Symmetry properties of positive solutions of parabolic equations on $\mathbb{R}^N$: I. Asymptotic symmetry for the Cauchy problem

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Abstract. We consider quasilinear parabolic equations on $\mathbb{R}^N$ satisfying certain symmetry conditions. We prove that bounded positive solutions decaying to zero at spatial infinity are asymptotically radially symmetric about a center. The asymptotic center of symmetry is not fixed a priori (and depends on the solution) but it is independent of time. We also prove a similar theorem on reflectional symmetry.

Contents

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1 Introduction and statement of the main results

In this paper we consider the Cauchy problem for quasilinear parabolic equations of the following form (using the summation convention)

$$u_t = A_{ij}(t,u,\nabla u)u_{x_i x_j} + f(t,u,\nabla u), \quad x \in \mathbb{R}^N,\, t>0, \tag{1.1}$$
$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N. \tag{1.2}$$

The nonlinearities A_{ij} and f are assumed to satisfy some regularity and ellipticity conditions and the initial function u_0 is taken in $C_0(\mathbb{R}^N)$. Here and below, $C_0(\mathbb{R}^N)$ stands for the space of continuous functions on $\mathbb{R}^N$ decaying to 0 at infinity. Whenever needed, we assume $C_0(\mathbb{R}^N)$ is equipped with the supremum norm.

We consider positive solutions $u(x,t)$ which are global (defined for all $t \geq 0$) bounded and which decay to 0 as $|x| \to \infty$ uniformly with respect to t. Our main theorem asserts that if the nonlinearities satisfy suitable symmetry conditions, then each such solution is asymptotically symmetric.

To simplify the discussion of our results and their relations to earlier theorems, we initially consider semilinear nonautonomous equations

$$u_t = \Delta u + f(t,u), \quad x \in \mathbb{R}^N,\, t>0. \tag{1.3}$$

Our assumptions on f are as follows

(S1) $f(t,u)$ is of class C^1 in u uniformly with respect to t, that is, f and f_u are continuous on $[0,\infty)\times\mathbb{R}$, and for each $M>0$

$$\lim_{\substack{0\leq u,v\leq M,\, t\geq 0\\ |u-v|\to 0}} |f_u(t,u) - f_u(t,v)| = 0,$$

(S2) $f(0,t) = 0 \quad (t \geq 0)$, and there is a constant $\gamma > 0$ such that

$$f_u(t,0) < -\gamma \quad (t \geq 0). \tag{1.4}$$

Consider a bounded solution of (1.3) satisfying

$$\sup_{t\geq 0} u(x,t) \to 0 \text{ as } |x| \to \infty. \tag{1.5}$$

By standard parabolic estimates, for such solution the orbit $\{u(\cdot,t) : t \geq 0\}$ is relatively compact in $C_0(\mathbb{R}^N)$. Consequently, the ω-limit set of u,

$$\omega(u) := \{\phi : \phi = \lim u(\cdot, t_n) \text{ for some } t_n \to \infty\}, \tag{1.6}$$

with the limit in $C_0(\mathbb{R}^N)$, is a nonempty compact subset of $C_0(\mathbb{R}^N)$ and one has

$$\lim_{t\to\infty} \operatorname{dist}_{C_0(\mathbb{R}^N)}(u(\cdot,t), \omega(u)) = 0.$$

In view of the last property, asymptotic symmetry of the solution u in naturally described in terms of the functions in $\omega(u)$. We have the following result to that effect.

Theorem 1.1. *Assume* (S1), (S2) *and let* u *be a positive bounded solution of* (1.3) *satisfying* (1.5)*. Then either* $u(\cdot,t) \to 0$ *in* $C_0(\mathbb{R}^N)$ *as* $t \to \infty$ *or else there exists* $\xi \in \mathbb{R}^N$ *such that for each* $\phi \in \omega(u)$*, the function* $x \mapsto \phi(x-\xi)$ *is radially symmetric and radially decreasing:*

$$\begin{gathered} \phi(x-\xi) = \tilde{\phi}(|x|) \quad (x \in \mathbb{R}^N), \\ \partial_r \tilde{\phi}(r) < 0 \quad (r = |x| > 0). \end{gathered}$$

Similar symmetry theorems for nonautonomous parabolic equations were previously known for bounded domains. For equations on $\mathbb{R}^N$, the asymptotic symmetrization was obtained earlier for time-independent nonlinearities $f = f(u)$. We discuss these results and their relation to Theorem 1.1 below in more details. Before that, to have a broader perspective, we include a brief account of much older symmetry results for elliptic equations.

The first such result is due to Gidas, Ni and Nirenberg. In [20] they proved that each positive solution u of the Dirichlet problem

$$\begin{aligned} \Delta u + f(u) = 0, &\qquad x \in \Omega, \\ u = 0, &\qquad x \in \partial\Omega, \end{aligned} \tag{1.7}$$

reflects the symmetry of the domain. Specifically, if Ω is a bounded domain in $\mathbb{R}^N$ which is convex in one direction, say $e_1 = (1, 0, \ldots, 0)$, and symmetric about a hyperplane orthogonal to that direction, say the hyperplane

$$\{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 = 0\},$$

then u is even in x_1 and decreasing in x_1 for $x_1 > 0$. If Ω is a ball, then u is radially symmetric and decreasing in the radial variable. The proof uses the method of moving hyperplanes which has its origins in the work of Alexandrov [1]; it was further developed by Serrin [38] in his work on radial symmetry in an overdetermined elliptic problem.

The hypotheses in [20] involved, in addition to the Lipschitz continuity of f, a smoothness condition on $\partial\Omega$, but the latter was later shown unnecessary (see [9], [15]). The result has also been generalized in other directions. In particular, in place of the semilinear equation, one can take a fully nonlinear equation $F(D^2u, Du, u) = 0$ satisfying suitable symmetry conditions (see [28]). Also systems of equations with structure that makes the comparison principle applicable have been considered (see [40]). The reader is referred to the surveys [5], [33] for more results and references.

The paper [20] has a sequel in [21] where equations on unbounded domains were considered. In particular, solutions of

$$\begin{aligned} \Delta u + f(u) = 0, &\qquad x \in \mathbb{R}^N, \\ u(x) \to 0, &\qquad \text{as } |x| \to \infty, \end{aligned} \tag{1.8}$$

are examined there. Radial symmetry (about some point) is proved for each positive solution which decays to 0 at infinity at a suitable rate. It has later been shown that a mere decay (with no specific rate) is sufficient for the symmetry if $f(0) = 0$ and f' is nonpositive near zero. The proof can be found in [30] and, under the stronger condition $f'(0) < 0$, in [29]. Both papers give a more general theorem dealing with fully nonlinear equations. Many other extensions of the symmetry results are available. For example, one can consider yet more general equations [39], special classes of elliptic systems [13], and different types of unbounded domains [7, 6, 8, 37]. We again refer the reader to the surveys [5], [33] for more details and references.

The method of moving hyperplanes is the most commonly used basic technique in all the above symmetry results. More recently, other techniques, relying on the variational structure of the problems in question, but not necessarily on the minimization properties of the solution, have been introduced. See [11] for a symmetry result based on the continuous Steiner symmetrization (a discussion of this technique and related ideas can also be found in the survey [25]).

We do mention related, though quite different, theorems where symmetry is shown to be a consequence of other properties of the solutions than positivity. For example, stability is in many cases known to imply some sort of symmetry, see [32, 34] and references therein.

Let us now discuss nonautonomous parabolic equations, first on bounded domains. The simplest case is

$$\begin{aligned} u_t &= \Delta u + f(t,u), && x \in \Omega, && t > 0, \\ u &= 0, && x \in \partial\Omega, && t > 0, \\ u &= u_0, && x \in \Omega, && t = 0. \end{aligned} \tag{1.9}$$

As above, assume Ω is convex in x_1 and symmetric about the hyperplane $\{x_1 = 0\}$. To avoid certain specific issues, assume $N \geq 2$. On f we only impose minor regularity assumptions (it is continuous and Lipschitz in u). To examine the asymptotic symmetry of a bounded positive solution u, we introduce its ω-limit set,

$$\omega(u) := \{\phi : \phi = \lim u(\cdot, t_n) \text{ for some } t_n \to \infty\},$$

(the limit is understood in $L^\infty(\Omega)$). Observe that it is the nonautonomous character of (1.9) that makes the symmetry problem interesting. If the equation is autonomous, $f = f(u)$, each element ϕ of $\omega(u)$ is a nonnegative solution of the corresponding elliptic problem and one gets the symmetry of ϕ trivially from elliptic results.

For (1.9), the symmetry problems was first addressed in [24]. Under slightly stronger regularity requirements, it was shown there that each positive bounded solutions is asymptotically symmetric: if $\phi \in \omega(u)$, then it is even in x_1,

$$\phi(-x_1, x') = \phi(x_1, x') \quad ((x_1, x') \in \Omega),$$

and, unless $\phi \equiv 0$, ϕ is decreasing in $x_1 > 0$. If Ω is a ball centered at the origin, then, similarly as in the elliptic case, each $\phi \in \omega(u)$ is radially symmetric and decreasing in the radial variable.

In an independent work, [2, 3], Babin proved the asymptotic symmetry for fully nonlinear autonomous parabolic equations. More recently, Babin and Sell [4] extended the symmetrization results to fully nonlinear nonautonomous equations on nonsmooth symmetric domains. A related result, the spatial symmetry for each time t of bounded positive solutions defined for all $t \in \mathbb{R}$, is also given in these papers (for time-periodic solutions of time-periodic equations the symmetry was proved in [16]).

