# Threshold solutions and sharp transitions for nonautonomous parabolic equations on $\mathbb{R}^N$

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**Abstract**

This paper is devoted to a class of nonautonomous parabolic equations of the form $u_t = \Delta u + f(t, u)$ on $\mathbb{R}^N$. We consider a monotone one-parameter family of initial data with compact support, such that for small values of the parameter the corresponding solutions decay to zero, whereas for large values they exhibit a different behavior (either blowup in finite time or locally uniform convergence to a positive constant steady state). We are interested in the set of intermediate values of the parameter for which neither of these behaviors occurs. We refer to such values as threshold values and to the corresponding solutions as threshold solutions. We prove that the transition from decay to the other behavior is sharp: there is just one threshold value. We also describe the behavior of the threshold solution: it is global, bounded, and asymptotically symmetric in the sense that all its limit profiles, as $t \to \infty$, are radially symmetric about the same center. Our proofs rely on parabolic Liouville theorems, asymptotic symmetry results for nonlinear parabolic equations, and theorems on exponential separation and principal Floquet bundles for linear parabolic equations.

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# 1 Introduction

We consider the Cauchy problem for nonautonomous parabolic equations of the following form

$$u_t = \Delta u + f(t,u), \quad x \in \mathbb{R}^N, t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N. \tag{1.2}$$

Here $u_0 \in L^\infty(\mathbb{R}^N)$ is a nonnegative function with compact support, and $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is $C^1$ in $u$ uniformly with respect to $t$ with $f(t,0) = 0$ and $f_u(t,0) \leq -\alpha$ ($t \geq 0$), where $\alpha$ is a positive constant. Our goal is to examine, in the context of nonautonomous problems, the structure and asymptotic behavior of threshold solutions. Such solutions appear in many situations where two types of robust behaviors of solutions are observed and one is interested in the way the transition from one to the other occurs when initial data are varied. In this paper we shall focus on two types of transitions: transition from extinction to propagation and transition from extinction to blowup.

To be more specific, consider a family of nonnegative initial data $u_0 = \psi_\mu \in L^\infty(\mathbb{R}^N)$, $\mu \geq 0$, such that $\psi_0 \equiv 0$, each $\psi_\mu$ has compact support, and the function $\mu \to \psi_\mu$ is monotone increasing and continuous in the $L^1$-norm (see hypothesis (IP2) in the next section). Denote by $u^\mu$ the solution of (1.1), (1.2) with $u_0 = \psi_\mu$. It is easy to prove (see Section 3) that if $\mu$ is small enough, then $u^\mu(\cdot, t) \to 0$ in $L^\infty(\mathbb{R}^N)$ as $t \to \infty$. Under further assumptions on $f$ and $\psi_\mu$, for large $\mu$ the solution $u^\mu$ exhibits a different behavior. We consider two classes of nonlinearities $f$ here. The first one is the class of bistable nonlinearities. For $f$ in this class, (1.1) has a positive constant steady state which attracts, locally uniformly, all solutions $u^\mu$ with large $\mu$. The second class of nonlinearities is such that all solutions $u^\mu$ with large $\mu$ blow up in finite time. For each of these two classes of $f$, we examine the *threshold set* which consists of those values of $\mu$ for which the corresponding solution $u^\mu$ does not decay to zero and does not exhibit the other behavior. Also, for the threshold values, we want to give a description of the behavior of the corresponding solutions $u^\mu$ (below, we refer to such solutions as *threshold solutions*).

To put this problem in perspective, let us recall available results on the autonomous problem

$$\begin{aligned} u_t &= \Delta u + f(u), && x \in \mathbb{R}^N,\ t > 0, \\ u(x, 0) &= \psi_\mu(x), && x \in \mathbb{R}^N. \end{aligned} \tag{1.3}$$

Here $f \in C^1[0, \infty)$, $f(0) = 0$ and $\psi_\mu$, $\mu \geq 0$, is a family of initial data as above.

First assume that $f : [0, \infty) \to \mathbb{R}$ a bistable nonlinearity: it has exactly two positive zeros $\gamma > \beta$ and satisfies the following conditions

$$f'(0) < 0,\ f'(\gamma) < 0,\ \text{and} \int_0^\gamma f(u)\, du > 0. \tag{1.4}$$

The relation $f'(0) < 0$ implies that for small $\mu$, $u^\mu(\cdot, t)$ converges to 0 in $L^\infty(\mathbb{R}^N)$ as $t \to \infty$. This may be true for all $\mu > 0$, depending on the growth of $-f(u)$ for large $u$ and the choice of the family $\psi_\mu$, $\mu \geq 0$, so we make an assumption to exclude this possibility. For example, fixing $\epsilon > 0$, assume that $\liminf_{\mu \to \infty} \psi_\mu(x) \geq \beta + \epsilon$ for all $x$ in a ball $B$. If the ball $B$ is sufficiently large, depending on $f$, then for large $\mu$, $u^\mu(\cdot, t)$ converges to $\gamma$ locally uniformly on $\mathbb{R}^N$ (see Section 3). Then, the threshold solutions $u^\mu$ are those that do not converge to any of the two stable steady states 0 and $\gamma$. One would like to understand how they behave and whether there is just one threshold value or a continuum of them. These are rather old problems. For bistable and other nonlinearities, including ignition-type nonlinearities which vanish for $u \approx 0$, they have been mentioned already in the papers of Kanel' [22] and Aronson and Weinberger [2], where they have been related to models in combustion (propagation versus extinction of flames) and population genetics (propagation of genes). For applications of the models perhaps the most interesting question is whether the transition from decay (extinction) to convergence to a positive steady state (propagation) is sharp in the sense that there is just one threshold value $\mu$, or if there is an interval of threshold values $\mu$. The former would mean that any intermediate behavior, neither extinction, nor propagation, is very exceptional; the latter, on the other had, would give rise to a persistent, hence observable, intermediate behavior. Mathematically, from a global perspective on the parabolic semiflow, it is also very interesting to learn what sort of behavior the solutions have to go through when $\mu$ increases from 0 to $\infty$. Also, examining $\omega$-limit sets of threshold solutions, one can often find interesting special solutions, such as equilibria in the autonomous case or spatially localized time-periodic solutions in case $f$ is periodic in $t$ (see Section 6 for an example).

Even in one space dimension, the problems concerning threshold values and threshold solutions are far from trivial and satisfactory answers have been given only recently. In [33], Zlatoš addressed the problems for general bistable equations (1.3) with $N = 1$, assuming that the functions $\psi_\mu$ are characteristic functions of an interval which is expanding with increasing $\mu$. He proved that the transition is sharp: there is exactly one threshold value $\mu_0 > 0$. Moreover, he proved that the corresponding solution $u^{\mu_0}$ converges to a *ground state*, a positive steady state of (1.3) decaying to 0 at $|x| = \infty$ (see [11, 12, 14, 16] for earlier results of this sort dealing with more specific one-dimensional problems). Generalizations and extensions of [33] were given by Du and Matano [10]. Using their new convergence result for equations on $\mathbb{R}$, they were able to treat very general families of initial data. In both [33] and [10], a weaker regularity condition is assumed ($f$ is Lipschitz, rather than $C^1$) and the conditions on the derivatives of $f$ at 0, $\beta$, $\gamma$ are replaced with the sign conditions $f < 0$ in $(0, \beta) \cup (\gamma, \infty)$ and $f > 0$ in $(\beta, \gamma)$. Also ignition nonlinearities are considered in [33, 10]. We remark that the proofs in these papers rely on one-dimensional techniques, notably the Sturmian intersection-comparison arguments, which do not apply if $N \geq 2$. (They do apply to some nonautonomous problems in 1D, however; in [14] such techniques were used in convergence results for threshold solutions of time-periodic parabolic problems on $\mathbb{R}$). Extensions of the results of [33, 10] to higher-dimensional problems with similarly general nonlinearities do not seem to be available (for a partial result see Remark (2) following Theorem 1 in [21]). For a $C^1$ bistable nonlinearity satisfying $f'(0) < 0$, a sharp transition result can be derived from a convergence theorem of [7], as we indicate below.

The second example we wish to discuss is similar to the first one, but the existence of a stable steady state $\gamma$ is replaced with the assumption that solutions $u^\mu$ with sufficiently large $\mu$ blow up in finite time. In this case, the threshold solutions are solutions that are global (that is, defined for all $t \geq 0$) and do not decay to 0 in $L^\infty(\mathbb{R}^N)$. For the nonlinearity $f(u) = -\lambda u + u^p$, where $\lambda > 0$ and $1 < p < p_S := (N+2)/(N-2)_+$ ($p_S = \infty$ if $N = 1$ or $N = 2$), the problem was considered in [9]; another specific class of nonlinearities was treated in [13]. It has been proved in these papers that if the family $\psi_\mu$, $\mu \geq 0$, is given by $\psi_\mu = \mu\psi$, for a fixed nonnegative function $\psi \not\equiv 0$ with compact support, then the threshold set consists of just one value. Moreover, the corresponding threshold solution is bounded and converges to a ground state. In [7], convergence to a ground state was proved for general autonomous nonlinearities $f$ satisfying $f(0) = 0$, $f'(0) < 0$. It says that whenever a global solution $u$ starts with compact support and is bounded in $L^\infty(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, with a finite $q$, then it converges to 0 or to a ground state. This in particular applies to threshold solutions if one can find such a priori bounds on them (for the nonlinearities considered in [9, 13] this is relatively easy). All these results rely in a crucial way on energy inequalities, using in particular the energy functional as a Lyapunov functional, and therefore these methods do not apply in nonautonomous problems.

Let us also remark, without elaborating, that there has been a lot of interest in threshold solutions for the pure power nonlinearity $f(u) = u^p$, with $p > 1$. This is mathematically a very interesting problem. Even though the trivial steady state is no longer asymptotically stable in $L^\infty$, it is stable in a different sense if $p > 1 + 2/N$. In particular, one can still prove that for small initial data which have compact support or which decay fast enough at $|x| = \infty$, the solutions of (1.3) decay to zero as $t \to \infty$. On the other hand, for large initial data the solutions blow up in finite time and there are solutions on the threshold between blowup and decay. The behavior of the threshold solutions has been extensively studied; see for example the recent monograph [32] for an account of available results.

Our study of threshold solutions for equation (1.1) has several motivations. In applications it is often desirable to take time variations of the environment into account and the underlying model then becomes nonautonomous. The problem concerning the transition from decay to propagation is equally meaningful and interesting in the nonautonomous setting. Mathematically, this problem is quite different, however. With general time dependence, the convergence of solutions to any special solution, like a ground state in the autonomous case, is not expected. So it is not a priori clear in what way, if any, the behavior of threshold solutions can be described. The study of the threshold set is also a lot more difficult than in autonomous problems. In many autonomous problems, energy estimates can be used to conclude that each threshold solution converges to a ground state [7]. The uniqueness of the threshold value is then relatively easy to prove using well-known properties of the ground states and local stability analysis. A key point is that each ground state is linearly unstable and it cannot attract two related solutions $u^\mu < u^\nu$. Also, it is impossible for two related solutions $u^\mu < u^\nu$ to converge to two different ground states, as such ground states would have to be pointwise related which is known to be impossible.

