Following up on the comment about the quadratic factors with complex roots at the bottom of p. 115:

Consider $\dfrac{x+3}{(4x^2+2)(x+2x^3)}$

Factoring the denominator, we obtain

$$(4x^2+2)(x+2x^3) = 2(2x^2+1)\cdot x(1+2x^2)$$

$$= 2x(2x^2+1)^2, \quad 2x^2+1=0 \text{ has only}$$

complex solutions.

Thus the correct form of the P. F. D. is

$$\frac{x+3}{(4x^2+2)(x+2x^3)} = \frac{x+3}{2x(2x^2+1)^2}$$

$$= \frac{A}{x} + \frac{Bx+C}{2x^2+1} + \frac{Dx+E}{(2x^2+1)^2}$$

A wrong form of P. F. D. would be

$$\frac{x+3}{(4x^2+2)(x+2x^3)} = \underbrace{\frac{Ax+B}{4x^2+2} + \frac{Cx+D}{x+2x^3}}_{\text{Wrong}}$$

It would likewise be wrong to write

$$\frac{x+3}{(4x^2+2)(x+2x^3)} \quad \left(= \frac{x+3}{(4x^2+2)x(1+2x^2)} \right)$$

$$\underbrace{= \frac{Ax+B}{4x^2+2} + \frac{C}{x} + \frac{Dx+E}{1+2x^2}}_{\text{WRONG}}$$

*

Additional Comment on Trig. Substitutions (sect. 7.3).

Note that these substitutions are formulated for $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$, i.e. the COEFFICIENT of x^2 is equal to ± 1. However, they are also used when the coefficient of x^2 does not equal ± 1: One needs to do a preparatory step in such a situation.

Thus in the Example $\int x^2(9-4x^2)^{3/2}dx$ $\left(= \int x^2(9-4x^2)\sqrt{9-4x^2}\,dx\right)$, starting on p. 66 in these Notes, I make the preparatory step of factoring out 4, i.e. $9-4x^2 = 4\left(\frac{9}{4}-x^2\right)$, so that

$$\int x^2(9-4x^2)^{3/2}dx = 8\int x^2\left(\frac{9}{4}-x^2\right)^{3/2}dx$$

(see p. 66), so that now (the coefficient of x^2) equals (-1).

There is another way to handle this, as in Example 6, p. 482 in the Book, namely (I'll do it for the Notes Example), $4x^2 = (2x)^2$, so we set $u = 2x$, $du = 2dx$, so that $\int x^2(9-4x^2)^{3/2}dx =$

$$\int \left(\frac{u}{2}\right)^2 (9-u^2)^{3/2}\cdot\frac{1}{2}du = \frac{1}{8}\int u^2(9-u^2)^{3/2}dx,$$

and the algebra is easier! So you

may prefer to use the Book method.

Another preparatory step for setting up a trig. substitution is Completing to a square. An Example is on pages 88, 89 in these Notes. Another Example is Example 7, p. 482 in the Book. A number of Examples are in the Exercises for sect. 7.3, p. 483.

A sneaky Example is #27, p. 483: Evaluate $\int \sqrt{x^2+2x}\,dx$:

$$= \int \sqrt{x^2+2x+1-1}\,dx = \int \sqrt{(x+1)^2-1}\,dx,$$

$u = x+1$, $du = dx$

$$= \int \sqrt{u^2-1}\,du, \quad u = \sec\theta, \; du = \sec\theta\tan\theta\,d\theta$$

$$= \int \underbrace{\sqrt{\sec^2\theta-1}}_{=\tan\theta}\,\sec\theta\tan\theta\,d\theta = \int \tan^2\theta\sec\theta\,d\theta$$

FINISH ON YOUR OWN

The last part of the section 7.4 is called
Rationalizing Substitutions, p. 492.

This concerns integrals involving an n-th root of an expression, in particular $\sqrt[n]{ax+b}$, i.e. the n-th root of a first degree polynomial: We set $u = \sqrt[n]{ax+b}$,

$$du = \frac{1}{n}(ax+b)^{\frac{1}{n}-1} \cdot a\,dx =$$

$$= \frac{a}{n}(ax+b)^{\frac{1-n}{n}} dx = \frac{a}{n}\Big(\underbrace{\sqrt[n]{ax+b}}_{u}\Big)^{1-n} dx$$

$$= \frac{a}{n} u^{1-n} dx, \text{ i.e.}$$

$$du = \frac{a}{n} u^{1-n} dx,$$

hence $\underline{dx = \frac{n}{a} u^{n-1} du}$

Some Examples, dealing with $\sqrt{x}$, are on pages 33 − 35 of these Notes

So when $n=2$: $dx = \frac{2}{a} u\,du$,

$n=3$: $dx = \frac{3}{a} u^2 du$, etc.