Unlike for elliptic equations, symmetry properties of solutions of parabolic equations on unbounded domains, in particular on $\mathbb{R}^N$, are much less understood. The fact that the center of symmetry is not fixed a priori, makes the symmetry problem more difficult. Already in the autonomous case, the problem is by no means trivial. Consider, for example, the Cauchy problem

$$\begin{aligned} u_t &= \Delta u + f(u), && x \in \mathbb{R}^N,\ t > 0, \\ u &= u_0, && x \in \mathbb{R}^N,\ t = 0, \end{aligned} \tag{1.10}$$

where f is of class C^1, $f(0) = 0$, $u_0 \in C_0(\mathbb{R}^N)$, and $u_0 > 0$. Assume the solution u is global, bounded, and satisfies (1.5). The question is whether it is asymptotically radially symmetric as $t \to \infty$ about some point. In contrast to equations on bounded domain, the answer is not immediate even when $\omega(u)$ is known to consist of steady states, each being symmetric about *some* center. It is not clear whether all the functions in $\omega(u)$ share the *same* center of symmetry. In fact, that is not true in general. A counterexample can be found in [36] where equations with $N \geq 11$, $f(u) = u^p$, and p sufficiently large are considered. The proof of the existence a solution with no asymptotic center of symmetry, as given there, depends on the fact that the steady states, in particular the trivial steady state, are stable in some weighted norms but are unstable in $L^\infty(\mathbb{R}^N)$ (see [22, 23, 35]). If, on the other hand, one makes the assumption $f'(0) < 0$, which in particular implies that $u \equiv 0$ is asymptotically stable in $L^\infty(\mathbb{R}^N)$, then bounded solutions satisfying (1.5) do symmetrize as $t \to \infty$: they actually converge to a symmetric steady state. This convergence result is proved in [12], under slightly stronger hypotheses (exponential decay of the solution at spatial infinity); for more specific nonlinearities proofs can also be found in [18], [14].

The symmetry problem for nonautonomous parabolic equations on $\mathbb{R}^N$, such as (1.3), does not seem to have been addressed previously. Techniques used in the above convergence results for $f = f(u)$, specifically, various energy estimates, are bound to autonomous equations. On the other hand, the method of moving hyperplanes, as applied in elliptic equations or parabolic equations on bounded domains, does not work the same way for (1.1). In a customary scenario, the (asymptotic) symmetry of a solution becomes obvious once the process of moving hyperplanes reaches its "limit". In the present case, however, by examining the limit case of the process we can only infer that *some* function in $\omega(u)$ is symmetric. To show the symmetry of *all* functions in $\omega(u)$, we consider the situation occurring when the process is pushed a little beyond the natural "limit". We are then led to investigating solutions of linearized equation that change sign for all t. Careful estimates on how the restriction of such a solution to a compact region "interacts" with spatial infinity are needed. Harnack inequality, maximum principles and constructions of subsolutions are among the basic tools employed in this analysis.

Let us now give precise formulations of our main results in the quasilinear case. The first theorem deals with reflectional symmetry. We fix a direction v, without loss of generality taken to be $v = (1, 0, \dots, 0)$, and assume that equation (1.1) is invariant under reflections in hyperplanes perpendicular to v. More specifically, for $\lambda \in \mathbb{R}$, let P_λ denote the reflection in the hyperplane $\{x \in \mathbb{R}^N : x_1 = \lambda\}$. We assume the following conditions on the functions $A_{ij}, f : [0, \infty) \times [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$.

(Q1) $A_{ij}(t, u, p)$, $f(t, u, p)$ are of class C^1 in u and $p = (p_1, \dots, p_N)$ uniformly with respect to t. This means that A_{ij}, f are continuous on $[0, \infty) \times [0, \infty) \times \mathbb{R}^N$ together with their partial derivatives $\partial_u A_{ij}$, $\partial_u f$, $\partial_{p_1} A_{ij}, \dots, \partial_{p_N} A_{ij}$, $\partial_{p_1} f, \dots, \partial_{p_N} f$; and if h stands for any of these partial derivatives, then for each $M > 0$ one has

$$\lim_{\substack{0 \le u, v, |p|, |q| \le M,\, t \ge 0 \\ |u-v|+|p-q| \to 0}} |h(t, u, p) - h(t, v, q)| = 0. \tag{1.11}$$

(Q2) $(A_{ij})_{i,j}$ is locally uniformly elliptic in the following sense: for each $M > 0$ there is $\alpha_0^M > 0$ such that

$$A_{ij}(t, u, p)\xi_i\xi_j \ge \alpha_0^M |\xi|^2 \qquad (\xi = (\xi_1, \dots \xi_N) \in \mathbb{R}^N, t \ge 0, u \in [0, M], |p| \le M). \tag{1.12}$$

(Q3) There is a constant $\gamma > 0$ such that

$$\partial_u f(t, 0, 0) < -\gamma \quad (t \ge 0); \tag{1.13}$$

(Q4) for each $(t, u, p) \in [0, \infty) \times [0, \infty) \times \mathbb{R}^N$ and $i, j = 1, \dots, N$ one has

$$A_{ij}(t, u, P_0 p) = A_{ij}(t, u, p), \quad f(t, u, P_0 p) = f(t, u, p),$$
$$A_{1j} \equiv A_{j1} \equiv 0 \ \text{ if } j \ne 1.$$

We consider global positive solutions of (1.1) satisfying the following boundedness and decay conditions:

$$u(x,t), |u_{x_i}(x,t)|, |u_{x_ix_j}(x,t)| < d_0 \quad (x \in \mathbb{R}^N, t > 0), \tag{1.14}$$

where d_0 is a positive constant, and

$$\lim_{|x|\to\infty} \sup\{u(x,t), |u_{x_i}(x,t)|, |u_{x_ix_j}(x,t)| : t > 0, i,j = 1,\dots,N\} = 0 \tag{1.15}$$

The ω-limit set, $\omega(u)$, of such a solution u is defined by (1.6). Again, one can show that $\omega(u)$ is a nonempty compact subset of $C_0(\mathbb{R}^N)$ and one has

$$\lim_{t\to\infty} \operatorname{dist}(u(\cdot,t), \omega(u)) = 0$$

with the distance in $C_0(\mathbb{R}^N)$.

Theorem 1.2. *Assume* (Q1)–(Q4). *Let u be a global positive solution of* (1.1) *satisfying* (1.14) *and* (1.15). *Then either $u(\cdot,t) \to 0$ in $L^\infty(\mathbb{R}^N)$ or else there exists $\lambda \in \mathbb{R}$ such that for each $\phi \in \omega(u)$ and each x in the halfspace $\{x : x_1 > \lambda\}$ one has*

$$\begin{aligned} &\phi(P_\lambda x) = \phi(x), \\ &\partial_{x_1}\phi(x) < 0. \end{aligned} \tag{1.16}$$

Note that the hypotheses that the derivatives of u are bounded for all $t > 0$, which we assume for simplicity, can be weakened to the boundedness for all sufficiently large t. This case is reduced to the one above by shifting the time interval. We further remark that if the functions A_{ij} and f are slightly more regular (Hölder continuous in t) then it is sufficient to assume the boundedness and decay of u and u_{x_i}. Also, under such a stronger assumption, one can simplify the proof of the theorem a little by employing "limit equations" of (1.1) as $t \to \infty$, similarly as in [4, 24]. As a corollary of Theorem 1.2, we obtain the following result on asymptotic radial symmetry.

Corollary 1.3. *Let* (Q1)–(Q3) *hold and let $A_{ij} \equiv 0$ if $i \neq j$ and*

$$A_{ii}(t,u,p) = A_{jj}(t,u,q), \quad f(t,u,p) = f(t,u,q)$$

whenever $|p| = |q|$. Let u be a positive solution of (1.1) *satisfying* (1.14) *and* (1.15). *Then either $u(\cdot,t) \to 0$ in $L^\infty(\mathbb{R}^N)$ or else there exists $\xi \in \mathbb{R}^N$ such that for each $\phi \in \omega(u)$ one has*

$$\begin{aligned} \phi(x-\xi) &= \tilde\phi(|x|) \quad (x \in \mathbb{R}^N), \\ \partial_r \tilde\phi(r) &< 0 \quad (r = |x| > 0). \end{aligned}$$

Proof. Assume u does not converge to 0. The equation is invariant under rotations around the origin. Using this and Theorem 1.2, we obtain that for each direction v there is a hyperplane Γ^v perpendicular to v such that all functions in $\omega(u)$ are symmetric with respect to the reflection in Γ^v and have negative derivative in direction v on the halfspace $\{x \cdot v > 0\}$. In particular, all $\phi \in \omega(u)$ assume their maximum at the intersection of the symmetry hyperplanes, which is necessarily a uniquely defined point $\xi \in \mathbb{R}^N$. The conclusion of the corollary readily holds for this point. □

The proof of Theorem 1.2 is given in Section 3. The same arguments, simplified at several places (see in particular the remark following the proof of Lemma 3.8), can be used in the semilinear case under the assumptions of Theorem 1.1. We omit the proof of this theorem. (Theorem 1.1 can also be derived from Corollary 1.3, although some work is needed to show that the stronger hypotheses follow from those in Theorem 1.1.)

In Section 2 we introduce the method of moving hyperplanes. Its application is facilitated by basic results on linear parabolic equations, maximum principles and Harnack inequality, which we also prepare in Section 2.

In Section 4 we discuss possible generalizations of our theorems.

Finally, we mention that in a forthcoming paper we will consider entire solutions (that is, solutions defined for all $t \in \mathbb{R}$) of similar parabolic problems. We establish their symmetry (for all t) and the symmetry of their unstable manifolds.

2 Reflection in hyperplanes and linear equations

In this preliminary section we prepare basic tools, moving hyperplanes, maximum principles and Harnack inequalities, for the proofs of the main theorems.

The following general notation is used throughout the paper. For a set $\Omega \subset \mathbb{R}^N$ and functions v and w on Ω, the inequalities $v \geq 0$ and $v > 0$ are always understood in the pointwise sense:

$$v(x) \geq 0,\ w(x) > 0 \quad (x \in \Omega).$$

For a function $z(x)$ we denote by z^+, z^- the positive and negative parts of

z, respectively:

$$z^+(x) = (|z(x)| + z(x))/2 \geq 0,$$
$$z^-(x) = (|z(x)| - z(x))/2 \geq 0.$$

If D_0 and D are domains in $\mathbb{R}^N$ the notation $D_0 \subset\subset D$ means $\overline{D}_0 \subset D$.

2.1 Reflections in hyperplanes

In applications of the method of moving hyperplanes, one considers a solution u of (1.1) together with another solution u^λ obtained by a reflection. In this subsection we introduce a linear equation satisfied by the difference of these solutions and examine its structure.