In contrast, in the time-dependent case the threshold solutions do not have limits in general and no local stability analysis seems to be useful. We use a new approach which depends on several key ingredients: asymptotic symmetry for parabolic equations on $\mathbb{R}^N$ [28], exponential separation for linear parabolic equations [19, 20], and, in the blowup case, also on universal a priori estimates for subcritical superlinear parabolic problems [31]. We apply the symmetry results to obtain an interesting information on the behavior of any threshold solution $u$: although $u(\cdot, t)$ itself may not converge to any limit as $t \to \infty$, its maximizer $\xi(t)$ does converge to a limit in $\mathbb{R}^N$. This is remarkable given that the dependence of $f$ on time is quite general. Having this information, we are able to reveal an instability property of the the threshold solutions $u^\mu$ themselves. Linearizing around $u^\mu$, we obtain a time-dependent linear parabolic equation. Using a principal Floquet bundle and exponential separation for that equation, we show a strong instability property of each threshold solution, leading eventually to the uniqueness of the threshold value.

Our method for proving the instability of time dependent solutions applies in a rather general context and it is of independent interest. In Section 5, we formulate a general instability theorem for localized solutions, not necessarily connected to any transition phenomena.

We state our results on the threshold set and threshold solutions in the next section, where we also give specific hypotheses characterizing the bistable case and the decay-blowup case. In the latter an important hypothesis concerns the growth of $f(t, u)$ as $u \to \infty$. Our results are valid only if the growth is Sobolev-subcritical, in fact, to accommodate a parabolic Liouville theorem, we need an even stronger growth restriction. Sections 3 - 5 contain the proofs of our main results. In Section 6, we briefly discuss different threshold behaviors in case some of our hypotheses are not satisfied and make additional comments on the results. In the appendix we have collected results on exponential separations and principal Floquet bundles that are relevant to the proof of the instability theorem of Section 5.

## 2 Main results

As indicated above, we shall consider two types of nonlinearities $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$. The following hypotheses and notation are common to both types:

(C1) $f$ is of class $C^1$ in $u$ uniformly with respect to $t$, that is, $f$ and $f_u$ are continuous on $[0, \infty) \times \mathbb{R}$, and for each $M > 0$

$$\lim_{\substack{0 \le u,v \le M,\, t \ge 0 \\ |u-v| \to 0}} |f_u(t,u) - f_u(t,v)| = 0.$$

(C2) For some constant $\alpha > 0$, one has

$$f(t,0) = 0, \ \sup_{t>0} f_u(t,0) < -\alpha.$$

Note that (C1), (C2) imply in particular that there is $\varsigma > 0$ such that

$$f_u(t,u) < -\alpha \quad (u \in [0,\varsigma], t > 0). \tag{2.1}$$

We set

$$g^I(u) := \inf_{t \ge 0} f(t,u), \quad g^S(u) := \sup_{t \ge 0} f(t,u). \tag{2.2}$$

It is not difficult to verify, using (C1), that $g^I$, $g^S$ are locally Lipschitz functions.

In the *bistable* case, we assume the existence of a positive constant $\gamma$ such that the following hypotheses hold (see Figure 1).

(BS1) $f(\cdot,\gamma) \equiv 0$, $g^S(u) < 0$ if $u > \gamma$, and $g^I(u) > 0$ if $u$ is sufficiently close to $\gamma$ and $u < \gamma$.

(BS2) One has

$$\int_0^\gamma g^I(\eta)\,d\eta > 0, \tag{2.3}$$

$$\int_0^u g^S(\eta)\,d\eta < 0 \quad (u \in (0,\beta]), \tag{2.4}$$

where

$$\beta := \sup\{u \in (0,\gamma) : g^I(u) = 0\} \tag{2.5}$$

(note that $\beta \in (0,\gamma)$ by (C1),(C2), and (BS1)).

Examples of specific nonlinearities satisfying these hypotheses can be found in Example 2.6 below.

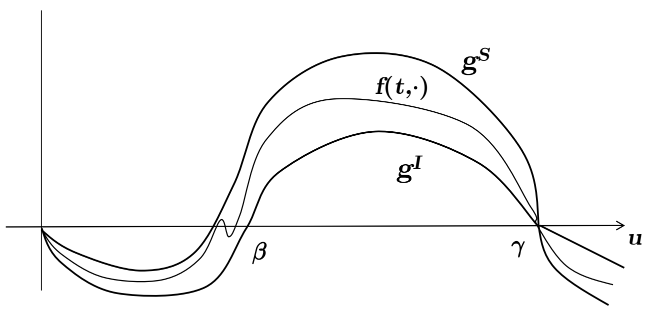


Figure 1: The bistable case.

In the *blowup* case, we assume the following hypotheses (see Figure 2).

(BL1) For each $t \geq 0$ one has $f(t,u) - \ell(t)u^p = o(u^p)$, as $u \to \infty$, uniformly in $t$, where $1 < p < p_{BV} := N(N+2)/(N-1)^2$ ($p_{BV} = \infty$ if $N = 1$) and $\ell$ is a uniformly continuous function on $[0,\infty)$ such that $0 < \inf_{t\geq 0} \ell(t) \leq \sup_{t\geq 0} \ell(t) < \infty$.

(BL2) Condition (2.4) holds, where $\beta$ is defined as in (2.5) with $\gamma := \infty$ (note that $\beta \in (0,\infty)$ by (C1),(C2), and (BL1)).

Specific nonlinearities satisfying the above hypotheses are given in Example 2.7 below.

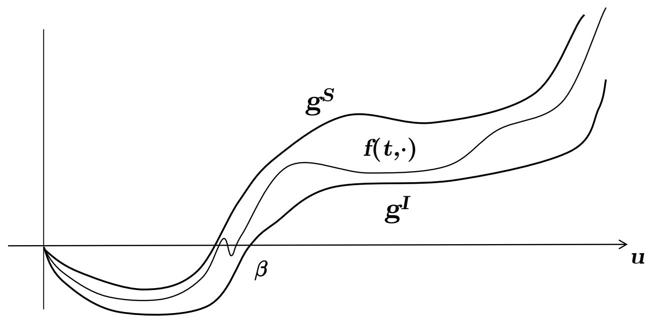


Figure 2: The blowup case.

The fact that $p > 1$ in the growth condition (BL1) guarantees that some solutions blow up in finite time (see Section 2.5). The upper bound $p < p_{BV}$ is too strong if $N > 1$ and it is assumed for technical reasons. Our results in the blowup case depend on a parabolic Liouville theorem, see Section 3, which has only been proved in this range of exponents, although it is likely to be valid for $p < p_S$, where $p_S = (N+2)/(N-2)_+$ ($p_S = \infty$ if $N = 1$ or $N = 2$) is the Sobolev critical exponent. The condition $p < p_{BV}$ can be replaced with $p < p_S$, if the initial data $\psi_\mu$ are assumed radially symmetric. In the supercritical case $p > p_S$, our results on threshold solutions are not valid, see Section 6.

We consider a family of initial data $\psi_\mu$, $\mu \in [0, \infty)$, with the following properties.

(I1) For each $\mu > 0$, $\psi_\mu$ is a nonnegative bounded measurable function on $\mathbb{R}^N$ with compact support $\operatorname{spt} \psi_\mu$ and $\psi_0 \equiv 0$.

(I2) The function $\mu \to \psi_\mu : [0, \infty) \to L^1(\mathbb{R}^N)$ is continuous and monotone increasing in the sense that if $\mu < \nu$, then $\psi_\lambda \leq \psi_\nu$ and there is a set of positive measure on which $\psi_\mu < \psi_\nu$.

By $u^\mu$ we denote the solution of (1.1), (1.2) with $u_0 = \psi_\mu$. It is assumed that the solution is defined on a maximal time interval $[0, T_\mu)$. We say that the solution is global if $T_\mu = \infty$. If $T_\mu < \infty$, then, as is well known, $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \to \infty$ as $t \to T_\mu$, i.e. the solution blows up in finite time.

Our results on sharp transition from decay to propagation and from decay to blowup are formulated in the following two theorems.

**Theorem 2.1.** *Assume* (C1), (C2), (BS1), (BS2), (I1), (I2). *Then either for each* $\mu > 0$ *one has*

$$\lim_{t\to\infty} \|u^\mu(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = 0, \tag{2.6}$$

*or else there exists* $\mu^*$ *such that the following statements hold.*

*(i) For each* $\mu \in (0, \mu^*)$ (2.6) *is valid.*

*(ii) For each* $\mu \in (\mu^*, \infty)$ *one has*

$$u^\mu(\cdot,t) \to \gamma, \text{ as } t \to \infty, \text{ uniformly on compact sets in } \mathbb{R}^N. \tag{2.7}$$

We remark that in the bistable case all solutions $u^\mu$ are global and bounded, since any constant greater than $\gamma$ is a supersolution of (1.1).

**Theorem 2.2.** *Assume* (C1), (C2), (BL1), (BL2), (I1), (I2). *Then either for each* $\mu > 0$ *the solution* $u^\mu$ *is global and satisfies* (2.6)*, or else there exists* $\mu^*$ *such that the following statements hold.*

*(i) For each* $\mu \in (0, \mu^*)$ *the solution* $u^\mu$ *is global and satisfies* (2.6).

*(ii) For each* $\mu \in (\mu^*, \infty)$ *the solution* $u^\mu$ *blows up in finite time.*

The next theorem describes the behavior of the threshold solution $u^{\mu^*}$ in both cases.

**Theorem 2.3.** *Assume that the hypotheses of Theorem 2.1 or Theorem 2.2 are satisfied. Further assume that* (2.6) *does not hold for all* $\mu > 0$ *and let* $\mu^*$ *be as in the corresponding theorem. Then the solution* $u^* := u^{\mu^*}$ *is global and it has the following properties.*

*(i) There are positive constants* $C$, $m$ *such that*

$$u^*(x,t) \le Ce^{-m|x|} \quad (x \in \mathbb{R}^N, t > 0).$$

*(ii)* $\liminf_{t\to\infty} u^*(x,t) > 0 \quad (x \in \mathbb{R}^N)$.

*(iii) There exists* $\xi \in \mathbb{R}^N$ *such that* $u^*(\cdot,t)$ *is asymptotically radially symmetric around* $\xi$ *and radially decreasing away from* $\xi$*:*

*(a)* $\lim_{t\to\infty}(u^*(x,t) - u^*(y,t)) = 0 \quad (x, y \in \mathbb{R}^N,\ |x-\xi| = |y-\xi|)$,

*(b)* $\limsup_{t\to\infty}(x-\xi)\cdot\nabla u^*(x,t) < 0 \quad (x \in \mathbb{R}^N \setminus \{\xi\})$.

As in [28], the asymptotic symmetry of the solution $u^*$ can also be formulated in terms of its $\omega$-limit set, $\omega(u^*)$. Set

$$\omega(u^*) := \{\phi : \phi = \lim u^*(\cdot, t_n) \text{ for some } t_n \to \infty\}, \tag{2.8}$$

with the limit in $L^\infty(\mathbb{R}^N)$. Then, with $\xi$ as in Theorem 2.3, all elements $\phi \in \omega(u^*)$ are radially symmetric about $\xi$ and radially decreasing away from $\xi$.

Note that statements (i)-(iii) of Theorem 2.3 and the fact that $\nabla u$ is bounded (which follows from (i) and parabolic estimates) in particular imply that for large enough $t$, $u(\cdot,t)$ has a maximizer $\xi(t)$ near $\xi$ and $\xi(t) \to \xi$ as $t \to \infty$.

In the following propositions, we give conditions on the family $\psi_\mu$, $\mu \geq 0$, which rule out the possibility of (2.6) holding for all $\mu > 0$.