The Substitution $u = \sqrt{x}$ was discussed on pages 33 − 35 (I wrote $t = \sqrt{x}$).

Some Examples on Rationalizing Substitutions.

Evaluate $\displaystyle\int \frac{1}{1+\sqrt[3]{2x+1}}\,dx$

We substitute $\quad u = \sqrt[3]{2x+1}$ $\left(= (2x+1)^{1/3}\right)$,

hence $\quad du = \frac{1}{3}\cdot 2\,(2x+1)^{-\frac{2}{3}}dx =$

$$= \frac{2}{3}\left((2x+1)^{1/3}\right)^{-2}dx = \frac{2}{3}\,u^{-2}dx,$$

i.e. $\quad du = \frac{2}{3}u^{-2}dx,$

hence $\quad dx = \frac{3}{2}\,u^{2}du$

$$= \int \frac{1}{1+u}\cdot\frac{3}{2}\,u^{2}du$$

You can finish on your own.

(Comment. One can either carry out a full-fledged long division, or one can write $\dfrac{u^2}{1+u} = \dfrac{u^2-1+1}{1+u} =$

$$= \frac{u^2-1}{1+u} + \frac{1}{1+u} = \frac{(u+1)(u-1)}{u+1} + \frac{1}{1+u}$$

$$= u-1+\frac{1}{1+u}, \text{ etc.}$$)

*

Evaluate $\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx$

Again, let $u = \sqrt[3]{x^2+1}$, hence

$$du = \tfrac{1}{3}\cdot 2x\,(x^2+1)^{-2/3}\,dx = \tfrac{2}{3}x\left((x^2+1)^{1/3}\right)^{-2}dx$$

$$= \tfrac{2}{3}x\,u^{-2}\,dx,$$

hence $\frac{3}{2}u^2\,du = x\,dx$

$$= \int \frac{x^2}{\sqrt[3]{x^2+1}}\,x\,dx$$

Moreover $u = \sqrt[3]{x^2+1} \Rightarrow x^2 = u^3-1$

$$= \int \frac{u^3-1}{u}\cdot\frac{3}{2}u^2\,du$$

$$= \frac{3}{2}\int u(u^3-1)\,du \quad \text{etc.}$$

$\ast$

(123)

Evaluate $\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$

We can start with $u=\sqrt{x}$, $(x=u^2)$

$du = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}\,dx = \frac{1}{2u}\,dx$,

$dx = 2u\,du$:

$$= \int \frac{\sqrt{1+u}}{u^2}\,2u\,du = \int \frac{\sqrt{1+u}}{u}\,du$$

Thus Next $\boxed{t=\sqrt{1+u}}$

$dt = \frac{1}{2}\cdot\frac{1}{\sqrt{1+u}}\,du = \frac{1}{2t}\,du$,

Hence $du = 2t\,dt$, Hence $t^2 = 1+u$,

$u = t^2-1$

$$= \int \frac{t}{t^2-1}\,2t\,dt = 2\int \frac{t^2}{t^2-1}\,dt =$$

$$= 2\int\left(\frac{t^2-1}{t^2-1} + \frac{1}{t^2-1}\right)dt$$

cont. on next page

$$= 2\int\left[1+\frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)\right]dt$$

$$= 2\left[t+\frac{1}{2}\left(\ln|t-1|-\ln|t+1|\right)\right]+C$$

$$= 2t+\ln|t-1|-\ln|t+1|+C$$

$$= 2\sqrt{1+u}+\ln\left|\sqrt{1+u}-1\right|-\ln\left|\sqrt{1+u}+1\right|+C$$

$$= 2\sqrt{1+\sqrt{x}}+\ln\left|\sqrt{1+\sqrt{x}}-1\right|-\ln\left|\sqrt{1+\sqrt{x}}+1\right|+C$$