For $R, \lambda \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, let

$$\begin{aligned} \mathbb{R}^N_\lambda &:= \{x \in \mathbb{R}^N : x_1 > \lambda\}, \\ \Gamma_\lambda &:= \partial \mathbb{R}^N_\lambda = \{x \in \mathbb{R}^N : x_1 = \lambda\}, \\ B(\xi, R) &:= \{x \in \mathbb{R}^N : |\xi - x| < R\}, \end{aligned} \tag{2.1}$$

As above, let P_λ denote the reflection in the hyperplane Γ_λ. For a function $z(x) = z(x_1, x')$ let z^λ and $V_\lambda z$ be defined by

$$\begin{aligned} z^\lambda(x) &= z(P_\lambda x) = z(2\lambda - x_1, x'), \\ V_\lambda z(x) &= z^\lambda(x) - z(x) \quad (x \in \mathbb{R}^N). \end{aligned} \tag{2.2}$$

Observe that under the symmetry assumption (Q4), if $u(x,t)$ is a solution of (1.1) then so is $u^\lambda(x,t) = u(P_\lambda x, t)$ for any $\lambda \in \mathbb{R}$. It follows that the function $v = V_\lambda u = u^\lambda - u$ satisfies the linear problem

$$v_t = a_{ij}(x,t) v_{x_i x_j} + b_i(x,t) v_{x_i} + c(x,t) v, \quad x \in \mathbb{R}^N,\ t > 0, \tag{2.3}$$
$$v = 0, \qquad x \in \Gamma_\lambda,\ t > 0, \tag{2.4}$$

where

$$
\begin{aligned}
a_{ij}(x,t) &= A_{ij}(t,u(x,t),\nabla u(x,t)),\\
b_i(x,t) &= \int_0^1 f_{p_i}(t,u(x,t),\nabla u(x,t)+s\nabla(u^\lambda(x,t)-u(x,t)))\,ds\\
&\quad + u^\lambda_{x_kx_\ell}(x,t)\int_0^1 A_{k\ell\,p_i}(t,u(x,t),\nabla u(x,t)+s\nabla(u^\lambda(x,t)-u(x,t)))\,ds,\\
c(x,t) &= \int_0^1 f_u(t,u(x,t)+s(u^\lambda(x,t)-u(x,t)),\nabla u^\lambda(x,t))\,ds\\
&\quad + u^\lambda_{x_kx_\ell}(x,t)\int_0^1 A_{k\ell\,u}(t,u(x,t)+s(u^\lambda(x,t)-u(x,t)),\nabla u^\lambda(x,t))\,ds.
\end{aligned}
$$

Sometimes it will be useful to indicate the dependence of the coefficients b_i, c on λ by writing b_i^λ and c^λ.

By (Q1), a_{ij}, $b_i = b_i^\lambda$ and $c = c^\lambda$ are continuous, in fact, uniformly continuous on $\mathbb{R}^N \times [1,T]$ for any $T > 1$. Moreover, by (1.14), they are bounded uniformly with respect to λ:

$$|a_{ij}(x,t)|, |b_i(x,t)|, |c(x,t)| < \beta_0 \quad (x \in \mathbb{R}^N,\ t > 0), \tag{2.5}$$

where β_0 is a constant independent of λ. From the ellipticity of A_{ij}, we get the uniform ellipticity of a_{ij}: there is a constant α_0, such that

$$a_{ij}(x,t)\xi_i\xi_j \geq \alpha_0|\xi|^2 \quad (\xi \in \mathbb{R}^N, x \in \mathbb{R}^N, t > 0). \tag{2.6}$$

Further observe that, by (1.13) and regularity of the functions A_{ij} and f (hypothesis (Q1)), we have $c^\lambda(x,t) < -\gamma$ whenever the values of u, u^λ and their first and second spatial derivatives at (x,t) are all sufficiently small. Hence, the decay condition (1.15) implies that there exists $\rho > 0$ such that

$$c^\lambda(x,t) < -\gamma \quad (t > 0, x \in \mathbb{R}^N, |x| \geq \rho, |P_\lambda x| \geq \rho). \tag{2.7}$$

Finally, note that the uniform continuity of the derivatives of A_{ij} and f (hypothesis (Q1)), in conjunction with (1.14), implies

$$\lim_{|\lambda-\mu|\to 0} \sup_{\substack{x\in\mathbb{R}^N,\, t>0,\\ i=1,\dots,N}} \left(|b_i^\lambda(x,t) - b_i^\mu(x,t)| + |c^\lambda(x,t) - c^\mu(x,t)|\right) = 0. \tag{2.8}$$

2.2 Basic estimates of solutions of linear equations

In this subsection we use the maximum principle and Harnack inequality to derive basic estimates of solutions of the linear equation (2.3). The relation of the coefficients a_{ij}, b_i, c to the nonlinear functions A_{ij} and f is irrelevant here. We consider a general linear equation

$$v_t = a_{ij}(x,t)v_{x_ix_j} + b_i(x,t)v_{x_i} + c(x,t)v, \quad (x,t) \in Q, \tag{2.9}$$

where Q is a cylindrical domain in $\mathbb{R}^{N+1}$ and the coefficients satisfy the following hypothesis.

(L1) a_{ij}, b_i, c are functions in $L^\infty(\mathbb{R}^N \times (0,\infty))$ satisfying (2.5), (2.6) for some positive constants β_0, α_0.

By a *solution* of (2.9) on Q we always mean a strong solution, that is, a function v in the Sobolev space $W^{2,1}_{N+1,loc}(Q)$ such that the equation is satisfied almost everywhere. We usually consider solutions with the additional property $v \in C(\overline{Q})$. We also use the concept of super and sub-solutions. A *supersolution* of (2.9) on Q is a function in $W^{2,1}_{N+1,loc}(Q)$ which satisfies the following inequality almost everywhere in Q:

$$v_t \geq a_{ij}(x,t)v_{x_ix_j} + b_i(x,t)v_{x_i} + c(x,t)v. \tag{2.10}$$

A *subsolution* is defined analogously.

We denote by $\partial_p Q$ the parabolic boundary of Q:

$$\partial_p Q = \{(x,t) \in \partial Q : t_Q \leq t < T_Q\},$$

where

$$t_Q = \inf\{t : (x,t) \in Q\}, \quad T_Q = \sup\{t : (x,t) \in Q\}.$$

The "side" and "bottom" parts of $\partial_p Q$ are defined by

$$\partial_s Q = \{(x,t) \in \partial Q : t_Q < t < T_Q\},$$
$$\partial_b Q = \{(x,t) \in \partial Q : t = t_Q\}.$$

The following lemma is a variant of the maximum principle. The proof of the maximum principle for strong solutions can be found in [31], for example.

Lemma 2.1. *Let Q be a domain in $\mathbb{R}^N \times (0,\infty)$ and let $v \in C(\overline{Q})$ be a solution of* (2.9) *on Q such that*

$$\lim_{(x,t)\in Q,\,|x|\to\infty} v(x,t) = 0. \tag{2.11}$$

Then for each $(x_0,t_0) \in Q$ one has

$$e^{-mt_0} v^{\pm}(x_0,t_0) \le \sup_{(x,t)\in\partial_p Q} e^{-mt} v^{\pm}(x,t),$$

where v^+ and v^- stand for the positive and negative parts of v, respectively, and $m = \sup_{(x,t)\in Q} c(x,t)$.

Remark 2.2. In the previous lemma, the statement regarding v^+ remains valid if v is a subsolution of (2.9) and the statement regarding v^- is valid if v is a supersolution.

We next state a version of Krylov-Safonov Harnack inequality, see [26, 19, 31]. It is commonly formulated for solutions on parabolic cylinders; one passes to general cylindrical domains in a standard way using chains of cylinders (cf. [17]).

Theorem 2.3. *Let d be a positive constant and $D \subset\subset D_1$ be bounded domains in $\mathbb{R}^N$ satisfying* $\mathrm{dist}(\overline{D}, \partial D_1) \ge d$. *For any $\vartheta > 0$ there exist $\nu > 0$, depending only on ϑ, α_0, β_0, D and d, such that if v is a nonnegative solution of* (2.9) *on $Q = D_1 \times (\tau_1,\tau_2)$ for some $\tau_2 > \tau_1 + 2\vartheta$ then*

$$v(x,s) \le \nu\, v(y,t) \ \text{ whenever } x,y \in \overline{D} \text{ and } \tau_1 + \vartheta \le s < s + \vartheta \le t < \tau_2.$$

We combine the previous two results in order to prove the following

Lemma 2.4. *In addition to* (L1), *assume that the functions a_{ij} are continuous. Let d be a positive constant and let D, D_1 be as in Theorem 2.3. There exists a constant $\kappa > 0$, depending only on D, d, α_0 and β_0, with the following property. If $\tau > 1$ and $v \in C(\overline{D}_1 \times (\tau-1,\tau+1])$ is a solution of* (2.9) *on $Q = D_1 \times (\tau-1,\tau+1)$ then*

$$v(x,\tau+1) \ge \kappa \|v^+(\cdot,\tau+\frac{1}{2})\|_{L^\infty(D)} - \sup_{\partial_p(D_1\times(\tau,\tau+1))} e^m v^- \quad (x \in \overline{D}), \tag{2.12}$$

where $m = \sup_{D_1\times(0,\infty)} c$.

Proof. Without loss of generality we may strengthen the hypotheses by requiring that D_1 has smooth boundary. Indeed, assume that the conclusion holds under the stronger hypotheses. Then, given general D_1 and v as in the lemma, we approximate D_1 by a sequence of smooth domains $\tilde{D}_1 \subset\subset D_1$ with $\operatorname{dist}(\overline{D}, \partial\tilde{D}_1) > d/2$. For each such domain (2.12) holds with D_1 replaced by $\tilde{D}_1$ (and with κ independent of $\tilde{D}_1$). Taking the limit, referring to continuity of v on $\overline{D}_1 \times (\tau - 1, \tau + 1)$, we obtain the desired conclusion.