**Proposition 2.4.** *Assume* (C1), (C2), (BS1), (BS2), (I1), (I2). *Then the following statements hold true.*

*(i) Given $\epsilon > 0$, there exists $R = R(\epsilon, f)$ such that if $\psi_\mu \geq \beta + \epsilon$ on a ball of radius $R$, then* (2.7) *holds.*

*(ii) If $f_u$ is bounded and there is a ball $B$ such that $\int_B \psi_\mu(x)\,dx \to \infty$ as $\mu \to \infty$, then* (2.7) *holds for all sufficiently large $\mu$.*

**Proposition 2.5.** *Assume* (C1), (C2), (BL1), (BL2), (I1), (I2). *If there is a ball $B$ such that $\int_B \psi_\mu(x)\,dx \to \infty$ as $\mu \to \infty$, then $u^\mu$ blows up in finite time for all sufficiently large $\mu$.*

The proofs of the above results are given in Sections 3 - 5.

We finish this section with examples of specific nonlinearities satisfying our hypotheses. In all these examples, $\lambda$ is a continuous function on $[0,\infty)$ satisfying $\lambda_1 \leq \lambda \leq \lambda_2$ for some positive constants $\lambda_1 \leq \lambda_2$. An elementary verification of our hypotheses in the examples is left to the reader.

**Example 2.6.** *Hypotheses* (C1), (C2), (BS1), (BS2) *are satisfied by the following nonlinearities.*

*(i)* $f(t,u) = \lambda(t)g(u)$, *where* $g$ *is a* $C^1$ *function on* $[0,\infty)$ *such that for some* $\gamma > \beta > 0$ *the following relations hold:*

$$g'(0) < 0,\ g(0) = g(\beta) = g(\gamma) = 0,$$
$$g < 0 \text{ in } (0,\beta),\ g > 0 \text{ in } (\beta,\gamma),$$
$$\lambda_2 \int_0^\beta g(\eta)\,d\eta + \lambda_1 \int_\beta^\gamma g(\eta)\,d\eta > 0.$$

*(ii)* $f(t,u) = u(u-\lambda(t))(\gamma - u)$, *where* $\gamma$ *is a constant satisfying* $\gamma > 2\lambda_2$.

**Example 2.7.** *Hypotheses* (C1), (C2), (BL1), (BL2) *are satisfied by the following nonlinearities.*

*(i)* $f(t,u) = \lambda(t)(-mu+u^p)$, *where* $m$ *is a positive constant,* $1 < p < p_{BV}$, *and the function* $\lambda$ *is uniformly continuous.*

*(ii)* $f(t,u) = -\lambda(t)u + m(t)u^p$, *where* $1 < p < p_{BV}$ *and* $m$ *is a uniformly continuous function on* $[0,\infty)$ *satisfying* $m_1 \le m \le m_2$ *for some positive constants* $m_1$, $m_2$ *such that*

$$\frac{\lambda_2}{\lambda_1} < \frac{p+1}{2}\frac{m_1}{m_2}.$$

# 3 A basic description of the threshold set

Henceforth we assume that either the hypotheses of Theorem 2.1 hold (the bistable case) or the hypotheses of Theorem 2.2 hold (the blowup case). In this section we prove that the threshold set, if nonempty, is a compact interval or a set consisting of just one value. This is a rather simple consequence of the comparison principle and continuity of solutions with respect to initial conditions. We need to be a little careful about the latter, as the family $\mu \to \psi_\mu$ is continuous in $L^1$-norm only. Recall that $T_\mu$ stands for the maximal existence time of the solution $u^\mu$. We also use the notation $u(\cdot,t;u_0,t_0)$ for the maximally defined solution of (1.1) with the initial condition $u(\cdot,t_0) = u_0$. If $t_0 = 0$, we often suppress the argument $t_0$. Thus, in particular, $u^\mu(\cdot,t) = u(\cdot,t;\psi_\mu)$. By $B(x,R)$ we denote the open ball in $\mathbb{R}^N$ centered at $x$ and having radius $R$.

**Lemma 3.1.** *Given* $\mu_0 \in [0,\infty)$ *and* $t \in (0,T_{\mu_0})$, *the map* $\mu \mapsto u^\mu(\cdot,t) \in L^\infty(\mathbb{R}^N)$ *is defined on a neighborhood of* $\mu_0$ *and is continuous at* $\mu_0$.

*Proof.* The existence, uniqueness, and continuous dependence on initial data in $L^\infty(\mathbb{R}^N)$ is well known, see for example [23]. It is therefore sufficient to prove that the statement is valid for all small $t = t_0 > 0$; the result then follows from the continuity properties of the composition $\mu \to u(\cdot, t; u^\mu(\cdot, t_0), t_0) = u^\mu(\cdot, t)$.

Note that by the monotonicity of $\mu \to \psi_\mu$ and the comparison principle we have $T_\mu \geq T_\nu$ and $u^\mu < u^\nu$ on $\mathbb{R}^N \times (0, T_\nu)$ if $\mu < \nu$. So we can fix positive constants $\delta_0$ and $c$ such that $T_\mu > \delta_0$ and $u^\mu \leq c$ on $\mathbb{R}^N \times (0, \delta_0]$ for all $\mu \approx \mu_0$. For any such $\mu$, the function $v := u^\mu - u^{\mu_0}$ is the solution of

$$\begin{aligned} &v_t = \Delta v + a(x,t)v, \quad x \in \mathbb{R}^N,\ t \in (0, \delta_0], \\ &v(\cdot, 0) = \psi_\mu - \psi_{\mu_0}, \end{aligned} \tag{3.1}$$

where

$$a(x,t) = \int_0^1 f_u(t, u^{\mu_0}(x,t) + s(u^\mu(x,t) - u^{\mu_0}(x,t)))\, ds.$$

Clearly, $a$ is continuous on $\mathbb{R}^N \times (0, \delta_0]$ and has its absolute value bounded by a constant independent of $\mu \approx \mu_0$. Hence for each $t_0 \in (0, \delta_0]$ and $q > 1$ (including $q = \infty$), there is a constant $C = C(q, t_0)$ such that

$$\|v(\cdot, t_0)\|_{L^q(\mathbb{R}^N)} \leq C_q \|v(\cdot, 0)\|_{L^1(\mathbb{R}^N)}. \tag{3.2}$$

This is a standard $L^1 - L^q$ estimate, which follows directly from Gaussian estimates on the Green's function of (3.1) (see for example [1, Sect. 7]). In view of (I2), this gives the desired continuity of $\mu \to u^\mu(\cdot, t_0)$. □

The above proof shows that Lemma 3.1 remains valid if $L^\infty(\mathbb{R}^N)$ is replaced with $L^q(\mathbb{R}^N)$, for any $q \in [1, \infty)$. Standard parabolic estimates then imply that for each multiindex $\kappa$ with norm $|\kappa| \leq 2$ and each $q \in (1, \infty)$, the map $\mu \mapsto D_x^\kappa u^\mu(\cdot, t) \in L^q(\mathbb{R}^N)$ is continuous as well.

We now introduce new notation. In the bistable case as well as in the blowup case,

$$M_0 := \{\mu > 0 : \lim_{t\to\infty} \|u^\mu(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = 0\}.$$

Further, in the bistable case

$$M_\gamma := \{\mu > 0 : u^\mu(\cdot, t) \to \gamma, \text{ as } t \to \infty, \text{ in } L^\infty_{loc}(\mathbb{R}^N)\}$$

and in the blowup case (recall that $\gamma = \infty$ in this case)

$$M_\gamma = M_\infty := \{\mu > 0 : u^\mu \text{ blows up in finite time}\}.$$

We define the threshold set $\mathcal{T}$ by

$$\mathcal{T} := (0, \infty) \setminus (M_0 \cup M_\gamma).$$

The following lemma is an easy consequence of the stability of the trivial solution.

**Lemma 3.2.** *$M_0$ is a (nonempty) open interval.*

*Proof.* It follows from (2.1) that if $u_0 = c$ is a small positive constant, then the ($x$-independent) solution $u(\cdot, t; c)$ of (1.1), (1.2) converges to 0, as $t \to \infty$. Fix such a constant $c$ and also fix some $t_0 > 0$. By Lemma 3.1, for small $\mu$ we have $u^\mu(\cdot, t_0) < u(\cdot, t_0; c)$, hence, by comparison, $\mu \in M_0$. This shows that $M_0 \neq \emptyset$. Comparison and (I2) imply that $M_0$ is an interval. Finally, if $\mu \in M_0$, then there is $t_1 > 0$ such that $u^\mu(\cdot, t_1) < c/2$. Then, by Lemma 3.1, for $\nu \approx \mu$ we have $u^\nu(\cdot, t_0) < c$ and consequently $\nu \in M_0$. This shows that $M_0$ is open. □

**Lemma 3.3.** *$M_\gamma$ is an unbounded interval or $M_\gamma = \emptyset$.*

*Proof.* To prove the conclusion it is sufficient to show that if $\nu > \mu \in M_\gamma$, then $\nu \in M_\gamma$. In the blowup case this follows directly from (I2) and comparison. Similarly, in the bistable case we obtain by comparison that $\liminf_{t\to\infty} u^\nu(x, t) \geq \lim_{t\to\infty} u^\mu(x, t) = \gamma$ uniformly for $x$ in any compact set. On the other hand, we also have $\limsup_{t\to\infty} u^\nu(x, t) \leq \gamma$ uniformly in $x$. This follows from (BS1) by comparison with a solution of the ODE $\dot{\zeta} = g^S(\zeta) \geq f(t, \zeta)$. Take $\zeta$ with an initial condition

$$\zeta(t_0) = \zeta_0 > \max\{\|u^\nu(\cdot, t_0)\|_{L^\infty(\mathbb{R}^N)}, \gamma\},$$

for some $t_0 > 0$. By (BS1), the ODE solution converges to $\gamma$ and this implies our claim. □

**Lemma 3.4.** *$M_\gamma$ is open, hence, if $M_\gamma$ is nonempty, then $\mathcal{T}$ is a compact interval or a set consisting of just one value.*

We give separate proofs in the bistable case and in the blowup case. In the bistable case, we use the following result.

**Lemma 3.5.** *Let $\beta$ be as in (2.5). Given any $\theta \in (\beta, \gamma)$ there exists a constant $R = R(f, \theta) > 0$ such that the solution of the problem*

$$\begin{aligned} &u_t = \Delta u + g^I(u), \quad x \in \mathbb{R}^N, t > 0, \\ &u(x,0) = \begin{cases} \theta \text{ for } |x| \le R, \\ 0 \text{ for } |x| > R, \end{cases} \end{aligned} \tag{3.3}$$

*satisfies $u(\cdot, t) \to \gamma$ in $L^\infty_{loc}(\mathbb{R}^N)$ as $t \to \infty$.*

For $N = 1$, the results of this form can be found in [10, 14, 15]. For any dimension, the proof of Lemma 3.5 is essentially contained in [2], although it is not apparent from the exposition there that our hypotheses are sufficient. The following comments clarify that.

*Proof of Lemma 3.5.* Assume first that $g^I$ is of class $C^1$. The proof of Lemma 5.1 in [2] shows that the conclusion Lemma 3.5 is valid, provided there exists $c > 0$ with the following property. The ODE

$$q'' + cq' + g^I(q) = 0$$

has a solution $q$ such that $q(0) = \theta$, $q'(0) = 0$, and for some $r_0 > 0$ one has $q' < 0$ in $(0, r_0]$ and $q(r_0) = 0$. By elementary phase plane analysis (cp. [2, Proposition 4.3]), such a solution exists for $c = 0$, and hence for $c \approx 0$, if condition (2.3) is satisfied (which we assume in the bistable case).