Fractions involving e^x. *

Evaluate $\displaystyle\int\frac{e^x}{e^{2x}+5e^x+4}\,dx$

We make the substitution $u=e^x$,

hence $du = e^x dx$

$$\int\frac{1}{(e^x)^2+5e^x+4}\,e^x dx$$

cont. on next page

$$= \int \frac{1}{u^2+5u+4}\,du$$

Perform P. F. D. :

$$\frac{1}{u^2+5u+4} = \frac{1}{(u+4)(u+1)} = \frac{A}{u+4} + \frac{B}{u+1}$$

$$= \frac{A(u+1) + B(u+4)}{(u+4)(u+1)} = \frac{(A+B)u + A+4B}{u^2+5u+4}$$

Hence $A+B=0$, $A+4B=1$

$$\Downarrow$$

$$A=-B \qquad -B+4B=1$$

$$3B=1$$

$$B=\frac{1}{3}$$

$$A=-\frac{1}{3}$$

$$= \int\left(-\frac{1}{3}\cdot\frac{1}{u+4} + \frac{1}{3}\cdot\frac{1}{u+1}\right)du$$

$$= -\frac{1}{3}\ln|u+4| + \frac{1}{3}\ln|u+1| + C$$

$$= -\frac{1}{3}\ln|e^x+4| + \frac{1}{3}\ln|e^x+1| + C$$

*

Evaluate $\int \frac{1}{e^{2x}+5e^x+4}\,dx$

This looks similar to the preceding Example (bottom half of p. 124), but since we don't the factor $e^x dx$, we need to proceed slightly differently, although there are $\underbrace{\text{different}}_{\text{slightly}}$ ways to write it up, which all come down to the same thing:

Method 1. $\int \frac{1}{e^{2x}+5e^x+4}\,dx =$

$$= \int \frac{e^x}{e^x(e^{2x}+5e^x+4)}\,dx$$

$u = e^x$, $du = e^x dx$

$$= \int \frac{1}{u(u^2+5u+4)}\,du \quad \text{etc.}$$

FINISH ON YOUR OWN

Method 2. $u = e^x$

Hence $du = e^x dx = u\,dx$

Hence $dx = \frac{1}{u} du$

Hence $\int \frac{1}{e^{2x}+5e^x+4} dx =$

$$= \int \frac{1}{u^2+5u+4} \cdot \frac{1}{u} du$$

etc.

Method 3. $u = e^x$

Hence $x = \ln u$

$dx = \frac{1}{u} du$ as in Method 2,

etc.

(Of course Methods 2, 3 are closely related. Method 2 simply amounts to finding $dx = d(\ln u)$ by implicit differentiation.)

*

Partial Fraction Decompositions allow to do more complicated problems on Integration by Parts:

Evaluate $\int \underline{x^2} \tan^{-1}x \, dx$:

$$= \int \underbrace{(\tan^{-1}x)}_{u} \underbrace{x^2 dx}_{dv}$$

$$= \underbrace{\tfrac{1}{3}x^3}_{v} \underbrace{\tan^{-1}x}_{u} - \int \overbrace{\underbrace{\tfrac{1}{3}x^3}_{v} \underbrace{\frac{1}{x^2+1}}_{du}}^{} dx$$

$$= \tfrac{1}{3}x^3 \tan^{-1}x - \tfrac{1}{3}\int \frac{x^3}{x^2+1} dx$$

$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3 \\ & x^3 + x \\ \hline & -x \end{array} \Rightarrow \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

$$= \tfrac{1}{3}x^3 \tan^{-1}x - \tfrac{1}{3}\int \left(x - \frac{x}{x^2+1}\right) dx$$

$$= \tfrac{1}{3}x^3 \tan^{-1}x - \tfrac{1}{6}x^2 + \tfrac{1}{6}\int \frac{2x}{x^2+1} dx$$

$$= \underline{\underline{\tfrac{1}{3}x^3 \tan^{-1}x - \tfrac{1}{6}x^2 + \tfrac{1}{2}\ln(x^2+1) + C}}$$