We proceed assuming the strengthened hypotheses. We write v as $v = v_1 + v_2$, where $v_1 = v - v_2$ and v_2 is the solution of (2.9) on $D_1 \times (\tau, \tau + 1)$ satisfying the following initial-boundary conditions:

$$v_2(x, \tau) = -\sigma \quad ((x, t) \in \partial_p(D_1 \times (\tau, \tau + 1)),$$

with $\sigma := \sup_{\partial_p(D_1 \times (\tau, \tau+1))} v^-$. The (unique) solvability of this boundary value problem follows from standard theorems (see [27, 31]) thanks to our stronger regularity assumption. Clearly, v_1 is a solution of (2.9) satisfying

$$v_1(x, t) = v(x, t) + \sigma \quad ((x, t) \in \partial_p(D_1 \times (\tau, \tau + 1)).$$

By the maximum principle, using also $v + \sigma \geq 0$ on $\partial_p(D_1 \times (\tau, \tau + 1))$, we have

$$\begin{aligned} -v_2(x, \tau + 1) &\leq e^m \sigma \quad (x \in D_1), \\ v_1(x, t) &\geq v(x, t) \quad (x \in D_1, t \in (\tau, \tau + 1)), \\ v_1(x, t) &\geq 0 \quad (x \in D_1, t \in (\tau, \tau + 1)). \end{aligned}$$

Applying Theorem 2.3 (with $\vartheta = 1/2$) to the nonnegative solution v_1, we obtain

$$\begin{aligned} v_1(x, \tau + 1) &\geq \nu v_1(y, \tau + 1/2) \\ &\geq \nu \max\{0, v(y, \tau + 1/2)\} = \nu v^+(y, \tau + 1/2) \quad (x, y \in \overline{D}). \end{aligned}$$

Therefore

$$v(x, \tau + 1) = v_1(x, \tau + 1) + v_2(x, \tau + 1) \geq \nu \|v^+(\cdot, \tau + \frac{1}{2})\|_{L^\infty(D)} - e^m \sigma,$$

as stated in (2.12). □

The purpose of the next lemma is to prepare a change of variables in (2.9), which makes the coefficient c negative in a thin slab $\lambda < x_1 < \lambda + \delta$ while not increasing it much elsewhere (cf. Remark 2.6 below). The observation that such transformations are possible on thin slabs or domains of small measure was used and proved in [9] and was attributed there to Varadhan.

Lemma 2.5. *Given positive constants Θ, ε, there exist $\delta > 0$ and a function g on $[0,\infty)$ with the following properties:*

$$g \in C^1[0,\infty) \cap C^2[0,\delta] \cap C^2[\delta,\infty), \tag{2.13a}$$

$$2 \geq g \geq \frac{1}{2}, \tag{2.13b}$$

$$g''(\xi) + \Theta(|g'(\xi)| + g(\xi)) \leq 0 \quad (\xi \in (0,\delta)), \tag{2.13c}$$

$$g''(\xi) + \Theta|g'(\xi)| - \varepsilon g(\xi) \leq 0 \quad (\xi \in (\delta,\infty)). \tag{2.13d}$$

Proof. Choose a smooth function ψ on $[0,1]$ such that

$$\begin{aligned} &\psi(0) = 1, \quad \psi'(0) = -1, \\ &\psi'(\xi) < 0 \quad (\xi \in [0,1)), \\ &\psi(1) = \frac{1}{2}, \quad \psi'(1) = \psi''(1) = 0. \end{aligned}$$

Next, fix a large enough k and a small enough $\delta > 0$ satisfying

$$k > 2\Theta, \quad \delta < \min\left\{1, \frac{1}{k}, \frac{\varepsilon}{2(4k^2 \max|\psi''| + 2k\Theta \max|\psi'|)}\right\}. \tag{2.14}$$

Define g by

$$g(\xi) = \begin{cases} 2 - k\xi^2 & (0 \leq \xi \leq \delta), \\ \tau_1 \psi(\frac{\tau_2}{\tau_1}(\xi - \delta)) & (\delta < \xi \leq \delta + \frac{\tau_1}{\tau_2}), \\ \frac{\tau_1}{2} & (\xi > \delta + \frac{\tau_1}{\tau_2}), \end{cases} \tag{2.15}$$

where $\tau_1 = 2 - k\delta^2$, $\tau_2 = 2k\delta$.

Verifying the matching, one easily checks that (2.13a) is satisfied. Relations (2.13b) follow from (2.14) and nonincrease of ψ. Finally, (2.13c), (2.13d) follow from (2.14) by simple computations which are left to the reader. □

Remark 2.6. When employing Lemma 2.5 below, we rely on the following observation. If v is a solution of (2.3) on $\{x : x_1 > \lambda\} \times (t_0, \infty)$, g is as in Lemma 2.5 with $\Theta = \beta_0/\alpha_0 + 1$, and $\tilde{g}(\xi) := g(\xi - \lambda)$, then $w = v/\tilde{g}$ is a solution of

$$w_t = a_{ij}(x,t)w_{x_i x_j} + \hat{b}_i(x,t)w_{x_i} + \hat{c}(x,t)v, \quad x \in \mathbb{R}^N_\lambda, t > t_0 \tag{2.16}$$

$$w = 0, \qquad x \in \Gamma_\lambda, t > t_0, \tag{2.17}$$

with

$$\begin{aligned} \hat{b}_i(x,t) &= b_i(x,t) + 2a_{11}(x,t)\frac{\tilde{g}'(x_1)}{\tilde{g}(x_1)}, \\ \hat{c}(x,t) &= \frac{c(x,t)\tilde{g}(x_1) + b_1(x,t)\tilde{g}'(x_1) + a_{11}(x,t)\tilde{g}''(x_1)}{\tilde{g}(x_1)}. \end{aligned} \tag{2.18}$$

For $x_1 \in [\lambda, \lambda + \delta)$, we have

$$\begin{aligned} \hat{c}(x,t) &\le a_{11}(x,t)\frac{\frac{\beta_0}{a_{11}(x,t)}(\tilde{g}(x_1) + |\tilde{g}'(x_1)|) + \tilde{g}''(x_1)}{\tilde{g}(x_1)} \\ &\le a_{11}(x,t)\frac{\frac{\beta_0}{\alpha_0}(\tilde{g}(x_1) + |\tilde{g}'(x_1)|) + \tilde{g}''(x_1)}{\tilde{g}(x_1)} \\ &= a_{11}(x,t)\frac{(\Theta - 1)(\tilde{g}(x_1) + |\tilde{g}'(x_1)|) + \tilde{g}''(x_1)}{\tilde{g}(x_1)} \le -\alpha_0. \end{aligned} \tag{2.19}$$

By a similar computation, for any $x_1 > \lambda$,

$$\hat{c}(x,t) \le c(x,t) + \alpha_0\varepsilon. \tag{2.20}$$

The above remarks remain valid, if v is a supersolution, rather than solution, of (2.3) on $\{x : x_1 > \lambda\} \times (t_0, \infty)$; w is then a supersolution of (2.16).

3 Proof of Theorem 1.2: moving hyperplanes

Throughout the section we assume the hypotheses (Q1)–(Q4) of Section 1 to be satisfied. We use the notation introduced in Sections 1 and 2.1.

Fix a positive global solution $u(x,t)$ of (1.1) satisfying the boundedness and decay conditions (1.14), (1.15). We assume that

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} > 0 \tag{3.1}$$

and prove the symmetry property of $\omega(u)$ stated in Theorem 1.2.

We start with basic regularity and positivity properties of u. Below $C_b^1(\mathbb{R}^N)$ stands for the Banach space of all functions that are continuous and bounded on $\mathbb{R}^N$ together with their first order derivatives. It is equipped with a standard C^1 supremum norm.

Lemma 3.1. *The solution u has the following properties:*

(i) The orbit $\{u(\cdot,t) : t \geq 1\}$ is relatively compact in $C_b^1(\mathbb{R}^N)$ and one has

$$\operatorname{dist}_{C_b^1(\mathbb{R}^N)}(u(\cdot,t), \omega(u)) \to 0 \text{ as } t \to \infty. \tag{3.2}$$

(ii) Given any ball $B \subset \mathbb{R}^N$, there exists a constant $k(B) > 0$ such that

$$u(x,t) \geq k(B) \quad (x \in \overline{B}, t \geq 1). \tag{3.3}$$

Proof. Statement (i) follows directly from (1.14), (1.15), and the definition of $\omega(u)$.

To prove (ii), first observe that (3.1) actually gives

$$\liminf_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} > 0. \tag{3.4}$$

Indeed, by (Q1), (Q3), we have $\partial_u f(t,u,0) < 0$ for all $t \geq 0$ and $u > 0$ sufficiently small. Thus all solutions of the ODE $\dot{\xi} = f(t,\xi,0)$ starting near zero converge to zero. Using the ODE solutions in comparison with u (note that $\xi - u$ satisfies a linear equation), we obtain that if $u(\cdot,t)$ is sufficiently close to 0 in $L^\infty(\mathbb{R}^N)$ for some t, then necessarily $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to \infty$. This being forbidden by (3.1), we obtain (3.4). It follows that there is a constant $s > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} > s \quad (t \geq 1).$$

By the decay condition (1.15), if the ball B is sufficiently large (which we may assume without loss of generality), then also

$$\sup\{u(x,t) : x \in B\} > s \quad (t \geq 1).$$

Now, since 0 is a steady state of (1.1), u can be viewed as a solution of a linear equation (2.9), as in Section 2.2. Applying the Harnack inequality, Theorem 2.3 with $D = B$ and D_1 equal to a larger ball, we obtain (3.3). □

We now set up the method of moving hyperplanes. Consider the statement

$$(\mathrm{S})_\lambda \qquad V_\lambda z = z^\lambda - z > 0 \text{ in } \mathbb{R}^N_\lambda \text{ for all } z \in \omega(u).$$

We carry out the proof of Theorem 1.2 in the following three steps.

STEP 1. $(\mathrm{S})_\lambda$ holds if λ is sufficiently large.

STEP 2. We set

$$\lambda_\infty = \inf\{\mu\colon (\mathrm{S})_\lambda \text{ holds for all } \lambda \geq \mu\}, \tag{3.5}$$

and prove that $\lambda_\infty > -\infty$ and $V_{\lambda_\infty} z \equiv 0$ for *some* $z \in \omega(u)$. Moreover, we prove that $\partial_{x_1} z < 0$ on $\mathbb{R}^N_{\lambda_\infty}$ for each $z \in \omega(u)$.