Now, to remove the extra requirement of $g^I$ being of class $C^1$, we replace $g^I$ with a $C^1$ function $\tilde{g}$ such that $\tilde{g} \le g^I$, $\tilde{g}(0) = \tilde{g}(\beta) = \tilde{g}(\gamma)$, $\tilde{g} > 0$ in $(\beta, \gamma)$ and $\int_0^\gamma \tilde{g}(\eta)\, d\eta > 0$. Then the above arguments apply to the equation $u_t = \Delta u + \tilde{g}(u)$ and since the solutions of this equation are subsolution of (3.3), the result follows. □

*Proof of Lemma 3.4 in the bistable case.* Fix $\theta \in (\beta, \gamma)$. As in the proof of Lemma 3.3, $\mu \in M_\gamma$ if $\liminf_{t \to \infty} u^\mu(x, t) \ge \gamma$, uniformly for $x$ in any compact set. By comparison and Lemma 3.5, this holds if for some $t_0 > 0$ one has

$$u^\mu(x, t_0) > \theta \quad (|x| \le R), \tag{3.4}$$

with $R = R(f, \theta)$. Conversely, it is obvious that for each $\mu \in M_\gamma$ there is $t_0 = t_0(\mu)$ such that 3.4 holds. Since for a fixed $t_0$, (3.4) is an open property of $\mu \in \mathbb{R}$, by Lemma 3.1, the desired conclusion follows. □

In the blowup case we use the following universal a priori estimate on global solutions.

**Lemma 3.6.** *Given $\delta > 0$ there is a constant $C = C(f, \delta)$ such that any global solution of* (1.1) *satisfies*

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C \quad (t \geq \delta). \tag{3.5}$$

This result is proved in [31], see Theorem 3.1 and its generalization in Section 6 of [31]. We remark that the conditions on the nonlinearity are somewhat stronger in [31], however, it is easy to verify that the arguments given there apply under hypothesis (BL1). This universal estimate is derived in [31] from the parabolic Liouville theorem which says that there is no positive solution of the equation $u_t = \Delta u + u^p$ defined for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$ (an entire solution). As of today, the Liouville theorem has only been proved for $p < p_{BV}$ [5], thus the restriction in our hypothesis (BL1). Alternatively, we could replace the condition $p < p_{BV}$ with the assumption that $p$ is such that the Liouville theorem holds. It is known that in the class of radially symmetric solutions the Liouville theorem holds in the Sobolev-subcritical range $1 < p < p_S = (N+2)/(N-2)_+$ (see [30, 31, 3]). Thus if the initial data $\psi_\mu$ are assumed radially symmetric, the condition $p < p_{BV}$ can be replaced with $p < p_S$.

*Proof of Lemma 3.4 in the blowup case.* We show that the set $[0, \infty) \setminus M_\infty$, consisting of those $\mu$ for which $u^\mu$ is global, is closed. Thus let $\mu_k \in [0, \infty) \setminus M_\infty$, $\mu_k \to \mu$. Assume that $u^\mu$ blows up in finite time. Then $\|u^\mu(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \to \infty$ as $t \to T_\mu$. Consequently, in view of Lemma 3.1, we can make the value of $\|u^{\mu_k}(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}$ arbitrarily large by taking $t$ close to $T_\mu$ and $k$ large. This, however, contradicts the universal estimate on the global solutions $u^{\mu_k}$, as given by Lemma 3.6. □

# 4 Behavior of the threshold solutions

In this section we use the hypotheses and notation of Section 3. We examine the behavior of any threshold solution, that is, the solution $u^\mu$ for any $\mu \in \mathcal{T}$.

## 4.1 Boundedness and spatial decay

We already know that each threshold solution $u$ is bounded (this is nontrivial in the blowup case only), see Lemma 3.6. Our next goal is to prove that

$u(x,t) \to 0$, as $|x| \to \infty$, uniformly with respect to $t > 0$. First we prove that $u$ is decreasing in $r = |x|$, for large enough $r$. One obtains this from the following lemma, taking $u_0 = \psi_\mu$, $\mu \in \mathcal{T}$.

**Lemma 4.1.** *Assume that* $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$, $\operatorname{spt} u_0$ *is compact, and the solution* $u$ *of* (1.1), (1.2) *is global. Given any unit vector* $e \in \mathbb{R}^N$, *one has*

$$e \cdot \nabla u(x,t) \begin{cases} < 0 & \text{if } x \cdot e > \lambda_+^e, \\ > 0 & \text{if } x \cdot e < \lambda_-^e,, \end{cases} \tag{4.1}$$

*where*

$$\begin{aligned} \lambda_+^e &= \sup\{x \cdot e : x \in \operatorname{spt} u_0\}, \\ \lambda_-^e &= \inf\{x \cdot e : x \in \operatorname{spt} u_0\}. \end{aligned} \tag{4.2}$$

*Proof.* This is proved by a standard argument using moving hyperplanes, see for example the Appendix in [7]. □

**Lemma 4.2.** *Let* $u_0$ *and* $u$ *be as in Lemma 4.1. Given a positive constant* $\theta$, *assume that there is a sequence* $(x_k, t_k) \in \mathbb{R}^N \times (0,\infty)$ *such that* $|x_k| \to \infty$ *and* $u(x_k,t_k) \geq \theta$ *for all* $k$. *Then there exist balls* $B_k \subset \mathbb{R}^N$, $k = 1,2,\ldots$ *such that* $|B_k| \to \infty$ *as* $k \to \infty$ *and*

$$u(x,t_k) \geq \theta \quad (x \in B_k). \tag{4.3}$$

*Proof.* Suppose that the conclusion is false. Then, passing to a subsequence, we may assume that any ball $B_k$ for which (4.3) holds has radius bounded above by a constant independent of $k$. Passing to a further subsequence, we may also assume that $x_k/|x_k| \to e_0 \in S^{N-1}$. Then $x_k \cdot e_0 \to \infty$ and we can find $\lambda_0$ and a neighborhood $S_0$ of $e_0$ in $S^{N-1}$ such that, with $\lambda_+^e$ as in (4.2), one has $\lambda_+^e < \lambda_0$ and $x_k \cdot e \to \infty$ for all $e \in S_0$. By Lemma 4.1, $u(\cdot,t_k)$ is decreasing along the line segment

$$L_{e,k} := \{x_k + se : s \in [\lambda_0 - x_k \cdot e, 0]\},$$

hence $u(\cdot,t_k) \geq u(x_k,t_k) \geq \theta$ on $L_{e,k}$. It is obvious that $\cup_{e\in S_0} L_{e,k}$ contains an arbitrarily large ball if $k$ is large, which is a contradiction. □

**Lemma 4.3.** *Let* $\beta$ *be as in* (2.5). *Given any* $\theta \in (\beta,\gamma)$, *there exists a constant* $\rho$ *such that for each* $\mu \in M_0 \cup \mathcal{T}$ *one has*

$$u^\mu(x,t) < \theta \quad (|x| \geq \rho, t > 0). \tag{4.4}$$

*Proof.* Suppose the statement is not true. Then there exist sequences $\mu_k \in M_0 \cup \mathcal{T}$ and $(x_k, t_k) \in \mathbb{R}^N \times (0,\infty)$ such that $|x_k| \to \infty$, and $u^{\mu_k}(x_k, t_k) \geq \theta$ for all $k$. By Lemma 4.2, we then have $u^{\mu_k}(\cdot, t_k) \geq \beta$ on a ball $B_k$ whose radius can be assumed arbitrarily large if $k$ is large enough. In the bistable case, we can now use a comparison with a spatial translation of the solution of (3.3), as in the proof of Lemma 3.4, to conclude that $\mu \in M_\gamma$, a contradiction. In the blowup case, we first find a $C^1$ function $\tilde{g}$ on $[0,\infty)$ such that $\tilde{g} \leq g^I$ everywhere and $\tilde{g}$ is bistable in the sense that it has exactly three zeros, $0 < \beta < \tilde{\gamma}$ in $[0,\infty)$, with $\beta$ is as in (2.5), such that $\tilde{g}'(0), \tilde{g}'(\tilde{\gamma}) < 0$ and $\int_0^{\tilde{\gamma}} \tilde{g}(u)\,du > 0$. In view of the growth condition (BL1), we can easily find such a function with an arbitrarily large $\tilde{\gamma}$. Consider now the solution $\tilde{u}$ of

$$\begin{aligned} &\tilde{u}_t = \Delta \tilde{u} + \tilde{g}(\tilde{u}), \quad x \in \mathbb{R}^N, t > 0, \\ &\tilde{u}(\cdot, 0) = u^{\mu_k}(\cdot, t_k). \end{aligned} \tag{4.5}$$

Using the previous argument for the bistable case, we obtain, if $k$ is sufficiently large, that $\tilde{u}(\cdot, t) \to \tilde{\gamma}$ in $L^\infty_{loc}(\mathbb{R}^N)$ as $t \to \infty$. Then, by comparison, $\liminf_{t\to\infty} u^\mu(x,t) \geq \tilde{\gamma}$ (uniformly on compact sets). Since $\tilde{\gamma}$ can be chosen arbitrarily large, we have a contradiction to the a priori bound on the global solutions $u^{\mu_k}$, as given in Lemma 3.6. □

**Remark 4.4.** Instead of referring to Lemma 3.6 in the previous proof, we could alternatively use a Kaplan-type estimate, which is independent of the Liouville theorem and applies for any $p > 1$. Let us sketch the argument. Using (C1), (C2), (BL1), it is easy to verify that there are positive constants $c_1$, $c_2$ such that $f(t,u) \geq c_1 u^p - c_2 u$, for all $t$ and $u$. Using this, one shows that the function $y(t) := \pi^{-N/2} \int_{\mathbb{R}^N} e^{-|x|^2} u^\mu(x,t)\,dx$ satisfies the inequality $y' \geq c_1 y^p - c_3 y$ for a suitable constant $c_3$. This inequality is obtained by multiplying (1.1) by $\pi^{-N/2} e^{-|x|^2}$, integrating by parts, and applying the Jensen's inequality, see for example [32, Theorem 17.1] for details. For a global solution $u^\mu$, the function $y(t)$ must clearly stay below the positive root of $y \mapsto c_1 y^p - c_3 y$, which gives the following integral a priori bound on $u^\mu$:

$$\int_{\mathbb{R}^N} e^{-|x|^2} u^\mu(x,t)\,dx \leq C_4 \quad (t \geq 0), \tag{4.6}$$

where $C_4$ is a constant independent of $\mu \in M_0 \cup \mathcal{T}$. Clearly this also supplies a contradiction in the above proof.

In a comparison argument below, we employ a ground state of the ODE

$$\varphi_{rr} + h(\varphi) = 0, \tag{4.7}$$

where $h$ is a $C^1$ function on $[0, \infty)$, such that

$$h(0) = 0,\ h'(0) < 0,\ \int_0^u h(\eta)\, d\eta < 0\ (u \in (0, \beta]), \int_0^\gamma h(\eta)\, d\eta > 0. \tag{4.8}$$

These relations, possibly with the exception of $h'(0) < 0$, are satisfied by $h = g^S$, see (C1)-(C2), and (BS1)-(BS2), respectively (BL1)-(BL2). The remaining relation $h'(0) < 0$ may not make sense, since $h = g^S$ is merely Lipschitz in general. However, in view of (C2), one can easily define a $C^1$ function $h$ satisfying all the indicated relations such that $h \geq g^S$. This makes this ODE suitable for a comparison with (1.1).