$\ast$

Evaluate $\int x \ln(x^2+2x+3)\,dx$:

the 2's cancel

$$= \int \underbrace{\left[\ln(x^2+2x+3)\right]}_{u} \underbrace{x\,dx}_{dv}$$

$$= \underbrace{\tfrac{1}{2}x^2}_{v} \underbrace{\ln(x^2+2x+3)}_{u} - \int \underbrace{\tfrac{1}{2}x^2}_{v} \cdot \underbrace{\frac{2x+2}{x^2+2x+3}}_{du}\,dx$$

$$= \tfrac{1}{2}x^2 \ln(x^2+2x+3) - \int \frac{x^3+x^2}{x^2+2x+3}\,dx$$

Thus we need to do division:

$$\begin{array}{r|l}
 & x - 1 \\
x^2+2x+3 & x^3+x^2 \\
 & \underline{x^3+2x^2+3x} \\
 & -x^2-3x \\
 & \underline{-x^2-2x-3} \\
 & -x+3
\end{array}$$

Thus we will evaluate

$$\int \frac{x^3+x^2}{x^2+2x+3}\,dx = \int \left(x-1+\frac{3-x}{x^2+2x+3}\right)dx$$

Continued next page

$$= \frac{1}{2}x^2 - x + \underbrace{\int \frac{3-x}{x^2+2x+3}\,dx}$$

It thus remains to evaluate

Similar problem was worked out on p. 103: Since x^2+2x+3 has no real roots, we need to

<u>complete to a square</u>:

$$x^2+2x+3 = (x^2+2x+1)+2 =$$
$$= (x+1)^2+2 = u^2+2$$

Thus we set $x+1 = u$, $dx = du$

Also $x = u-1$. Thus

$$\int \frac{3-x}{x^2+2x+3}\,dx = \int \frac{3-(u-1)}{u^2+2}\,du$$

$$= \int \frac{4-u}{u^2+2}\,du = 4\int \frac{1}{u^2+2}\,du - \int \frac{u}{u^2+2}\,du$$

$$= 4 \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\ln(u^2+2) + C$$

$$= 2\sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) - \frac{1}{2}\ln(x^2+2x+3) + C$$

Hence finally from the bottom of p. 129 and top of p. 130, we obtain

$$\int \frac{x^3+x^2}{x^2+2x+3}\,dx$$

$$= \frac{1}{2}x^2 - x + 2\sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) - \frac{1}{2}\ln(x^2+2x+3) + C$$

Thus (from top p. 129),

$$\int x\ln(x^2+2x+3)\,dx$$

$$= \frac{1}{2}x^2\ln(x^2+2x+3) - \frac{1}{2}x^2 + x - 2\sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

$$+ \frac{1}{2}\ln(x^2+2x+3) + C$$

*

Evaluate $\int x\ln(x^2+2x-3)\,dx$

Now the expression x^2+2x-3 can be factored: $=(x+3)(x-1)$

Thus $\int x\ln(x^2+2x-3)\,dx =$

$$\int x\ln[(x+3)(x-1)]\,dx$$

continued on next page

$$= \int x[\ln(x+3) + \ln(x-1)]\,dx$$

$$= \int x\ln(x+3)\,dx + \int x\ln(x-1)\,dx$$

Now a good way is to first do a substitution (for each of the two integrals separately); compare with the Exercise #41, p. 469.

Thus to evaluate $\int x\ln(x+3)\,dx$ we set $u = x+3$, $du = dx$, $x = u-3$. Hence

$$= \int (u-3)\ln u\,du = \int u\ln u\,du - 3\int \ln u\,du$$

We know $\int \ln u\,du = u\ln u - u + C$,

$$\int u\ln u\,du = \frac{1}{2}u^2\ln u - \int \frac{1}{2}u^2 \cdot \frac{1}{u}\,du$$

$$= \frac{1}{2}u^2\ln u - \frac{1}{4}u^2 + C$$

$$= \frac{1}{2}u^2\ln u - \frac{1}{4}u^2 - 3u\ln u + 3u + C$$

$$= \boxed{\frac{1}{2}(x+3)^2\ln(x+3) - \frac{1}{4}(x+3)^2 - 3(x+3)\ln(x+3) + 3(x+3) + C}$$

Similarly, to evaluate $\int x \ln(x-1)\,dx$,
we set $u = x-1$, $du = dx$, $x = u+1$

Hence $\int x\ln(x-1)\,dx = \int (u+1)\ln u\,du$

$= \int u \ln u\,du + \int \ln u\,du$

$= \frac{1}{2}u^2 \ln u - \frac{1}{4}u^2 + u\ln u - u + C$

$= \boxed{\frac{1}{2}(x-1)^2 \ln(x-1) - \frac{1}{4}(x-1)^2 + (x-1)\ln(x-1) - x + 1 + C}$

Thus to obtain the final answer, i.e. $\int x \ln(x^2+2x-3)\,dx$, we add the two answers in the box above, and in the box at the bottom of p. 132

<u>Comment.</u> The method, of dealing with integrals which contain expressions such as $\ln(x+3)$, by using the substitution $u = x+3$, was already described earlier on pages 36, 37.