STEP 3. We prove that $V_{\lambda_\infty} z \equiv 0$ for *all* $z \in \omega(u)$. Assuming the contrary, we find a contradiction by examining the function $V_\lambda u$ for $\lambda < \lambda_\infty$, $\lambda \approx \lambda_\infty$.

All these steps depend on properties of the function $v = V_\lambda u$ viewed as a solution of the linear problem (2.3), (2.4). We assume that constants β_0, α_0, γ and ρ are fixed such that the coefficients of (2.3) satisfy relations (2.5), (2.6) and (2.7). Note also that the coefficients are continuous. Denoting

$$G_\lambda = B(0,\rho) \cup P_\lambda B(0,\rho), \tag{3.6}$$

condition (2.7) can be rewritten as

$$c^\lambda(x,t) < -\gamma \quad (t > 0, x \in \mathbb{R}^N_\lambda \setminus G_\lambda). \tag{3.7}$$

The following lemma gives a useful criterion for $(\mathrm{S})_\lambda$ to hold. It will be used in Steps 1 and 2.

Lemma 3.2. *There exists a constant $\delta_1 > 0$ independent of λ with the following property. Statement $(\mathrm{S})_\lambda$ holds provided $v = V_\lambda u$ satisfies*

$$v(x,t) > 0 \quad (x \in D_0, t \geq t_0) \tag{3.8a}$$

$$\liminf_{t\to\infty} \|v(\cdot,t)\|_{L^\infty(D_0)} > 0, \tag{3.8b}$$

for some $t_0 > 0$ and some domain $D_0 \subset \mathbb{R}^N_\lambda$ such that

$$D_0 \supset G_\lambda \cap \{x \in \mathbb{R}^N_\lambda : x_1 \geq \lambda + \delta_1\}. \tag{3.9}$$

Proof. We show that the conclusion holds for $\delta_1 = \delta$, where δ is as in Lemma 2.5 with $\Theta = \beta_0/\alpha_0 + 1$ and $\varepsilon = \gamma/2\alpha_0$.

Let g be as in that lemma and $\tilde{g}(\xi) = g(\xi - \lambda)$. We modify equation (2.3), satisfied by v, as in Remark 2.6. Set $w = v/\tilde{g}$. Then w satisfies (2.16), (2.17) with

$$\begin{aligned}&\hat{c}(x,t) \leq -\alpha_0 \quad \text{if } x_1 \in [\lambda, \lambda + \delta_1] \text{ and}\\ &\hat{c}(x,t) \leq c^\lambda(x,t) + \alpha_0\varepsilon = c^\lambda(x,t) + \frac{\gamma}{2} \quad (x \in \mathbb{R}^N_\lambda, t > 0).\end{aligned}$$

Combining these estimates with (3.7), we obtain

$$\hat{c}(x,t) \leq -\hat{\gamma} = -\min\{\alpha_0, \gamma/2\} < 0,$$

whenever $x \in \mathbb{R}^N_\lambda \setminus G_\lambda$ or $\lambda < x_1 \leq \lambda + \delta_1$.

Let us now assume that (3.8) and (3.9) hold. Then, we have

$$\hat{c}(x,t) \leq -\hat{\gamma} \quad (x \in \mathbb{R}^N_\lambda \setminus D_0).$$

Since $w = v/\tilde{g} > 0$ in $D_0 \times (t_0, \infty)$, any connected component Q of the set $\{w < 0\}$ is contained in $x \in \mathbb{R}^N_\lambda \setminus D_0$. Clearly, $w = 0$ on $\partial_s Q$ for any such component and $w(x,t) \to 0$ as $|x| \to \infty$. Hence, applying the maximum principle, Lemma 2.1, to w we obtain

$$\|v^-(\cdot,t)\|_{L^\infty(\mathbb{R}^N_\lambda)} \leq 2\|w^-(\cdot,t)\|_{L^\infty(\mathbb{R}^N_\lambda)} \to 0 \text{ as } t \to \infty. \tag{3.10}$$

Now let D, D_1 be any domains with $D_0 \subset D \subset\subset D_1 \subset\subset \mathbb{R}^N_\lambda$. According to Lemma 2.4, for each $\tau > 1$ we have

$$v(x,\tau+1) \geq \kappa\|v^+(\cdot,\tau+\frac{1}{2})\|_{L^\infty(D)} - e^{\beta_0} \sup_{\partial_p(D_1\times(\tau,\tau+1))} e^{\beta_0}v^- \quad (x \in D)$$

for some constant $\kappa > 0$. This, together with (3.10) and (3.8b), imply

$$\liminf_{t\to\infty} v(x,t) > \varsigma \quad (x \in D)$$

for some constant $\varsigma = \varsigma(D, D_1) > 0$. Since $v = V_\lambda u$, we obtain, as a consequence, that $V_\lambda z \geq \varsigma$ on D for each $z \in \omega(u)$. Since D can be taken arbitrarily large, the conclusion of the lemma follows. $\square$

3.1 Step 1: Large λ

Lemma 3.3. *There exists $\lambda_1 \in \mathbb{R}$ such that* $(\mathrm{S})_\lambda$ *holds for all $\lambda > \lambda_1$.*

Proof. If λ is sufficiently large, then

$$\mathbb{R}^N_\lambda \cap G_\lambda = P_\lambda B(0,\rho). \tag{3.11}$$

Moreover, since

$$\inf\{|x| : x \in P_\lambda B(0,\rho)\} \to \infty \text{ as } \lambda \to \infty,$$

the decay assumption (1.15) implies that for λ sufficiently large, $\lambda > \lambda_1$ say, (3.11) holds together with

$$u(y,t) < \frac{k(B(0,\rho))}{2} \quad (y \in P_\lambda B(0,\rho), t > 0)$$

where $k(B(0,\rho))$ is as in Lemma 3.1(ii). Consequently, for $\lambda > \lambda_1$ we have

$$u(x,t) - u(P_\lambda x, t) > \frac{k(B(0,\rho))}{2} \quad (x \in B(0,\rho), t > 0),$$

or, equivalently,

$$V_\lambda u(x,t) = u(P_\lambda x, t) - u(x,t) > \frac{k(B(0,\rho))}{2} \quad (x \in P_\lambda B(0,\rho), t > 0).$$

By Lemma 3.2 (with $D_0 = P_\lambda B(0,\rho)$), this and (3.11) imply that $(\mathrm{S})_\lambda$ holds for $\lambda > \lambda_1$. $\square$

3.2 Step 2: $\lambda = \lambda_\infty$

Let λ_1 be as in Lemma 3.3 and λ_∞ as in (3.5).

Lemma 3.4. *The following statements hold:*

(i) $-\infty < \lambda_\infty \le \lambda_1$.

(ii) $V_{\lambda_\infty} z \ge 0 \quad (z \in \omega(u))$.

(iii) There exists $\hat{z} \in \omega(u)$ such that $V_{\lambda_\infty} \hat{z} \equiv 0$.

Proof. (i) For any fixed x we have $|P_\lambda x| \to \infty$ as $\lambda \to -\infty$. Therefore, by Lemma 3.1(ii) and (1.15), $V_\lambda u(x,t) < 0$ if λ is sufficiently large negative. Clearly, $(\mathrm{S})_\lambda$ does not holds for such λ which proves $\lambda_\infty > -\infty$. The relation $\lambda_\infty \le \lambda_1$ is trivial.

(ii) This statement is obvious since $V_\lambda z(x) \to V_{\lambda_\infty} z(x)$ as $\lambda \searrow \lambda_\infty$.

(iii) In view of compactness of $\omega(u)$, the statement readily follows from the claim that for each bounded domain $D \subset \mathbb{R}^N_{\lambda_\infty}$ there is $z \in \omega(u)$ such that $V_{\lambda_\infty} z \equiv 0$ on D. We prove the claim by contradiction. Assume it is not true for some D. Then, since $\omega(u)$ is compact, we have

$$\|V_{\lambda_\infty} z\|_{L^\infty(D)} \ge 2b \quad (z \in \omega(u)),$$

for some $b > 0$. Consequently, for large t, we also have

$$\|V_{\lambda_\infty} u(\cdot,t)\|_{L^\infty(D)} \ge b. \tag{3.12}$$

This of course remains valid if D is enlarged. We make it so large that for each $\lambda \le \lambda_\infty$ sufficiently close to λ_∞ we have

$$D \supset G_\lambda \cap \{x : x_1 \ge \lambda + \delta_1\}, \tag{3.13}$$

where δ_1 is as in Lemma 3.2. Now, statement (ii) yields

$$\lim_{t\to\infty} \|(V_{\lambda_\infty} u)^-(\cdot,t)\|_{L^\infty(\mathbb{R}^N_{\lambda_\infty})} = 0.$$

This trivially remains valid if $\mathbb{R}^N_{\lambda_\infty}$ is replaced by a bounded domain D_1 with $D \subset\subset D_1$. Hence, using (3.12), an application of Lemma 2.4 to the solution $v = V_{\lambda_\infty} u$ yields a constant $\kappa > 0$ such that for each sufficiently large t one has

$$V_{\lambda_\infty} u(x,t) > \kappa b/2 \quad (x \in \overline{D}). \tag{3.14}$$

Since ∇u is bounded, (3.14) implies that for each $\lambda \le \lambda_\infty$ sufficiently close to λ_∞ we have

$$V_\lambda u(x,t) > \kappa b/4 \quad (x \in \overline{D}). \tag{3.15}$$

This, together with (3.13) and Lemma 3.2, imply that $(\mathrm{S})_\lambda$ holds for each $\lambda \le \lambda_\infty$ sufficiently close to λ_∞, contradicting the definition of λ_∞. This contradiction proves our claim and thereby completes the proof. ☐

We complete Step 2 by proving the decreasing property of $z \in \omega(u)$.