**Lemma 4.5.** *Let $h$ be a $C^1$ function on $[0, \infty)$ satisfying* (4.8)*. There is a solution of* (4.7) *on $[0, \infty)$ such that $\varphi(0) \in (\beta, \gamma)$, $\varphi_r < 0$ on $(0, \infty)$, and $\varphi(r) \to 0$, as $r \to \infty$, exponentially.*

*Proof.* The result follows by standard and elementary phase plane analysis, using the first integral $\varphi_r^2/2 + H(\varphi)$, $H(u) = \int_0^u h(\eta)\, d\eta$ (cp. [2, Proposition 4.3]). The fact that the convergence of $\varphi$ to 0 is exponential follows from the hyperbolicity of the equilibrium $(0, 0)$ of the planar system corresponding to (4.7). $\square$

We are now ready to show the exponential spatial decay of threshold solutions.

**Lemma 4.6.** *For each $\mu \in \mathcal{T}$ there exist positive constants $C$, $m$, and $\rho$ such that*

$$u^\mu(x, t) \leq C e^{-m|x|} \quad (|x| > \rho\,, t > 0).$$

*Proof.* Fix any $\mu \in \mathcal{T}$. Choose a $C^1$ function $h \geq g^S$ satisfying (4.8) and let $\varphi$ be as in Lemma 4.5. Set $\theta = \varphi(0)$. Let $\rho$ be such that the conclusion of Lemma 4.3 holds and, in addition, $\operatorname{spt} \psi_\mu$ is contained in the ball $B(0, \rho)$. Define

$$\Phi(x) = \varphi(|x| - \rho) \quad (x \in \mathbb{R}^N, |x| \geq \rho).$$

This radially symmetric function satisfies

$$\Delta\Phi(x) + h(\Phi(x)) = \\ \varphi_{rr}(r - \rho) + \frac{N-1}{r}\varphi_r(r - \rho) + h(\varphi(r - \rho)) < 0 \quad (r = |x| > \rho),$$

since $\varphi_r < 0$. Hence $\Phi$ is a supersolution of equation (1.1) in the exterior of $B(0,\rho)$. Moreover, $\Phi > 0 \equiv \psi_\mu$ in $\mathbb{R}^N \setminus B(0,\rho)$ and, by Lemma 4.3,

$$\Phi(x) = \varphi(0) = \theta > u^\mu(x,t) \quad (|x| = \rho, t > 0).$$

Therefore, by comparison,

$$u^\mu(x,t) \le \Phi(x) \quad (|x| \ge \rho, t > 0),$$

which proves the lemma. □

## 4.2 Asymptotic symmetrization: proof of Theorem 2.3

At this point, we do not know whether the threshold set, if nonempty, is an interval or a single value. However, we are already able to prove that *each* threshold solution $u^*$ has the properties stated in Theorem 2.3:

**Proposition 4.7.** *Assume that the hypotheses of Theorem 2.1 or Theorem 2.2 are satisfied and let $\mu \in \mathcal{T}$. Then the solution $u^* := u^\mu$ is global and it has the properties (i)-(iii) listed in Theorem 2.3.*

*Proof.* We know that $u^*$ is bounded by Lemma 3.6. Lemma 4.6 gives exponential spatial decay of $u^*(x,t)$, uniform in time. These properties combined give (i). Further, since $\mu \notin M_0$, $\|u^*(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}$ is bounded below by a positive constant $c$ (namely, the constant $c$ as in the proof of Lemma 3.2). Also, in view of (i), for $R$ sufficiently large we have

$$c \le \|u^*(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \|u^*(\cdot,t)\|_{L^\infty(B(0,R))}. \tag{4.9}$$

Now, since $f(0,t) = 0$, we can write (1.1) as

$$u_t = \Delta u + (f(t,u)/u)u, \quad x \in \mathbb{R}^N, t > 0, \tag{4.10}$$

and view it as a linear equation in which the coefficient $f(t,u(x,t))/u(x,t)$ is bounded (when the solution $u$ is bounded). The Harnack inequality and (4.9) then imply that for each $x \in \mathbb{R}^N$ the function $t \mapsto u(x,t)$ is bounded below by a positive constant, possibly depending on $x$, hence (ii) holds. For solutions with these properties, the asymptotic symmetry statement (iii) is a reformulation of the conclusion of Theorem 1.1 in [28]. □

### 4.3 Proofs of Propositions 2.4, 2.5

Statement (i) of Proposition 2.4 is proved in the proof of Lemma 3.4. To prove statement (ii), assume that $f_u$ is bounded. Writing equation (1.1) as (4.10), we note that $|f(t, u(x,t))/u(x,t)|$ is bounded, for any solution $u = u^\mu$, by a constant $M$ independent of $\mu$. This implies, by comparison, that for all $x \in \mathbb{R}^N$ and $t > 0$ one has $e^{Mt}u^\mu(x,t) \geq v(x,t)$, where $v$ is the solution of the heat equation $v_t = \Delta v$ with the initial condition $\psi_\mu$. By the integral representation of $v$, it is clear that given any balls $B$, $B_1$ in $\mathbb{R}^N$, there is a constant $C = C(B, B_1)$ such that

$$e^M u^\mu(x,1) \geq v(x,1) \geq C \int_B \psi_\mu(y)\,dy \quad (x \in B_1).$$

We can now apply the arguments of the proof of Lemma 3.4 again to obtain statement (ii).

Proposition 2.5 follows from Remark 4.4. More specifically, the a priori bound (4.6) implies that if $u^\mu$ is global, then on any ball $B$ the integral $\int_B \psi_\mu(x)\,dx$ is bounded above by a universal constant independent of $\mu$. This readily implies the conclusion of Proposition 2.5.

# 5 Instability of threshold solutions and sharp transitions

In this section we prove Theorems 2.1, 2.2. Crucial to that aim is the following instability result concerning localized solutions of (1.1). This theorem is of independent interest and we will prove it in a more general setting than that of our main results; note in particular that none of the hypotheses (BS1), (BS2), (BL1), (BL2) is assumed in it.

**Theorem 5.1.** *Assume conditions* (C1), (C2) *and let* $u_0^* \in L^\infty(\mathbb{R}^N)$*, be such that the solution* $u^* := u(\cdot,\cdot;u_0^*)$ *is bounded,* $u^*(x,t) \to 0$ *as* $x \to \infty$ *uniformly with respect to* $t > 0$*, and* $\liminf_{t\to\infty} \|u^*(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} > 0$*. Then for each* $\rho$ *there is a positive constant* $d$ *such that the following statement holds true. For any* $u_0 \in L^\infty(\mathbb{R}^N) \setminus \{u_0^*\}$ *such that* $0 \leq u_0 \leq u_0^*$ *or* $u_0 \geq u_0^*$ *and the solution* $u(\cdot,t;u_0)$ *is global, one has*

$$\liminf_{t\to\infty} |u^*(x,t) - u(x,t;u_0)| \geq d \quad (x \in B(0,\rho)). \tag{5.1}$$

Before proving this result, we show how it implies Theorems 2.1, 2.2.

*Proof of Theorems 2.1, 2.2.* We only need to prove that the threshold set $\mathcal{T}$ does not contain any interval $[\mu_1, \mu_2]$ with $\mu_1 < \mu_2$. Suppose it does: $[\mu_1, \mu_2] \subset \mathcal{T}$. For each $\nu \in \mathcal{T}$, Theorem 5.1 applies to the threshold solution $u^* := u^\nu$ (the hypotheses of Theorem 5.1 are satisfied by Proposition 4.7). Fixing $\rho = 1$, Theorem 5.1 in particular implies that there exists a constant $\bar{d}(\nu) > 0$ such that

$$\liminf_{t\to\infty} |u^\nu(0,t) - u^\mu(0,t)| \geq \bar{d}(\nu) \tag{5.2}$$

for each $\mu \in [\mu_1, \mu_2]$, $\mu \neq \nu$.

Pick any finite sequence $\nu_1 < \nu_2 < \cdots < \nu_k$ in $[\mu_1, \mu_2]$. Using (5.2), we obtain

$$\begin{aligned}
&\liminf_{t\to\infty}(u^{\mu_2}(0,t) - u^{\mu_1}(0,t)) = \\
&\liminf_{t\to\infty}(u^{\mu_2}(0,t) - u^{\nu_k}(0,t) + u^{\nu_k}(0,t) - u^{\nu_{k-1}}(0,t) + \dots \\
&\qquad\qquad\qquad\qquad\qquad\qquad + u^{\nu_1}(0,t) - u^{\mu_1}(0,t)) \\
&\qquad \geq \bar{d}(\nu_k) + \cdots + \bar{d}(\nu_1).
\end{aligned} \tag{5.3}$$

Since $k$ and $\nu_1, \dots, \nu_k$ can be chosen arbitrarily, for each $m > 0$ there can be at most finitely many values $\nu$ with $\bar{d}(\nu) > 1/m$ (otherwise we could make the last sum in (5.3) arbitrarily large, contradicting the boundedness of $u^{\mu_2}$, $u^{\mu_1}$). Hence the set

$$\bigcup_{m=1}^{\infty} \{\nu \in [\mu_1, \mu_2] : \bar{d}(\nu) > \frac{1}{m}\}$$

is at most countable. But this set clearly covers $[\mu_1, \mu_2]$ and we have a contradiction. □

Our proof of Theorem 5.1 is based on some results on exponential separations and principal Floquet bundles for linear parabolic equations. For the reader's convenience we included the relevant definitions and theorems in the appendix. We also prove there the following result. Let $u^*$ be as in Theorem 5.1 and let

$$a^*(x,t) := f_u(u^*(x,t),t) \quad (x \in \mathbb{R}^N, t \geq 0). \tag{5.4}$$

We consider the linear Dirichlet problem

$$\begin{aligned} v_t &= \Delta v + \tilde{a}(x,t)v, \quad x \in B(0,R), t > 0, \\ v &= 0, \quad x \in \partial B(0,R), t > 0, \end{aligned} \tag{5.5}$$

where $\tilde{a}$ is a small perturbation of $a^*$.

**Lemma 5.2.** *Assume* (C1), (C2). *Let $u^*$ be as in Theorem 5.1 and $a^*$ as in* (5.4)*. Then there are positive constants $R$, $\omega$, $\epsilon$, $C_0$ with the following property. For each bounded measurable function $\tilde{a}$ on $B(0,R) \times (0,\infty)$ satisfying $\|\tilde{a} - a^*\|_{L^\infty(B(0,R)\times(0,\infty))} \le \epsilon$ there exists a positive solution $\varphi$ of* (5.5) *such that*

$$\|\varphi(\cdot,t)\|_{L^\infty(B(0,R))} \ge C_0 e^{\omega(t-s)} \|\varphi(\cdot,s)\|_{L^\infty(B(0,R))} \quad (t \ge s > 0) \tag{5.6}$$

*for some positive constant $C_0$.*

This lemma gives a sort of linear instability of $u^*$. The reason for its validity is, briefly, as follows. The linear equation $v_t = \Delta v + a^*(x,t)v$ on $\mathbb{R}^N \times (0,\infty)$ has a localized sign-changing solution $v = u^*_{x_1}$ whose $L^\infty$-norm is bounded below by a positive constant. An exponential separation result then implies that there must be a positive exponentially growing solution of this equation. Then there is also a positive exponentially growing solution of the Dirichlet problem for that equation on a sufficiently large ball. The same remains valid if the coefficient $a^*$ is perturbed slightly. See the appendix for details.