*

We can also evaluate more integrals of fractions involving trigonometric functions:

Evaluate $\int \frac{\cos x}{\sin^3 x + \sin^2 x} dx$:

Since $\cos x = (\sin x)'$, we substitute

$u = \sin x$, $du = \cos x \, dx$

Hence $\int \frac{\cos x}{\sin^3 x + \sin^2 x} dx = \int \frac{1}{u^3 + u^2} du$

$$\frac{1}{u^3+u^2} = \frac{1}{u^2(u+1)} = \frac{A}{u^2} + \frac{B}{u} + \frac{C}{u+1}$$

$$= \frac{A(u+1) + Bu(u+1) + Cu^2}{u^2(u+1)} =$$

$$= \frac{(B+C)u^2 + (A+B)u + A}{u^2(u+1)}$$

Hence $\underline{\underline{A=1}}$, $B+C=0$, $A+B=0$

$\underline{\underline{B}} = -A = \underline{\underline{-1}}$, $\underline{\underline{C}} = -B = \underline{\underline{1}}$

Hence $\displaystyle\int \frac{1}{u^3+u^2}\,du$

$$= \int\left(\frac{1}{u^2} - \frac{1}{u} + \frac{1}{u+1}\right)du$$

$$= -\frac{1}{u} - \ln|u| + \ln|u+1| + C$$

$$= \boxed{-\frac{1}{\sin x} - \ln|\sin x| + \ln|\sin x + 1| + C}$$

$*$

Evaluate $\displaystyle\int \frac{\sin 2x}{\sin^4 x + \sin^3 x}\,dx$:

We use the identity $\sin 2x = 2\sin x\cos x$, hence

$$= \int \frac{2\sin x\cos x}{\sin^4 x + \sin^3 x}\,dx$$

$$= 2\underbrace{\int \frac{\cos x}{\sin^3 x + \sin^2 x}\,dx}_{\text{Evaluated in the box above}}$$

Evaluated in the box above

$*$

Question. Describe the steps that need to be taken to evaluate

$$\int \frac{x^6+2x^3+5}{x(4x^2+2)(x+2x^3)}\,dx$$

In particular specify carefully the relevant form of Partial Fraction Decomposition. Do not carry out any of the calculations (for any of the steps), in particular do not calculate any of the coefficients of the P.F.D.

Solution. The Denominator

$$= x(4x^2+2)(x+2x^3)$$

$$= 2x(2x^2+1)\,x(1+2x^2)$$

$$= \underbrace{2x^2\underbrace{(2x^2+1)^2}_{\text{Degree } 4}}_{\text{Degree } 6}$$

Thus both the Numerator and the Denominator have Degree $= 6$, hence we need to first carry out long Division, which yields
$$\frac{x^6+2x^3+5}{x(4x^2+2)(x+2x^3)}$$
$$= P_1(x) + \frac{P_2(x)}{2x^2(2x^2+1)^2}$$
where $P_1(x)$ is a Polynomial and $P_2(x)$ has degree < 6.

Since $2x^2+1$ has no real roots, we do not factor any further.

P.F.D. for $\dfrac{P_2(x)}{2x^2(2x^2+1)^2}$:
$$= \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{2x^2+1} + \frac{Ex+F}{(2x^2+1)^2}$$

$\ast$

ANOTHER LOOK AT TRIG. INTEGRALS

We will make some additional comments on how we handle the main cases of the trig. INTEGRALS, which we have already described on pages 42 — 55.

The first case, i.e. product of powers of $\sin x$, $\cos x$, where at least one of the powers is a positive odd integer, i.e. we have $\sin^n x$, or $\cos^n x$ where n is odd. This is discussed on p. 42, and see also p. 53.
So we have already discussed it, so what is there to add?
So consider the case of $\cos^n x$, where n odd. The thing to REMEMBER is that we will use the substitution

$\boxed{u = \sin x}$ (see bottom half of p. 42).

Similarly, if we have $\sin^n x$, n odd, we will use the substitution

$\boxed{u = \cos x}$.

Of course the reason for this can be understood by looking at the calculation on p. 42.