Proposition 3.5. *For each $z \in \omega(u)$ one has $\partial_{x_1} z < 0$ in $\mathbb{R}^N_{\lambda_\infty}$.*

Proof. For each $z \in \omega(u)$ and $\lambda > \lambda_\infty$, we have $V_\lambda z > 0$ in $\mathbb{R}^N_\lambda$ and $V_\lambda z = 0$ for $x_1 = \lambda$. Hence

$$0 \le \partial_{x_1} V_\lambda z \,\Big|_{\,x_1=\lambda} = -2\partial_{x_1} z \,\Big|_{\,x_1=\lambda}.$$

This shows that z is monotone nonincreasing in x_1. Actually, z is strictly decreasing. Indeed, if not, then there exists $x' \in \mathbb{R}^{N-1}$ such that $x_1 \mapsto z(x_1, x')$ is constant on an interval. Taking λ in that interval, we arrive at a contradiction with $V_\lambda z > 0$.

We now prove that $\partial_{x_1} z(x^0) < 0$ for any $x^0 \in \mathbb{R}^N_{\lambda_\infty}$. Fix such an x^0 and let r be so small that $B(x^0, 3r) \subset \mathbb{R}^N_{\lambda_\infty}$. By the decreasing property proved above, for each $z \in \omega(u)$ there is a constant $\varpi > 0$ such that $\|\partial_{x_1} z\|_{L^\infty(B(x^0,2r))} > \varpi$. In view of compactness of $\omega(u)$ in $C^1_b(\mathbb{R}^N)$ (see Lemma 3.1), we may assume that ϖ is independent of $z \in \omega(u)$. It follows that if $h_0 \in (0, r)$ is sufficiently small, then for each $z \in \omega(u)$ and each $h \in (0, h_0)$ one has $\|d^h z\|_{L^\infty(B(x^0,r))} > \varpi$, where

$$d^h z = \frac{z(x_1, x') - z(x_1 + h, x')}{h}.$$

Fix any such $h \in (0, h_0)$. Clearly,

$$\|d^h u(\cdot, t)\|_{L^\infty(B(x^0,r))} > \varpi$$

for all sufficiently large t and any $h \in (0, h_0)$. Also, since $d^h z > 0$ on $\overline{B}(x^0, 2r)$ for each $z \in \omega(u)$, we have, by compactness, $d^h u(\cdot, t) > 0$ on $\overline{B}(x^0, 2r)$ for each sufficiently large t. Now, $d^h u(\cdot, t)$ is a solution of a linear equation (2.9) whose coefficients satisfy (2.5) and (2.6) with constants β_0 and α_0 independent of h. It follows from Harnack inequality that for all sufficiently large t we have

$$d^h u(x, t + 1) \ge \nu \|d^h u(\cdot, t)\|_{L^\infty(B(x^0,r))} > \nu\varpi \quad (x \in B(x^0, r)),$$

with some constant ν independent of h. Thus,

$$d^h z(x) \ge \nu\varpi \quad (x \in B(x^0, r),\, z \in \omega(u)).$$

Taking the limit as $h \to 0$, we obtain in particular that $\partial_{x_1} z(x^0) < -\nu\varpi < 0$. This completes the proof. $\square$

3.3 Step 3: $\lambda < \lambda_\infty$, $\lambda \approx \lambda_\infty$

Our aim in this step is to prove that $V_{\lambda_\infty} z \equiv 0$ for all $z \in \omega(u)$. We first use the method of moving hyperplanes starting with λ near $-\infty$. Proceeding analogously as in the steps above, we obtain the following result (cp. Lemma 3.4 and Proposition 3.5).

Lemma 3.6. *There exists $\lambda_\infty^- \in (-\infty, \lambda_\infty)$ with the following properties.*

(i) $V_{\lambda_\infty^-} z \leq 0 \quad (z \in \omega(u))$.

(ii) $V_\lambda z < 0 \quad (z \in \omega(u), \lambda < \lambda_\infty^-)$.

(iii) There exists $\bar{z} \in \omega(u)$ such that $V_{\lambda_\infty^-} \bar{z} \equiv 0$.

By Lemma 3.6(i), Lemma 3.4(ii) and Proposition 3.5, it is clear that Theorem 1.2 will be proved once we show

Lemma 3.7. $\lambda_\infty^- = \lambda_\infty$.

In the proof of this equality, the following lemma is crucial. Its meaning is roughly as follows. For $\lambda > \lambda_\infty$ we know, by $(\mathrm{S})_\lambda$, that if $x \in \mathbb{R}^N_\lambda$, then the function $t \mapsto V_\lambda u(x,t)$ is bounded below by a positive constant. This may no longer be true for $\lambda = \lambda_\infty$, however, using the next lemma we will show that for large t, $V_\lambda u(x,t)$ stays above the exponential function $e^{-\epsilon t}$ with arbitrarily small $\epsilon > 0$. The same is true for each $\lambda < \lambda_\infty$, $\lambda \approx \lambda_\infty$, as long as $V_\lambda u(x,t)$ stays positive. To prove these lower estimates, we use a subsolution provided by Lemma 3.8.

Lemma 3.8. *Given any domain $D_0 \subset\subset \mathbb{R}^N_{\lambda_\infty}$ and any $\theta > 0$ there exist $\lambda_2 < \lambda_\infty$, $t_0 > 0$, domain D and a function $\varphi : \overline{D} \times [t_0, \infty) \to \mathbb{R}$ with the following properties:*

(i) $D_0 \subset\subset D \subset\subset \mathbb{R}^N_{\lambda_\infty}$,

(ii) φ is C^2 in x and C^1 in t on $\overline{D} \times [t_0, \infty)$,

(iii) $\varphi > 0$ in $D_0 \times (t_0, \infty)$,

(iv) $\varphi < 0$ on $\partial D \times (t_0, \infty)$,

(v) one has

$$\frac{\|\varphi^+(\cdot,t)\|_{L^\infty(D)}}{\|\varphi^+(\cdot,s)\|_{L^\infty(D)}} \geq Ce^{-\theta(t-s)} \quad (t \geq s \geq t_0), \tag{3.16}$$

for some constant $C > 0$ independent of t and s,

(vi) for each $\lambda \in [\lambda_2, \lambda_\infty]$, φ is a (strict) subsolution of (2.3) *on $D \times (t_0, \infty)$:*

$$\varphi_t < a_{ij}(x,t)\varphi_{x_i x_j} + b_i^\lambda(x,t)\varphi_{x_i} + c^\lambda(x,t)\varphi, \quad x \in D,\, t > t_0.$$

The proof of this lemma is given at the end of the section.

Proof of Lemma 3.7. The reader may find useful to recall the meaning of γ and G_λ, see (3.6) and (3.7).

The proof is by contradiction. Suppose $\lambda_\infty^- < \lambda_\infty$. Let $\hat{z}, \bar{z} \in \omega(u)$ be as in Lemmas 3.4 and 3.6. Then for each $\lambda \in (\lambda_\infty^-, \lambda_\infty)$ the following holds:

$$V_\lambda \hat{z} < 0 \quad (x \in \mathbb{R}^N_\lambda), \tag{3.17a}$$

$$V_\lambda \bar{z} > 0 \quad (x \in \mathbb{R}^N_\lambda). \tag{3.17b}$$

Indeed, since $V_{\lambda_\infty}\hat{z} \equiv 0$ and $\hat{z}$ is decreasing in x_1 for $x_1 > \lambda_\infty$ (see Proposition 3.5), (3.17a) holds for each $\lambda < \lambda_\infty$. Similarly, (3.17b) holds for each $\lambda > \lambda_\infty^-$.

Fix $\delta > 0$ is as in Lemma 2.5 with

$$\Theta = \beta_0/\alpha_0 + 1, \quad \varepsilon = \frac{\gamma}{2\alpha_0}. \tag{3.18}$$

Choose a domain $D_0 \subset\subset \mathbb{R}^N_{\lambda_\infty}$ such that the following inclusion holds for $\lambda = \lambda_\infty$:

$$G_\lambda \cap \{x \in \mathbb{R}^N : x_1 \geq \lambda + \delta\} \subset\subset D_0. \tag{3.19}$$

Clearly, this is still valid if $\lambda \leq \lambda_\infty$ is close enough to λ_∞, say if $\lambda \in (\lambda_3, \lambda_\infty]$, for some $\lambda_3 < \lambda_\infty$.

Let $\lambda_2 < \lambda_\infty$ and D be as in Lemma 3.8 with

$$\theta := \min\{\gamma/2, \alpha_0\}.$$

Fix any λ satisfying $\max\{\lambda_3, \lambda_2\} < \lambda < \lambda_\infty$, and set $v = V_\lambda u$.

By (3.17), there is a constant $q > 0$ such that

$$\begin{aligned} V_\lambda \hat{z} &< -q \quad (x \in \overline{D}), \\ V_\lambda \bar{z} &> q \qquad (x \in \overline{D}). \end{aligned} \tag{3.20}$$

Since $\hat{z}, \bar{z} \in \omega(u)$, there are sequences $t_n < \hat{t}_n$ such that

$$u(\cdot, t_n) \to \bar{z}, \quad u(\cdot, \hat{t}_n) \to \hat{z}$$

with convergence in $C_b^1(\mathbb{R}^N)$. For each large n we have, by (3.20),

$$\begin{aligned} v(\cdot, \hat{t}_n) &< -q \quad (x \in \overline{D}), \\ v(\cdot, t_n) &> q \qquad (x \in \overline{D}). \end{aligned} \tag{3.21}$$

It follows that there exists $T_n \in (t_n, \hat{t}_n)$ such that

$$v(x,t) > 0 \quad ((x \in \overline{D},\, t \in [t_n, T_n)), \tag{3.22a}$$

$$v(\cdot, T_n) \text{ vanishes somewhere on } \partial D. \tag{3.22b}$$

We claim that (3.22) have the following consequences:

(C1) $T_n - t_n > 2$ for all n large enough,

(C2) $\sup_{t \in [t_n, T_n]} e^{\theta(t-t_n)} \|v^-(\cdot, t)\|_{L^\infty(\mathbb{R}^N_\lambda)} \to 0$ as $n \to \infty$,

(C3) there is a constant $C_0 > 0$ such that

$$\inf_{t \in [t_n, T_n]} e^{\theta(t-t_n)} \|v^+(\cdot, t)\|_{L^\infty(D)} \geq C_0 \text{ for all } n \text{ large enough.}$$