*Proof of Theorem 5.1.* We start with some preliminary remarks. First we note that without loss of generality we may assume that there is a constant $\gamma$ such that

$$\gamma > \|u^*\|_{L^\infty(\mathbb{R}^N\times(0,\infty))} \text{ and } f(t,\gamma) \le 0 \quad (t \ge 0). \tag{5.7}$$

Indeed, if this is not satisfied, we can easily modify $f(t,u)$, by making it smaller for large values of $u$ only (away from the range of $u^*$), so as to achieve the new condition. The solutions below $u^*$ and $u^*$ itself are not affected by this modification. On the other hand the solutions of the modified equation which are above $u^*$ are subsolutions of the original equation, hence it is clearly sufficient to prove the result for the modification. We proceed assuming that (5.7) is satisfied. It is further sufficient to consider initial data satisfying

$u_0 \leq \gamma$ only: the validity of (5.1) for some $u_0$ implies it for any larger $u_0$, by comparison.

Let $R$, $\omega$, $\epsilon$, and $C_0$ be as in Lemma 5.2. Using condition (C1), we find a constant $\delta > 0$ such that

$$\left| \frac{f(t,\bar{u}) - f(t,u)}{\bar{u} - u} - f_u(t,u) \right| < \epsilon \quad (u, \bar{u} \in [0,\gamma], 0 < |u - \bar{u}| \leq \delta). \tag{5.8}$$

Now let $u_0$ be as in Theorem 5.1. As remarked above, we may without loss of generality assume that $u_0 \leq \gamma$. For brevity we denote $u(x,t) := u(x,t;u_0)$. We only consider the case $u_0 \geq u_0^*$, the case $u_0 \leq u_0^*$ is analogous. By (5.7), $u(\cdot,t) \leq \gamma$ for all $t > 0$.

Define

$$\tilde{a}(x,t) = \begin{cases} \dfrac{f(t,u(x,t)) - f(t,u^*(x,t))}{u(x,t) - u^*(x,t)} & \text{on } \{(x,t) \in \bar{B}(0,R) \times (0,\infty) : \\ & \qquad 0 < |u(x,t) - u^*(x,t)| \leq \delta\}, \\ a^*(x,t) \ \ (= f_u(u^*(x,t),t)) & \text{elsewhere in } \bar{B}(0,R) \times (0,\infty). \end{cases} \tag{5.9}$$

Clearly, this is a bounded measurable function satisfying

$$\|\tilde{a} - a^*\|_{L^\infty(B(0,R)\times(0,\infty))} < \epsilon.$$

Thus there exists a positive solution $\varphi$ of (5.5) satisfying (5.6). We first use this to show that the relation $\|u^*(\cdot,t) - u(\cdot,t)\|_{L^\infty(B(0,R))} \leq \delta$ does hold on any unbounded time interval $[\tau,\infty) \subset (0,\infty)$. Suppose it does. Then, by the definition of $\tilde{a}$, $u - u^*$ solves the equation

$$v_t = \Delta v + \tilde{a}(x,t)v, \quad x \in B(0,R), t \geq \tau \tag{5.10}$$

(the same equation as in (5.5)). Since $u - u^*$ is positive and $\varphi$ vanishes on $\partial B(0,R) \times (0,\infty)$, a comparison argument shows that for a sufficiently small $q > 0$ one has $\delta \geq u - u^* > q\varphi$ on $B(0,R) \times [\tau,\infty)$. However, this is impossible by (5.6).

Thus we either have

$$\liminf_{t\to\infty} \|u^*(\cdot,t) - u(\cdot,t)\|_{L^\infty(B(0,R))} \geq \delta \tag{5.11}$$

or there are intervals $(\tau,T)$, with arbitrarily large $\tau$, such that

$$\|u^*(\cdot,t) - u(\cdot,t)\|_{L^\infty(B(0,R))} \begin{cases} = \delta & \text{for } t = \tau, \\ < \delta & \text{for } t \in (\tau,T) \\ = \delta & \text{for } t = T. \end{cases} \tag{5.12}$$

Assume first that the latter occurs and consider any such interval $(\tau, T]]$ assuming $\tau > 1$. Note that, by the definition of $\tilde{a}$, $u - u^*$ solves (5.10) on $[\tau, T]$.

We first claim that there is $\sigma > 0$ independent of $u_0$ and $\tau$ such that

$$\|u(\cdot, t) - u^*(\cdot, t)\|_{L^\infty(B(0,R))} \geq \delta/2 \quad (t \in [\tau, \tau + \sigma]). \tag{5.13}$$

This readily follows from the fact that $u$ and $u^*$ are bounded, $\zeta$-Hölder functions on $\mathbb{R}^N \times [\tau, \infty)$ ($\zeta \in (0, 1)$), with $\zeta$-Hölder norms bounded by a constant independent of $u_0$. The latter follows from (1.1) by parabolic estimates and the relations $0 \leq u^* \leq u \leq \gamma$.

If $T \leq \tau + \sigma$, we just use estimate (5.13). Otherwise, we continue observing that $u - u^*$ is a positive solution of the linear equation

$$v_t = \Delta v + b(x, t)v, \quad x \in \mathbb{R}^N, t > 0, \tag{5.14}$$

with

$$b(x, t) = \int_0^1 f_u(t, u^*(x, t) + s(u(x, t) - u^*(x, t)))\, ds.$$

Clearly $b$ is continuous and, since $0 \leq u^* \leq u \leq \gamma$, $b$ has its absolute value bounded by a constant $K$ independent of $u_0$. Therefore it is legitimate to use the Harnack inequality to infer that for some constant $\kappa$ independent of $u_0$ and $\tau$ (recall that $\tau > 1$) one has

$$\begin{aligned} u(x, \tau + \sigma) - u^*(x, \tau + \sigma) &\geq \kappa \|u(\cdot, \tau) - u^*(\cdot, \tau)\|_{L^\infty(B(0,R))} \\ &\geq \kappa\delta \quad (x \in B(0, R)). \end{aligned} \tag{5.15}$$

Consequently, since $u - u^*$ and $\varphi$ solve the same linear equation on $B(0, R) \times (\tau, T)$ and $u - u^* > 0 = \varphi$ on $\partial B(0, R) \times (\tau, T)$, we obtain by comparison that

$$\begin{aligned} u(x, t) - u^*(x, t) &\geq \kappa\delta \frac{\varphi(x, t)}{\|\varphi(\cdot, \tau + \sigma)\|_{L^\infty(B(0,R))}} \\ &\geq C_0 \kappa\delta e^{\omega(t - \tau - \sigma)} \\ &\geq C_0 \kappa\delta \quad (x \in B(0, R), t \in [\tau + \sigma, T]). \end{aligned} \tag{5.16}$$

Combining this estimate with (5.13), we conclude that for all sufficiently large $t$ one has

$$\|u^*(\cdot, t) - u(\cdot, t)\|_{L^\infty(B(0,R))} \geq d_0 := \min\{\delta/2, C_0 \kappa\delta\}. \tag{5.17}$$

This conclusion trivially holds as well if alternative (5.11) takes place.

Using Harnack inequality, (viewing $u-u^*$ as a positive solution of (5.14)), we conclude that for each $\rho > R$ there is $M = M(\rho)$ such that

$$u(x,t)-u^*(x,t) \geq M\|u(\cdot,t-1)-u^*(\cdot,t-1)\|_{L^\infty(B(0,R))} \quad (x \in B(0,\rho), t > 2). \tag{5.18}$$

In combination with (5.17), this implies that (5.1) holds with $d = Md_0$. The proof of the Theorem is now complete. □

# 6 Discussion and examples

We start the discussion with a comment on time-periodic equations (1.1). Assume that $f(\cdot+\tau,\cdot) \equiv f$ for some $\tau > 0$. Further assume that conditions (C1), (C2) are satisfied, and either (BS1)-(BS2) or (BL1)-(BL2) are satisfied. We show how our results on threshold solutions can be used to find positive $\tau$-periodic solutions $\bar{u}$ of (1.1) which are localized in space in the sense that $\bar{u}(x,t) \to 0$ as $|x| \to \infty$, uniformly in $t$ (any time-periodic solution with this property is necessarily radially symmetric in $x$ around some $\xi \in \mathbb{R}^N$ and radially decreasing away from $\xi$, see [29]). Take a family of radially symmetric initial conditions such that (I1), (I2) hold and $\mathcal{T} \neq \emptyset$, see Propositions 2.4, 2.5 for sufficient conditions. As we are taking radial initial data, in the blowup case the condition $p < p_{BV}$ can be replaced with $p < p_S$ (see the remarks following (BL2)). It can be proved that the threshold solution $u^*(\cdot,t)$ approaches a $\tau$-periodic solution $\bar{u}$: $\|u^*(\cdot,t) - \bar{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to \infty$ (the solution $\bar{u}$ is automatically spatially localized by Theorem 2.3). Since we are dealing with radially symmetric solutions, this convergence result can be proved using standard intersection-comparison arguments as in [6, Theorem 6.1], [14, Lemma 3.5]. In fact, in [6, Theorem 6.1], the convergence of the threshold solution to a periodic solution is proved for the nonlinearity $f(t,u) = m(t)(u^p - u)$, with $m$ periodic and bounded above and below by positive constants. The same arguments apply in the present more general setting.

Our next remark concerns the assumption on the exponent $p$ in (BL1). Using the example $f(t,u) = f(u) := u^p - u$, we illustrate that our results are not valid if $p > p_S$. Indeed, consider the family $\psi_\mu = \mu\psi$, where $\psi \not\equiv 0$ is a nonnegative continuous function on $\mathbb{R}^N$ with compact support and, as above, let $u^\mu$ be the solution of (1.1), (1.2) with $u_0 = \psi_\mu$. As in Lemma 3.2,

$M_0$ is an open interval. A Kaplan estimate, as in Remark 4.4, shows that for large $\mu$ the solution $u^\mu$ blows up in finite time, hence, as in Lemma 3.3, $M_\infty$ is an unbounded interval. Let us now define the threshold set a little differently than before:

$$\mathcal{T} := (0, \infty) \setminus (M_0 \cup M_\infty^o),$$

where $M_\infty^o$ stands for the interior of $M_\infty$. Clearly, $\mathcal{T}$ is a nonempty compact interval or a single value and its definition coincides with the one in Section 3 if $M_0$ is open. Here is a difference from the subcritical case: for any $\mu^* \in \mathcal{T}$, the solution $u^* := u^{\mu^*}$ is unbounded. Indeed, assume that $u^*$ is bounded. Then the arguments of Section 4 apply and they show that $u^*$ has the properties (i)-(iii) of Theorem 2.3. Then, by [7], $u^*(\cdot, t)$ converges, as $t \to \infty$, to a ground state - a positive steady state of (1.1) which decays to 0 at $|x| = \infty$. However, in the supercritical case, $p > p_S$, there is no ground state (see [4, Section 2]) and we have a contradiction. It is not a trivial matter to determine if the threshold solution can actually be global unbounded. However, in more specific situations this problem can be settled with the method of [8]. Specifically, assume that

$$p_S < p < p_{JL} := \begin{cases} \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N > 10, \\ \infty & \text{if } N \leq 10, \end{cases}$$

and $\psi$ is a radially symmetric and radially nonincreasing function. It appears that in this situation one can adapt arguments of [8] to conclude that $u^*$ has to blow up in finite time. Then, obviously there is just one threshold value $\mu^*$ and it belongs to $M_\infty$. Hence $M_\infty$ is closed and there is no intermediate behavior between blowup and decay.