<u>The 2nd Case</u>, powers of $\sin x$, $\cos x$, both powers even.

<u>The thing to Remember</u> is that we use the formulas

$$\boxed{\sin^2 x = \tfrac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)}$$

at the very start, multiply everything out, and see if we have any terms with an odd power

of either $\cos 2x$ or $\sin 2x$.
With such terms we can now do the First Case. With the terms that are a product of even powers of $\cos 2x$, $\sin 2x$, we again apply the half-angle formulas, i.e.,

$$\boxed{\cos^2 2x = \frac{1}{2}(1+\cos 4x),\ \sin^2 2x = \frac{1}{2}(1-\cos 4x),}$$

etc. See p. 43, p. 68 – 70,

p. 82–85.

The 3-rd case, product of powers of $\sec x$, $\tan x$, with the power of $\sec x$ even.

The thing to REMEMBER is that we have to use the substitution $\boxed{u = \tan x.}$

The Reason for this can be seen at the bottom of p. 44.

The 4-th Case, product of powers of $\sec x$, $\tan x$, where powers of $\tan x$ is odd — can assume power of $\sec x$ also odd, since if even we can switch to the 3-rd Case.
The Thing to REMEMBER is that we use the substitution $\boxed{u = \sec x}$. To see how it works look at page 45.

We don't have such succinct comments for the Remaining cases:
But we add as follows:

Even power of $\sec x$, no $\tan x$:
This is basically the case 3 (bottom p. 140):

Evaluate $\int \sec^6 x \, dx$:

$= \int \sec^4 x \sec^2 x \, dx =$

$= \int (1+\tan^2 x)^2 \sec^2 x \, dx$

$= \int (1 + 2\tan^2 x + \tan^4 x) \sec^2 x \, dx$

$= \int (\sec^2 x + 2\tan^2 x \sec^2 x + \tan^4 x \sec^2 x) dx$

which is the third case (bottom p. 140), i.e. $u = \tan x$.

Likewise even power of $\tan x$ (no sec) is related to case 3; for this see p. 48.

Odd powers of $\tan x$ (no $\sec x$) are explained on p. 46, 47.

Finally the Remaining Cases are $\int \tan^m x \sec^n x \, dx$ where m is even and n is odd, as stated on p. 49.

The discussion (Examples) is on pages 49 — 52.

A part of this case is when $m=0$ (0 is even), which is the case

$$\int \sec^n x\,dx \quad \text{where } n \text{ is odd.}$$

The cases such as $\int \tan^4 x \sec^3 x\,dx$ are reduced to cases of the form $\int \sec^n x\,dx$, n odd, as explained on pages 49—52.

The thing to REMEMBER about $\int \sec^3 x\,dx$ is that it is handled by a trick similar to evaluating the integrals

$$\int e^{ax}\sin bx\,dx, \int e^{ax}\cos bx\,dx.$$

*

More on P.F.D.

Let us return to the Example in the Comment on pages 114, 115. Thus let us evaluate

$$\int \frac{1}{(4x-2)(1-4x^2)}\,dx$$

On p. 115 we factored the denominator as

$$(4x-2)(1-4x^2) = -2(2x-1)^2(1+2x)$$

and we wrote the form of P.F.D. as

$$\frac{1}{(4x-2)(1-4x^2)} = -\frac{1}{2(2x-1)^2(1+2x)}$$

$$= \frac{A}{(2x-1)^2} + \frac{B}{2x-1} + \frac{C}{1+2x}$$

$$= \frac{A(1+2x) + B(2x-1)(2x+1) + C(2x-1)^2}{(2x-1)^2(1+2x)} =$$

$$= \frac{A(1+2x) + B(4x^2-1) + C(4x^2-4x+1)}{(2x-1)^2(1+2x)}$$

$$= \frac{x^2(4B+4C) + x(2A-4C) + A-B+C}{(2x-1)^2(1+2x)}$$

which equals

$$-\frac{1}{2(2x-1)^2(1+2x)} = \frac{-\frac{1}{2}}{(2x-1)^2(1+2x)},$$

i.e. we must make sure that the denominators are <u>identical</u> before we compare the coefficients of the various powers of x:

$$4B+4C=0, \; 2A-4C=0, \; A-B+C=-\frac{1}{2}$$

Hence $B=-C$, $A=2C$, thus substituting into the third equation, we obtain

$$2C+C+C=-\frac{1}{2} \Rightarrow C=-\frac{1}{8}$$

Hence $A=-\frac{1}{4}$, $B=\frac{1}{8}$ and thus

$$\frac{1}{(4x-2)(1-4x^2)} = -\frac{1}{4}\cdot\frac{1}{(2x-1)^2} + \frac{1}{8}\cdot\frac{1}{2x-1} - \frac{1}{8}\cdot\frac{1}{1+2x}$$