To verify (C1), we first find a sequence v_0^n of smooth nonnegative functions with compact support in $\mathbb{R}^N_\lambda$ such that $v_0^n \to V_\lambda \bar{z}$ in $L^\infty(\mathbb{R}^N_\lambda)$. This can be achieved by suitably molifying $V_\lambda \bar{z} > 0$. Let v^n be the solution of the problem:

$$\begin{aligned} v^n_t &= a_{ij}(x,t) v^n_{x_i x_j} + b_i^\lambda(x,t) v^n_{x_i} + c^\lambda(x,t) v^n, & x \in \mathbb{R}^N_\lambda,\, t > t_n, \\ v^n &= 0, & x \in \Gamma_\lambda,\, t > t_n, \\ v^n(x, t_n) &= v_0^n, & x \in \mathbb{R}^N_\lambda \end{aligned}$$

(cf. [31, Theorem 7.17]). Then v^n and v solve the same equation and boundary condition, and they both decay to zero as $|x| \to \infty$. By the maximum principle, for $t \in [t_n, t_n + 2]$ we have $v_n \geq 0$ and also

$$\begin{aligned} \|v^n(\cdot,t) - v(\cdot,t)\|_{L^\infty(\mathbb{R}^N_\lambda)} \\ \leq e^{\beta_0(t-t_n)} \|v_0^n - v(\cdot,t_n)\|_{L^\infty(\mathbb{R}^N_\lambda)} \to 0 \quad \text{as } n \to \infty. \quad (3.23) \end{aligned}$$

Now, by (1.14) and (1.1), u_t (and hence v_t) is bounded. Therefore, (3.21) implies that there is $\vartheta > 0$ such that for each sufficiently large n

$$v(x,t) > q/2 \quad (x \in \overline{D}, t \in [t_n, t_n + 2\vartheta]). \quad (3.24)$$

By (3.23), the same is true if v is replaced by v^n for any sufficiently large n. Applying Harnack inequality to v_n, we obtain

$$v_n(x,t) \geq C_3 \quad (x \in \overline{D}, t \in [t_n + 2\vartheta, t_n + 2]),$$

for some $C_3 > 0$ independent of n. Combining this with (3.23), we consequently obtain

$$v(x,t) \geq C_3/2 \quad (x \in \overline{D}, t \in [t_n + 2\vartheta, t_n + 2]).$$

Thus $v(\cdot,t) > 0$ on $\overline{D}$ for $t \in [t_n, t_n+2]$, in particular $T_n > t_n + 2$. (The same arguments can be used to show that $T_n - t_n \to \infty$, but this is not needed.)

We next show (C2). Let g be as in Lemma 2.5 with Θ and ε as in (3.18). Recall that we have chosen δ as in that lemma.

Consider the function $w = v/\tilde{g}$, with $\tilde{g}(\xi) := g(\xi - \lambda)$. Then w satisfies the linear problem (2.16), (2.17), with coefficients specified in Remark 2.6. In particular, we have $\hat{c}(x,t) \leq -\alpha_0$ if $\lambda \leq x_1 \leq \lambda + \delta$. Also, by (3.7) and (2.20),

$$\hat{c}(x,t) < -\gamma + \varepsilon\alpha_0 = -\frac{\gamma}{2} \quad (x \in \mathbb{R}^N_\lambda \setminus G_\lambda).$$

Thus, if $x \in \mathbb{R}^N_\lambda \setminus D \subset \mathbb{R}^N_\lambda \setminus D_0$, we have, by (3.19),

$$\hat{c}(x,t) \leq -\min\{\gamma/2, \alpha_0\} = -\theta.$$

Since $v(\cdot,t) > 0$ in D for $t \in [t_n, T_n]$, we have $\hat{c} \leq -\theta$ on any connected component of $\{(x,t) : t \in [t_n, T_n], w < 0\}$. Therefore, by the maximum principle, for each $t \in [t_n, T_n]$ we have

$$\begin{aligned} e^{\theta(t-t_n)} \|v^-(\cdot,t)\|_{L^\infty(\mathbb{R}^N_\lambda)} &\leq 2e^{\theta(t-t_n)} \|w^-(\cdot,t)\|_{L^\infty(\mathbb{R}^N_\lambda)} \\ &\leq 2\|w^-(\cdot,t_n)\|_{L^\infty(\mathbb{R}^N_\lambda)} \leq 4\|v^-(\cdot,t_n)\|_{L^\infty(\mathbb{R}^N_\lambda)} \to 0 \end{aligned}$$

(the convergence follows from $v(\cdot, t_n) \to V_\lambda \bar{z} > 0$). This proves (C2).

Finally, we verify (C3). For that we use the subsolution φ as in Lemma 3.8. Denote $\eta_n := \|\varphi^+(\cdot, t_n)\|_{L^\infty(D)}$ and $\psi := q\varphi/\eta_n$ (q is as in (3.21)). Then, if t is large enough, v and ψ are, respectively, a solution and a subsolution of the same linear equation on $D \times [t_n, T_n]$ and they satisfy the following relations:

$$\begin{aligned}\psi(x,t) &< 0 \le v(x,t) \quad (x \in \partial D,\, t \in [t_n, T_n]),\\ \psi(x,t_n) &\le q \le v(x,t_n) \quad (x \in \overline{D}).\end{aligned}$$

Therefore, by the maximum principle, for each $t \in [t_n, T_n]$

$$\sup_{x\in D} v(x,t) \ge \sup_{x \in D} \psi^+(x,t) = q\frac{\|\varphi^+(\cdot,t)\|_{L^\infty(D)}}{\eta_n} \ge qCe^{-\theta(t-t_n)}$$

(see (3.16)). This proves (C3).

We now complete the proof of the lemma by showing that (C1)-(C3) lead to a contradiction. For this we apply Lemma 2.4. Choose any bounded domain $D_1 \subset \mathbb{R}^N_\lambda$ such that $D \subset\subset D_1$. Using the conclusion of Lemma 2.4 with $\tau = T_n - 1$ we find constants κ and m independent of n such that

$$v(x,T_n) \ge \kappa \|v^+(\cdot, T_n - \frac{1}{2})\|_{L^\infty(D)} - e^m \|v^-\|_{L^\infty(\mathbb{R}^N_\lambda \times (T_n - 1, T_n))} \quad (x \in \overline{D}).$$

By (C1) - (C3), this inequality implies that for each $x \in \overline{D}$

$$\begin{aligned} v(x,T_n) &\ge e^{-\theta(T_n - t_n)}\big(C_0 e^{\theta/2}\kappa - e^m e^{\theta(T_n-t_n)}\|v^-\|_{L^\infty(\mathbb{R}^N_\lambda \times (T_n-1,T_n))}\big)\\ &> e^{-\theta(T_n-t_n)} C_0 e^{\theta/2}\kappa/2 > 0\end{aligned}$$

if n is sufficiently large. We have thus derived a contradiction to (3.22b), which completes the proof. □

It remains to prove Lemma 3.8.

Proof of Lemma 3.8. Suppose $\theta > 0$ and $D_0 \subset\subset \mathbb{R}^N_{\lambda_\infty}$ are given. Without loss of generality we shall assume that θ is so small that

$$\theta < \max\{1, \frac{\gamma}{2}\},$$

where γ is as in (2.7).

We first find a function φ which serves as a subsolution for (2.3) on $\mathbb{R}^N_\mu \times (t_0, \infty)$ for each $\lambda \approx \lambda_\infty$. Here $\mu > \lambda_\infty$ is close to λ_∞ and t_0 is sufficiently large. We construct φ using the function $v := V_\mu u$. Specifically, we set

$$\varphi(x,t) = e^{-\theta t} v^\alpha(x,t) + s\left(-e^{-\theta t}(x_1 - \mu)^\beta\right) = w_1 + s w_2, \tag{3.25}$$

where $\alpha > 1 > \beta$ and $s > 0$ are to be determined (this construction was partly inspired by [10, Proof of Proposition 5.1]). Then we choose a domain D such that all the statements in Lemma 3.8 are satisfied.

In computations below we take

$$\lambda_\infty - \zeta \le \lambda \le \lambda_\infty \text{ and } \mu = \lambda_\infty + \zeta$$

where $\zeta \in (0,1)$ is sufficiently small. How small it has to be is specified by a condition below and by the requirement that

$$D_0 \subset\subset \mathbb{R}^N_\mu.$$

To simplify the notation, we let $b_i = b_i^\mu$, $c = c^\mu$, $\tilde{b}_i = b_i^\lambda$, $\tilde{c} = c^\lambda$ and

$$\begin{aligned} Mw &= a_{ij}(x,t) w_{x_i x_j} + b_i(x,t) w_{x_i}, \\ \tilde{M}w &= a_{ij}(x,t) w_{x_i x_j} + \tilde{b}_i(x,t) w_{x_i}, \\ Lw &= Mw + c(x,t)w, \quad \tilde{L}w = \tilde{M}w + \tilde{c}(x,t)w. \end{aligned}$$

By (2.8), we have

$$\sup_{\substack{x\in\mathbb{R}^N,\, t>0,\\ i=1,\dots,N}} \left((|\tilde{b}_i(x,t) - b_i(x,t)| + |\tilde{c}(x,t) - c(x,t)|\right) \le \delta,$$

where $\delta = \delta(\zeta) \to 0$ as $\zeta \to 0$. Also, since $0 < \zeta < 1$, using (2.7) we find $\varrho > 0$ such that

$$\tilde{c}(x,t), c(x,t) < -\gamma \quad (|x| \ge \varrho, t > 0). \tag{3.26}$$

Recall that, as in (2.6) and (2.5), α_0, β_0 are the ellipticity constant and bound on the coefficients, respectively, for both L and $\tilde{L}$. Note that v satisfies $\partial_t v - Mv = cv$.