Let us now discuss the possibility of taking more general families of initial data in our results. The main reason for assuming that each $\psi_\mu$ has compact support is Lemma 4.1, which gives the monotonicity of $u^\mu(\cdot, t)$ in $r = |x|$ outside a fixed ball independent of $t$. This lemma is proved using the method of moving hyperplanes which can be applied under more general assumptions. For example, one can assume that there is $\lambda_0 > 0$ such that for any direction $e \in S^{N-1}$, any $\mu \in (0, \infty)$, and any $\lambda > \lambda_0$ one has

$$\varphi_\mu(P_\lambda^e x) - \varphi_\mu(x) \geq 0 \quad (x \in \mathbb{R}^N, x \cdot e > \lambda), \tag{6.1}$$

where $P^e_\lambda$ is the reflection in the hyperplane $\{x : x \cdot e = \lambda\}$. This assumption can be used to replace the assumption of $\operatorname{spt}\psi_\mu$ being compact in our hypotheses. The results have to be modified so as to say that the the threshold solution decays (not necessarily exponentially) as $|x| \to \infty$, uniformly with respect to $t > 0$. Perhaps other hypotheses, like specific asymptotic expansions of the initial data at $\infty$, can be considered as well. Note, however, that the mere radial decay of $\psi_\mu$ outside a fixed ball is not sufficient for our arguments. It is clear that the initial data cannot be too general, otherwise the threshold set may be an interval. The simplest example is found with a bistable nonlinearity $f(t, u) = f(u)$ which has an interval $J$ of zeros between the two asymptotically stable zeros 0 and $\gamma$ (note that this is not ruled out in the hypotheses (BS1), (BS2)). For constant initial data $\psi_\mu \equiv \mu$, the whole interval $J$ is contained in the threshold set. It might be an interesting problem to find "sharp" conditions on the initial families which rule out the occurrence of a "fat threshold".

We finish the discussion with an example where a "fat threshold" between decay and blowup occurs due to a violation of condition (2.4). Similar examples can be found for the threshold between decay and locally uniform convergence to a positive constant.

Consider the equation

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\, t > 0. \tag{6.2}$$

Here $f$ is a smooth function, with superlinear polynomial growth $f(u) = u^p + o(u^p)$, as $u \to \infty$, such that $f(0) = 0 > f'(0)$ and $f$ has exactly three positive zeros $\beta > \gamma_0 > \beta_0$, all of them simple, and

$$\int_{\beta_0}^{\gamma_0} f(\eta)\, d\eta > \int_0^{\beta_0} f(\eta)\, d\eta + \int_{\gamma_0}^{\beta} f(\eta)\, d\eta$$

(see Figure 3). Since for $N = 1$ we have $p_{BV} = \infty$, the hypotheses (C1), (C2), (BL1) are clearly satisfied, but the hypothesis (BL2) is not satisfied. Note that in the interval $[0, \gamma_0]$, we have a bistable nonlinearity. Choose $\theta \in (\beta_0, \gamma_0)$. By Lemma 3.5, there is $R > 0$ such that if $u_0$ is a continuous function satisfying $0 \le u_0 < \gamma_0$ everywhere and $u_0 \ge \theta$ on $(-R, R)$, then the solution of (6.2) with $u(\cdot, 0) = u_0$ converges to $\gamma_0$ in $L^\infty_{loc}(\mathbb{R})$, as $t \to \infty$.

Now let $\psi_\mu$, $\mu \ge 0$, be a family of continuous functions satisfying (I1), (I2) such that

$$\lim_{\mu\to\infty} \int_{-R}^{R} \psi_\mu(x)\, dx = \infty$$

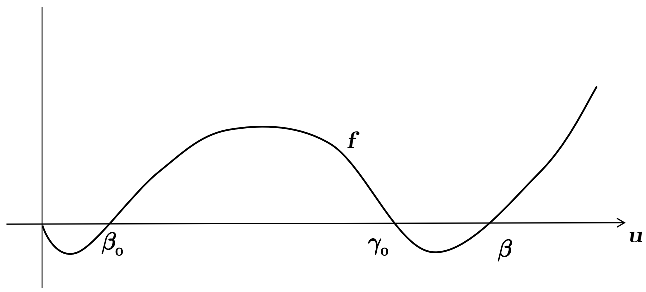


Figure 3: The graph of f.

and there is an interval $J \subset (0, \infty)$ such that

$$\psi_\mu(x) < \gamma_0 \quad (x \in \mathbb{R}, \mu \in J), \qquad \psi_\mu(x) \geq \theta \quad (x \in (-R, R), \mu \in J). \quad (6.3)$$

Then, as in Proposition 2.5, $M_\infty \neq \emptyset$ and (6.3) clearly implies $J \subset \mathcal{T}$, so that the threshold set is "fat". An interested reader may enjoy the exercise to give the following complete description of the threshold set in this example. There are $0 < \mu_1 < \mu_2 < \infty$ such that $\mathcal{T} = [\mu_1, \mu_2]$ and, as $t \to \infty$,

(i) $u^{\mu_1}(\cdot, t) \to \varphi_0$ in $L^\infty(\mathbb{R})$, where $\varphi_0$ is a ground state of (6.2) (this is also the ground state of the bistable equation in $[0, \gamma_0]$),

(ii) for each $\mu \in (\mu_1, \mu_2)$ one has $u^\mu(\cdot, t) \to \gamma_0$ in $L^\infty_{loc}(\mathbb{R})$,

(iii) $u^{\mu_2}(\cdot, t) \to \varphi_1$ in $L^\infty_{loc}(\mathbb{R})$, where $\varphi_1$ is a steady state of (6.2) such that $\varphi_1 > \gamma_0$ and $\varphi_1(x) \to \gamma_0$, as $x \to \pm\infty$ ($\varphi_1$ is a "ground state" of the equation in $[\gamma_0, \infty]$).

# 7 Appendix: exponential separation and principal Floquet bundle

In this section we recall several results from [19, 20] on exponential separations and principal Floquet bundles for linear parabolic equations, and use them to prove Lemma 5.2.

Let $a$ be a bounded measurable function on $\mathbb{R}^N \times \mathbb{R}$ satisfying the following condition.

(A) There are positive constants $d_0$, $\alpha$, and $\rho_0$ such that $\|a\|_{L^\infty(\mathbb{R}^N\times\mathbb{R})} \le d_0$ and $a(x,t) \le -\alpha$ for almost all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ with $|x| \ge \rho_0$.

For $\tau \ge -\infty$, we consider the linear equation

$$v_t = \Delta v + a(x,t)v, \quad x \in \mathbb{R}^N, t \ge \tau, \tag{7.1}$$

and its adjoint equation

$$-v_t = \Delta v + a(x,t)v, \quad x \in \mathbb{R}^N, t \ge \tau. \tag{7.2}$$

We also consider the following Dirichlet problem on balls with large radii $R$:

$$\begin{aligned} &v_t = \Delta v + a(x,t)v, \quad x \in B(0,R), t > \tau, \\ &v = 0, \quad x \in \partial B(0,R), t > 0. \end{aligned} \tag{7.3}$$

Solutions of (7.1), (7.2), or (7.3) with $\tau = -\infty$ are referred to as *entire solutions*. We denote by $v(\cdot,t;s,u_0) \in L^\infty(\mathbb{R}^N)$, $t \ge s$, the solution of (7.1) with the initial condition $v(\cdot,s) = u_0 \in L^\infty(\mathbb{R}^N)$.

It is known (see [20], [25], or [26]) that for each $R$ there is a unique positive entire solution $\varphi_R$ of (7.3) such that $\|\varphi_R(\cdot,0)\|_{L^\infty(B(0,R))} = 1$. It will be important to consider exponents $\lambda \in \mathbb{R}$ such the following estimate holds

$$\frac{\|\varphi_R(\cdot,t)\|_{L^\infty(B(0,R))}}{\|\varphi_R(\cdot,s)\|_{L^\infty(B(0,R))}} \ge C\, e^{\lambda(t-s)} \quad (t \ge s \ge \tau) \tag{7.4}$$

for some constant $C$ (the maximal exponent $\lambda$ for which this is true is called the lower principal Lyapunov exponent of (7.3)).

**Theorem 7.1.** *Assume that* (A) *holds and there is* $R > 0$ *such that for some* $C > 0$ *and* $\lambda > -\alpha$ *condition* (7.4) *holds with* $\tau = -\infty$. *Then the following statements are valid.*

*(i) There exist positive entire solutions* $\varphi$, $\psi$ *of* (7.1) *and* (7.2), *respectively, such that for all* $(x,t) \in \mathbb{R}^N \times \mathbb{R}$

$$\frac{\varphi(x,t)}{\|\varphi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} \le c_1\, e^{-\sqrt{\epsilon_0}|x|}, \quad \frac{\psi(x,t)}{\|\psi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} \le c_1\, e^{-\sqrt{\epsilon_0}|x|} \tag{7.5}$$

*and for all $t \geq s$*

$$\frac{\|\varphi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}}{\|\varphi(\cdot,s)\|_{L^\infty(\mathbb{R}^N)}} \geq c_2\, e^{\lambda\,(t-s)} \quad \text{and} \quad \frac{\|\psi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}}{\|\psi(\cdot,s)\|_{L^\infty(\mathbb{R}^N)}} \leq c_3\, e^{-\lambda\,(t-s)}. \quad (7.6)$$

*Here, $\epsilon_0 := \lambda + \alpha > 0$ and $c_1$, $c_2$, $c_3$ are positive constants determined only by the dimension $N$ and the quantities $d_0$, $\rho_0$, $\alpha$, $C$, $\lambda$, $R$ appearing in* (A) *and* (7.4)*.*

*(ii) For each $t \in \mathbb{R}$ the sets*

$$X^1(t) := \operatorname{span}\{\varphi(\cdot,t)\},$$
$$X^2(t) := \{v \in L^\infty(\mathbb{R}^N) : \int_{\mathbb{R}^N} \psi(x,t)v(x) = 0\} \quad (t \in \mathbb{R})$$

*are closed subspaces of $L^\infty(\mathbb{R}^N)$ which are invariant under* (7.1) *in the following sense: if $i \in \{1,2\}$ and $v_0 \in X^i(s)$, then $v(\cdot,t;s,v_0) \in X^i(t)$ $(t \geq s)$. Moreover, $X^1(t)$, $X^2(t)$ are complementary subspaces of $L^\infty(\mathbb{R}^N)$:*

$$L^\infty(\mathbb{R}^N) = X^1(t) \oplus X^2(t) \quad (t \in \mathbb{R}). \quad (7.7)$$

*(iii) There are constants $K, \vartheta > 0$ determined only by $N$, $d_0$, $\rho_0$, $\alpha$, $C$, $\lambda$, $R$ such that for any $v_0 \in X^2(s)$ one has*

$$\frac{\|v(\cdot,t;s,v_0)\|_{L^\infty(\mathbb{R}^N)}}{\|\varphi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} \leq K\, e^{-\vartheta\,(t-s)} \frac{\|v_0\|_{L^\infty(\mathbb{R}^N)}}{\|\varphi(\cdot,s)\|_{L^\infty(\mathbb{R}^N)}} \quad (t \geq s). \quad (7.8)$$

We refer to the collection of the one-dimensional spaces $X^1(t)$, $t \in \mathbb{R}$, as the *principal Floquet bundle* of (7.1) and to $X^2(t)$, $t \in \mathbb{R}$, as its *complementary Floquet bundle.* Property (ii) is an *exponential separation* between these two bundles. The existence of the Floquet bundles with exponential separation extends in a natural way properties of the principal eigenvalue of time-independent (or time-periodic) parabolic problems. On bounded domains, results analogous to Theorem 7.1 were first proved in [24, 27] and have since been significantly improved and generalized, see [20, 17, 18, 26] and references therein. On $\mathbb{R}^N$, one may not have exponential separation even in the autonomous case: $a = a(x)$, as the presence of the essential spectrum complicates matters. One needs an extra condition, such as the existence of eigenvalues above the top of the essential spectrum for the autonomous case or the assumption in the previous theorem, to guarantee the existence of the principal Floquet bundle with exponential separation (see [19] for more on this).

