, 23]) reveals that the chain recurrent set consists of τ-periodic solutions. Once it is known that $\omega(u)$ consists of τ-periodic solutions, one can apply known convergence criteria tailored to such a situation [13, 42]. These give a sufficient condition, formulated in terms of the linearization along solutions in $\omega(u)$, for $\omega(u)$ to actually consists of just one periodic solution. The sufficient condition can be verified using a version of the symmetry result for the linearized equation given by Theorem 3.5, see [23] for details.

In connection with the above sketch, let us mention an interesting open problem. What makes the proof of Theorem 7.1 rather complicated is that even though we know that the positive solution u is asymptotically symmetric, we do not know if its asymptotic behavior is determined by a solution of the radial problem. This is only obtained at the end, after a rather involved analysis of the radial problem. Had it been known that u symmetrizes with an *asymptotic symmetric phase*, that is, that $\|u(\cdot,t)-z(\cdot,t)\|_{L^\infty} \to 0$ for some symmetric solution z, the convergence of u would immediately follow from known convergence results for radial solutions. Whether such an asymptotic phase exists in general and, if not, what are sufficient conditions for its existence is a very interesting problem that has not been addressed.

Another, probably closely related, problem concerns the rate of symmetrization, that is, the rate with which a positive solution converges to the space of symmetric functions. For equations on bounded domains, sufficient conditions for the rate to be exponential are given in [6]. It appears that for problems $\mathbb{R}^N$ the exponential rate can be established as well, although no detailed proof is available yet.

From other applications of symmetry results and related ideas, let us mention Liouville-type theorems. Those assert, that for a specific class of elliptic or parabolic equations on the whole space or a subspace there are no nontrivial nonnegative solutions. A well-known example is the Liouville theorem for the elliptic equation

$$\Delta u + u^p = 0 \quad x \in \mathbb{R}^N,$$

where p is a subcritical Sobolev subcritical exponent: $1 < p < (N+2)/(N-2)_+$ (see [39, 21]). Chen and Li's proof of this result, see [21], uses symmetry: they show that that any hypothetical positive solution would have to be radially symmetric around any origin in $\mathbb{R}^N$. This implies that no such solution can exist. In parabolic equations, the scenario symmetry-then-nonexistence has also been shown to work in some situations. For example, in [64] the following problem on the halfspace $\mathbb{R}^N_0$ is considered under some conditions on the exponent $p > 1$.

$$\begin{aligned} u_t - \Delta u &= u^p, \qquad && x \in \mathbb{R}^N_0, t \in \mathbb{R}, \\ u &= 0, && x \in \partial\mathbb{R}^N_0, t \in \mathbb{R}, \end{aligned}$$

Expanding on an idea of [29], a Liouville theorem for this problem is proved, after first establishing the monotonicity in x_1 of each hypothetical positive solution. This is but an example of symmetry results for a parabolic equation on an unbounded proper subdomain of $\mathbb{R}^N$. Such equations have not been studied systematically from the symmetry point of view.

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