With w_1 introduced by (3.25), we compute (for $x \in \mathbb{R}^N_\mu$)

$$\begin{aligned}
e^{\theta t}(\partial_t w_1 - \tilde{L} w_1) &= \alpha v^{\alpha-1}(\partial_t v - \tilde{M} v) - \alpha(\alpha-1) v^{\alpha-2} \tilde{a}_{ij} v_{x_i} v_{x_j} - v^\alpha(\theta + \tilde{c}) \\
&\le \alpha v^{\alpha-1}(\partial_t v - M v) + \alpha v^{\alpha-1}(M - \tilde{M}) v \\
&\qquad\qquad - \alpha_0 \alpha(\alpha-1) v^{\alpha-2} |\nabla v|^2 - v^\alpha(\theta + \tilde{c}) \\
&\le v^\alpha(\alpha c - \theta - \tilde{c}) + \alpha v^{\alpha-1} \delta \sqrt{N} |\nabla v| - \alpha_0 \alpha(\alpha-1) v^{\alpha-2} |\nabla v|^2 \\
&\le v^\alpha((\alpha-1)\beta_0 - \theta + \delta) + \alpha v^{\alpha-1} \delta \sqrt{N} |\nabla v| - \alpha_0 \alpha(\alpha-1) v^{\alpha-2} |\nabla v|^2 \\
&= v^{\alpha-2}\big((\alpha-1)\beta_0 - \theta + \delta) v^2 + \alpha \delta \sqrt{N} v |\nabla v| - \alpha_0 \alpha(\alpha-1) |\nabla v|^2\big)
\end{aligned}$$

Using $|\nabla v| v \le \sigma^2 |\nabla v|^2/2 + v^2/(2\sigma^2)$ with $\sigma^2 = 2\alpha_0(\alpha-1)$, we obtain

$$e^{\theta t}(\partial_t w_1 - \tilde{L} w_1) \le v^\alpha \left((\alpha-1)\beta_0 - \theta + \delta + \frac{\delta^2 N}{4\alpha_0(\alpha-1)} \right). \tag{3.27}$$

Choose $\alpha > 1$ so close to 1 that $-\theta + (\alpha-1)\beta_0 < -3\theta/4$. If $\zeta > 0$ is sufficiently small, then $\delta = \delta(\zeta)$ satisfies

$$\delta < \max\left\{ \frac{\theta}{2}, \sqrt{\frac{\theta \alpha_0(\alpha-1)}{2N}} \right\}. \tag{3.28}$$

For such ζ we have

$$e^{\theta t}(\partial_t w_1 - \tilde{L} w_1) \le -\frac{\theta}{8} v^\alpha. \tag{3.29}$$

Next, for $w_2 = -e^{-\theta t}(x_1 - \mu)^\beta$ we have the following (assuming $x_1 > \mu$):

$$\begin{aligned}
&e^{\theta t}(\partial_t - \tilde{L}) w_2 \\
&\qquad \le (\theta + \tilde{c})(x_1 - \mu)^\beta + \tilde{b}_1 \beta (x_1 - \mu)^{\beta-1} + a_{11}\beta(\beta-1)(x_1 - \mu)^{\beta-2} \\
&\qquad \le (x_1 - \mu)^{\beta-2}\left((\theta + \tilde{c})(x_1 - \mu)^2 + \beta_0 \beta (x_1 - \mu) + \alpha_0 \beta(\beta-1)\right)
\end{aligned} \tag{3.30}$$

(since $\beta < 1$). We now continue with two kinds of estimates. First, for $|x| > \varrho$, relations (3.26), (3.30) and $\theta < \gamma/2$, yield

$$\begin{aligned}
&e^{\theta t}(\partial_t - \tilde{L}) w_2 \\
&\qquad \le (x_1 - \mu)^{\beta-2}\left(-\frac{\gamma}{2}(x_1 - \mu)^2 + \beta_0 \beta (x_1 - \mu) + \alpha_0 \beta(\beta-1)\right) \\
&\qquad \le (x_1 - \mu)^{\beta-2}\left(-\frac{\gamma}{2}(x_1 - \mu)^2 + \frac{\beta_0^2 \beta}{4(1-\beta)\alpha_0}(x_1 - \mu)^2\right) \\
&\qquad = (x_1 - \mu)^\beta\left(-\frac{\gamma}{2} + \frac{\beta_0^2 \beta}{4(1-\beta)\alpha_0}\right).
\end{aligned}$$

This is negative, provided $\beta > 0$ is sufficiently small. Fix any such β. From (3.30) we further obtain

$$e^{\theta t}(\partial_t - \tilde{L})w_2 \\ \le (x_1-\mu)^{\beta-2}\big((1+\beta_0)(x_1-\mu)^2 + \beta_0\beta(x_1-\mu) + \alpha_0\beta(\beta-1)\big) < 0$$

if $x_1 - \mu \le d$, where $d = d(\alpha_0, \beta_0, \beta) > 0$ is a sufficiently small constant.

Summarizing, we have shown that, with α, β and d chosen as above, and with each $\zeta > 0$ sufficiently small, we have, independently of $s > 0$,

$$e^{\theta t}(\partial_t - \tilde{L})\varphi = e^{\theta t}(\partial_t - \tilde{L})(w_1 + sw_2) < 0,$$

whenever $|x| > \varrho$ or $x_1 - \mu < d$. In the remaining region, $\mathcal{E} := \{x \in \mathbb{R}^N_\mu : x_1 - \mu \ge d, |x| \le \varrho\}$, we have

$$e^{\theta t}(\partial_t - \tilde{L})(w_1 + sw_2) < -\frac{\theta}{8}v^\alpha + sK,$$

for some constant K. Since $\mu > \lambda_\infty$, there are $C_1 > 0$ and $t_0 > 0$ such that $v(\cdot,t) = V_\mu u(\cdot,t) \ge C_1$ on $\mathcal{E}$ for each $t \ge t_0$. Therefore, if $s > 0$ is sufficiently small, $(\partial_t - \tilde{L})\varphi$ is negative in $\mathcal{E}$ and hence everywhere is $\mathbb{R}^N_\mu$. Making s yet smaller and t_0 larger, if necessary, we achieve that, in addition,

$$e^{\theta t}\varphi(x,t) = v^\alpha(x,t) - s(x-\mu)^\beta > C_2 \quad (x \in D_0, t \ge t_0). \tag{3.31}$$

This means that statement (iii) of Lemma 3.8 is satisfied and, setting $\lambda_2 = \lambda_\infty - \zeta$, statement (vi) is satisfied for any domain $D \subset \mathbb{R}^N_\mu$. Now, since $\partial_{x_1}u$ (hence $\partial_{x_1}v$) is bounded and $v = 0$ for $x_1 = \mu$, we have $v(x,t) \le C_3(x_1 - \mu)$ for some constant C_3. This implies, since $\alpha > 1 > \beta$, that there is $d_1 > 0$ such that $\varphi < 0$ if $x_1 - \mu < d_1$. On the other hand, if $x_1 - \mu \ge d_1$, then by the decay condition (1.15), we have $\varphi(x,t) < 0$ if $|x|$ is large. We can thus choose a bounded domain $D \subset\subset \mathbb{R}^N_\mu$ such that $D_0 \subset\subset D$ and $\varphi < 0$ on $\partial D \times [t_0, \infty)$. For any such domain, statements (i)-(iv) and (vi) are clearly satisfied. Since v is bounded, $e^{\theta t}\|\varphi(\cdot,t)\|_{L^\infty(D)}$ is bounded. This and (3.31) imply statement (v). □

We remark that when carrying out the proof for the semilinear equation (1.3), the construction of the subsolution φ can be simplified. It is sufficient to take $\alpha = 1$, $\beta = 0$ in (3.25).

4 Generalizations

In this section, we discuss possible generalization of our main theorem.

Taking symmetry theorems in elliptic equations as a model, it is natural to address the asymptotic symmetry question for equation with a more general structure. Specifically, consider the equation

$$u_t = F(t, x, u, Du, D^2u), \quad x \in \mathbb{R}^N,\ t > 0, \tag{4.1}$$

where $F(t, x, u, p, H)$ is a function of $t \in [0, \infty)$, $x \in \mathbb{R}^N$, $u \in [0, \infty)$, $p = (p_1, \dots, p_N) \in \mathbb{R}^N$, and $H = (H_{11}, H_{12}, \dots, H_{NN}) \in R^{N^2}$. Assume that F is of class C^1 in (u, p, H) uniformly with respect to $x \in \mathbb{R}^N$ and $t \geq 0$, that the equation is uniformly parabolic, and that

$$F_u(t, x, u, 0, 0) \leq 0 \tag{4.2}$$

if $t \geq 0$, $|x|$ is sufficiently large and $u \geq 0$ is sufficiently small. Assume further that F is radially symmetric in x: $F = F(t, |x|, u, p, H)$ and nonincreasing in $r = |x|$, and for each admissible values of the arguments and each $i_0 \neq j_0$ we have

$$\begin{aligned} F(t, x, p_1, \dots, p_{i_0-1}, -p_{i_0}, p_{i_0+1}, \dots, p_N,& \\ H_{11}, \dots, -H_{i_0 j_0}, \dots, -H_{j_0 i_0}, \dots, H_{NN}) &= F(t, x, u, p, H). \end{aligned}$$

This is the structure considered in [30] in the elliptic case. It is natural to address the question whether the asymptotic radial symmetry result holds in this more general setting. Let us comment on some difficulties that arise when we attempt to apply our method. The structure of (4.1) is more general, compared to the quasilinear equations considered in the previous sections, in the following regards:

a) the explicit spatial dependence is allowed,

b) the equation is fully nonlinear, and

c) the strict negativity condition (Q3) is relaxed.

We do not see a way of dealing with the generalization c). Even in the simplest case of the semilinear equation (1.3), the strict negativity condition (Q3) is needed for our method to work.

Replacing (4.2) by a stronger relation similar to (Q3), it is still not clear whether our proof can be modified to be applicable to (4.1). The main problem is in the construction of a subsolution φ; Lemma 3.8 appears difficult to generalize. Other difficulties that arise because of a) or b) do not seem to be so essential. For example, the spatial inhomogeneity causes that the function $V_\lambda u$ is a supersolution of the underlying linear equation, rather than a solution as in the homogeneous case. For this reason, all arguments relying on Harnack inequality would have to be modified. This can be dealt with without major trouble, as the weak Harnack inequality for supersolutions (see [31]) would serve our purposes equally well.

We leave the possibility of extending our symmetrization theorem to equations of the form (4.1) open.

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