Theorem 7.1 is proved in [19]. Note that the hypotheses in [19] are different in that $\alpha = 0$ in (A) and then it is required that $\lambda > 0$ in (7.4). Using the usual transformation $\tilde{v} = e^{\alpha t} v$, which transforms (7.1) to $\tilde{v}_t = \Delta \tilde{v} + (a(x,t)+\alpha)\tilde{v}$, we bring the present setting to that of [19]. Theorem 7.5 is then obtained by a mere reformulation of Theorems 2.1 and 2.2 of [19]. This also applies to other results of [19] quoted below.

**Remark 7.2.** It will be useful to observe that the space $X_2(s)$ contains all $v_0 \in L^\infty(\mathbb{R}^N)$ such that the solution $v(\cdot,t;s,v_0)$ is not eventually positive or eventually negative in some fixed ball $B$. More specifically, $v_0$ necessarily belongs to $X_2(s)$ if there exist a ball $B$ and a sequence $(x_k,t_k) \in B \times (s,\infty)$ such that $t_k \to \infty$ and $v(x_k,t_k;s,v_0) = 0$. To prove this, write $v_0$ as $v_0 = q\phi(\cdot,s) + \tilde{v}_0$, where $q \in \mathbb{R}$ and $\tilde{v}_0 \in X_2(s)$. We verify that in fact $q = 0$. Assume $q > 0$ ($q < 0$ is ruled out analogously). For $t \geq s$, we have $v(\cdot,t;s,v_0) = q\phi(\cdot,t) + v(\cdot,t;1,\tilde{v}_0)$. Consider the function

$$\begin{aligned} \frac{v(x,t;s,v_0)}{\|\phi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} &= \frac{q\phi(x,t)}{\|\phi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} + \frac{v(x,t;s,\tilde{v}_0)}{\|\phi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} \\ &\geq \frac{q\phi(x,t)}{\|\phi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} - K\, e^{-\vartheta\,(t-s)} \frac{\|\tilde{v}_0\|_{L^\infty(\mathbb{R}^N)}}{\|\varphi(\cdot,s)\|_{L^\infty(\mathbb{R}^N)}}. \end{aligned} \tag{7.9}$$

For $x \in \bar{B}$, the first function in (7.9) is bounded below by a positive constant (see Lemma 5.3 in [19]). Therefore, $v(\cdot,t;s,v_0) > 0$ on $B$ for all sufficiently large $t$, which is a contradiction.

The following result is a reformulation of [19, Theorem 2.4].

**Theorem 7.3.** *Assume that* (A) *holds. Assume further that for some* $\tau \in \mathbb{R}$ *there exists a (possibly sign-changing) solution* $\phi$ *of* (7.1) *such that* $\phi(\cdot,t) \in L^\infty(\mathbb{R}^N)$ *for all* $t \geq \tau$ *and for some constants* $C_0 > 0$ *and* $\lambda_0 > -\alpha$ *one has*

$$\frac{\|\phi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}}{\|\phi(\cdot,s)\|_{L^\infty(\mathbb{R}^N)}} \geq C_0\, e^{\lambda_0\,(t-s)} \quad (t \geq s \geq \tau). \tag{7.10}$$

*Then for each* $\lambda < \lambda_0$ *there exist* $R > 0$ *and* $C > 0$ *such that*

$$\frac{\|\varphi_R(\cdot,t)\|_{L^\infty(B(0,R))}}{\|\varphi_R(\cdot,s)\|_{L^\infty(B(0,R))}} \geq C\, e^{\lambda\,(t-s)} \quad (t \geq s \geq \tau). \tag{7.11}$$

*The constants* $R$ *and* $C$ *depend only on* $\lambda_0 - \lambda$, $N$, $d_0$, $\rho_0$, $C_0$, $\lambda_0$, *and* $\alpha$.

Finally, we recall a perturbation result for (7.3). It concerns the problem

$$\begin{aligned} v_t &= \Delta v + \tilde{a}(x,t)v, \quad x \in B(0,R), t > \tau, \\ v &= 0, \quad x \in \partial B(0,R), t > 0, \end{aligned} \tag{7.12}$$

where $\tilde{a}$ is a small perturbation of $a$. We denote by $\tilde{\varphi}_R$ the unique positive entire solution of this equation such that $\|\tilde{\varphi}_R(\cdot,0)\|_{L^\infty(B(0,R))} = 1$. The following result is a special case of [20, Proposition 8.3].

**Theorem 7.4.** *Given $R > 0$ and $a \in L^\infty(B(0,R)\times\mathbb{R})$ assume that $\varphi_R$ satisfies (7.4) with $\tau = \infty$ for some $\lambda \in \mathbb{R}$ and $C > 0$. Then for each $\eta > 0$ there exist positive constants $\epsilon_\eta$ and $C_\eta$ with the following property. If $\tilde{a} \in L^\infty(B(0,R)\times\mathbb{R})$ and $\|\tilde{a} - a\|_{L^\infty(B(0,R)\times\mathbb{R})} < \epsilon_\eta$, then*

$$\frac{\|\tilde{\varphi}_R(\cdot,t)\|_{L^\infty(B(0,R))}}{\|\tilde{\varphi}_R(\cdot,s)\|_{L^\infty(B(0,R))}} \ge C_\eta\, e^{(\lambda-\eta)(t-s)} \quad (-\infty < s \le t < \infty) \tag{7.13}$$

We are now ready to prove Lemma 5.2.

*Proof of Lemma 5.2.* Let $u^*$ and $a^*$ be as in the hypotheses of the lemma. In order to apply the above results, we need to choose an extension $a$ of the function $a^*$ to $\mathbb{R}^N\times\mathbb{R}$. At the first step, we use any extension satisfying (A1), for example set $a \equiv a^*$ on $\mathbb{R}^N\times[0,\infty)$ and $a \equiv -\alpha$ on $\mathbb{R}^N\times(-\infty,0)$. Condition (A) is then satisfied, since by (2.1) and the uniform spatial decay of $u^*(x,t)$ we have $a^*(x,t) = f_u(t,u^*(x,t)) < -\alpha$ if $|x|$ is sufficiently large.

Set $\phi^* := u^*_{x_1}$. Clearly, $\phi^*$ is a solution of the linear equation $v_t = \Delta v + a^*(x,t)v$ on $\mathbb{R}^N\times(0,\infty)$. Boundedness of $u^*$ and standard parabolic estimates imply that the norm $\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}$ is bounded on $[1,\infty)$. Moreover, the assumptions on $u^*$ (see Theorem 5.1) clearly imply that this norm also stays above a positive constant for $t \in [1,\infty)$. Consequently, $\phi = \phi^*$ satisfies (7.10) with $\tau = 1$, $\lambda_0 = 0$, and some constant $C_0 > 0$. Then, by Theorem 7.3, for $\lambda := -\alpha/2$ there is $R$ such that $\varphi_R$ satisfies (7.11) with $\tau = 1$ and some $C > 0$.

We now redefine $a(x,t)$ for $t < 0$ in order for (7.11) to be valid with $\tau = -\infty$. Before doing so, we claim such a change of $a$ has no effect on the validity of condition (7.11) with $\tau = 1$, only the constant $C$ may have to be made smaller (the entire solution $\varphi_R$ itself of course changes on the time interval $(0,\infty)$ even if $a$ is modified for $t < 0$ only). In fact, (7.11) with $\tau = 1$ remains valid, possibly with a smaller $C > 0$, if we replace $\varphi_R$ with any other positive solution of problem (7.3) on $B(0,R)\times(0,\infty)$. This readily follows from the fact that for any two positive solutions $v_1$, $v_2$ of (7.3) on $B(0,R)\times(0,\infty)$, the function $\sup_{x\in B(0,R)}(v_2(x,t)/v_1(x,t))$ is nonincreasing in $t$ and takes finite values for $t>0$ (see Corollary 3.2 and Theorem 2.1 in [20]).

Our new definition of the function $a$ is as follows. We keep $a\equiv a^*$ on $\mathbb{R}^N\times[0,\infty)$ and $a\equiv-\alpha$ on $(\mathbb{R}^N\setminus B(0,R))\times(-\infty,0)$. On $B(0,R)\times(-\infty,0)$ we set $a\equiv\lambda_1$, where $\lambda_1$ is the principal eigenvalue on the Dirichlet Laplacian on $B(0,R)$. With this definition we have $\varphi_R\equiv\bar\varphi$ on $B(0,R)\times(-\infty,0)$, where $\bar\varphi>0$ is the eigenfunction corresponding to $\lambda_1$ with $\|\bar\varphi\|_{L^\infty(B(0,R))}=1$. Indeed, this function, continued for $t\geq 0$ as the solution of (7.3) with the initial condition $v(\cdot,0)=\bar\varphi$, is an entire positive solution, hence it coincides with $\varphi_R$ by uniqueness. We already know that (7.11) holds for $\tau=1$ (with $\lambda=-\alpha/2<0$). Then it also holds for $\tau=0$, since

$$C_2\leq\|\varphi_R(\cdot,t)\|_{L^\infty(B(0,R))}\leq C_3\quad(t\in[0,1])$$

for some positive constants $C_2$, $C_3$. Using the explicit form of $\varphi_R$, one now easily shows that (7.11) also holds for $\tau=-\infty$ with a suitable constant $C$.

The above conclusion verifies the hypotheses of Theorem 7.1. Let $\phi$ be as in that theorem. We show that $\phi$ grows exponentially for $t\geq 1$:

$$\frac{\|\phi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}}{\|\phi(\cdot,s)\|_{L^\infty(\mathbb{R}^N)}}\geq C\,e^{\vartheta(t-s)}\quad(t\geq s\geq 1),\tag{7.14}$$

where $\vartheta>0$ is as in (7.8). To show this, we use again the solution $\phi^*=u^*_{x_1}$. By the assumptions on $u^*$ (see Theorem 5.1) $u^*(\cdot,t)$ cannot be monotone in $x_1$ on $B(0,R_1)\times(0,\infty)$ if $R_1$ is large enough. Hence $\phi^*(\cdot,t)$ changes sign in $B(0,R_1)$ for all $t>0$ and therefore, by Remark 7.2, $\phi^*(\cdot,1)\in X_2(1)$. Using (7.8) with $v_0=\phi^*(\cdot,1)$, and recalling that the norm of $v(\cdot,t;1,v_0)=\phi^*(\cdot,t)$ in $L^\infty(\mathbb{R}^N)$ is bounded below by a positive constant, we obtain (7.14).

From Theorem 7.3, we now obtain that, possibly after making $R$ larger, (7.11) holds with $\lambda=\vartheta/2$. Lemma 5.2 now follows directly from Theorem 7.4 (we extend the functions $\tilde a$ in Lemma 5.2 by setting $\tilde a\equiv a$ on $B(0,R)\times(-\infty,0)$). □

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