Hence $\displaystyle\int \frac{1}{(4x-2)(1-4x^2)}\,dx$

$$= \int \Big(\underbrace{-\frac{1}{4}\,\frac{1}{(2x-1)^2}}_{} + \frac{1}{8}\cdot \underbrace{\frac{1}{2x-1}}_{} - \frac{1}{8}\cdot \underbrace{\frac{1}{1+2x}}_{u=1+2x} \Big)\,dx$$

Substitution $u = 2x-1$

If you can not carry out these substitutions mentally, you should write out the details

$$= -\frac{1}{4}\cdot\frac{1}{2}\left(-\frac{1}{2x-1}\right) + \frac{1}{8}\cdot\frac{1}{2}\ln|2x-1|$$

$$-\frac{1}{8}\cdot\frac{1}{2}\ln|1+2x| + C$$

$$= \frac{1}{8(2x-1)} + \frac{1}{16}\ln|2x-1| - \frac{1}{16}\ln|1+2x| + C$$

$\ast$

One more Example on P. F. D.

Evaluate $\displaystyle\int \frac{x+2}{x^4-2x^3+x^2}\,dx$

We factor the denominator:

$$x^4-2x^3+x^2 = x^2(x^2-2x+1) = x^2(x-1)^2$$

Thus the P. F. D. has the form

$$\frac{x+2}{x^4-2x^3+x^2} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x-1)^2} + \frac{D}{x-1}$$

Putting on the common denominator,

$$= \frac{A(x-1)^2 + Bx(x-1)^2 + Cx^2 + Dx^2(x-1)}{x^2(x-1)^2}$$

$$= \frac{A(x^2-2x+1) + B(x^3-2x^2+x) + Cx^2 + D(x^3-x^2)}{x^2(x-1)^2}$$

$$= \frac{(B+D)x^3 + (A-2B+C-D)x^2 + (-2A+B)x + A}{x^2(x-1)^2}$$

Hence $B+D=0$, $A-2B+C-D=0$, $-2A+B=1$, $A=2$

Thus $A=2$, $B=1+2A=5$, $D=-B=-5$,

$C=-A+2B+D=-2+10-5=3$,

i.e. $A=2$, $B=5$, $C=3$, $D=-5$

Hence $\displaystyle\int \frac{x+2}{x^4-2x^3+x^2}\,dx$

$$= \int \left(\frac{2}{x^2} + \frac{5}{x} + \frac{3}{(x-1)^2} - \frac{5}{x-1} \right) dx$$

$$= -\frac{2}{x} + 5\ln|x| - \frac{3}{x-1} + 5\ln|x-1| + C$$

*

Another Idea for Rationalizing:

Evaluate $\displaystyle\int \frac{x+2}{\sqrt{x+3}-\sqrt{x+1}}\,dx$

We multiply both the denominator and numerator by $\sqrt{x+3}+\sqrt{x+1}$:

$$= \int \frac{(x+2)(\sqrt{x+3}+\sqrt{x+1})}{(\sqrt{x+3}-\sqrt{x+1})(\sqrt{x+3}+\sqrt{x+1})}\,dx$$

$$= \int \frac{(x+2)(\sqrt{x+3}+\sqrt{x+1})}{(\sqrt{x+3})^2-(\sqrt{x+1})^2}\,dx$$

Cont. on next page.

$$= \int \frac{(x+2)(\sqrt{x+3}+\sqrt{x+1})}{\underbrace{(x+3)-(x+1)}_{=2}}\,dx$$

$$= \frac{1}{2}\int (x+2)(\sqrt{x+3}+\sqrt{x+1})\,dx$$

$$= \underbrace{\frac{1}{2}\int (x+2)\sqrt{x+3}\,dx}_{\text{Substitute } u=x+3} + \underbrace{\frac{1}{2}\int (x+2)\sqrt{x+1}\,dx}_{\text{Substitute } u=x+1}$$

Finish on your own.