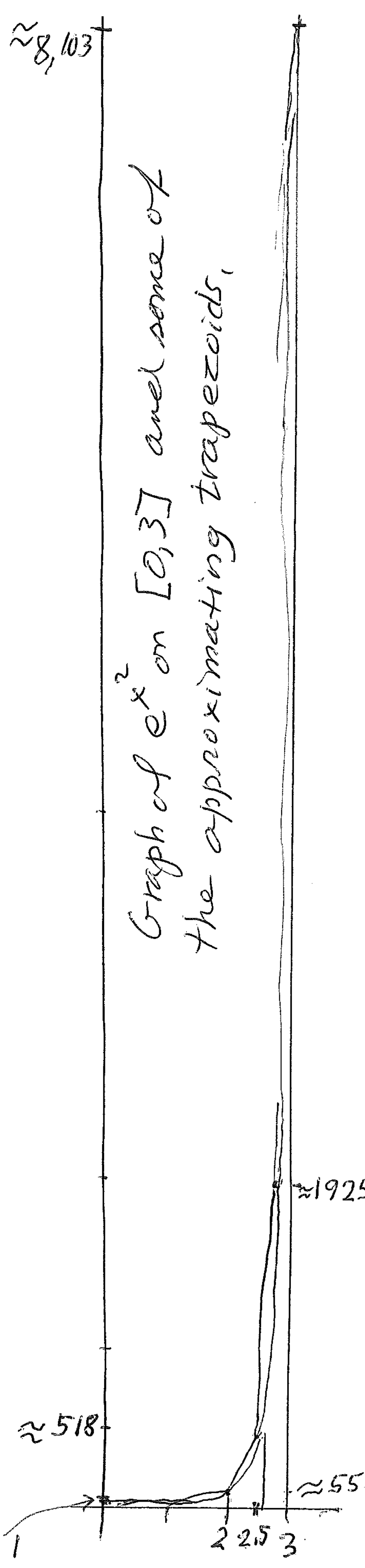
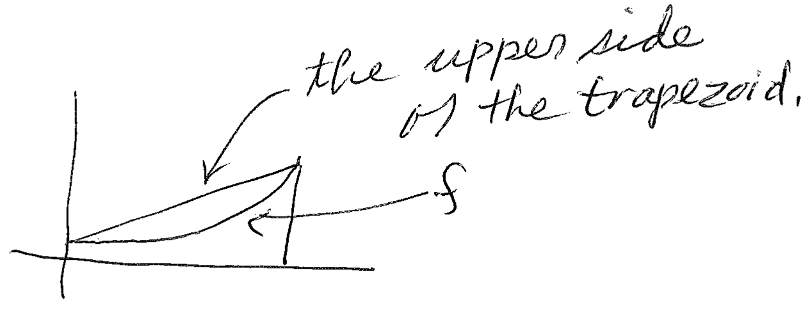
It is interesting to evaluate some of the approximations using a calculator. Thus for the integral $\int_0^3 e^{x^2} dx$, the formula for M_6 is on p. 155, and T_6 and S_6 are on p. 156. Plugging everything into the calculator we find $M_6 \approx 1{,}055.8$, $T_6 \approx 2{,}319.1$, $S_6 \approx 1{,}722.3$.

A very accurate value of $\int_0^3 e^{x^2} dx$ can be found using MATHEMATICA, keeping only two decimal places, we find $\int_0^3 e^{x^2} dx \simeq 1{,}444.55$.

Note that T_6 overshoots the most while M_6 is significantly less than the correct value. One can find an explanation for this by examining the graph of e^{x^2}. We shall show an explanation for the Trapezoidal Rule which is the easiest to picture.



The point of this picture is on each of the intervals of the partition the function e^{x^2} grows much faster near the end of such an interval than at its beginning. This produces the effect that the area of the approximating trapezoid is a bit larger than the area under the corresponding part of the graph:
A magnified picture of this effect



The trapezoid area is quite a bit larger than under the graph.

$\approx 55 \approx e^{2^2}$, i.e. e^{x^2} at 2 equals ≈ 55,
but e^{x^2} at 3 equals $\approx 8{,}103$.

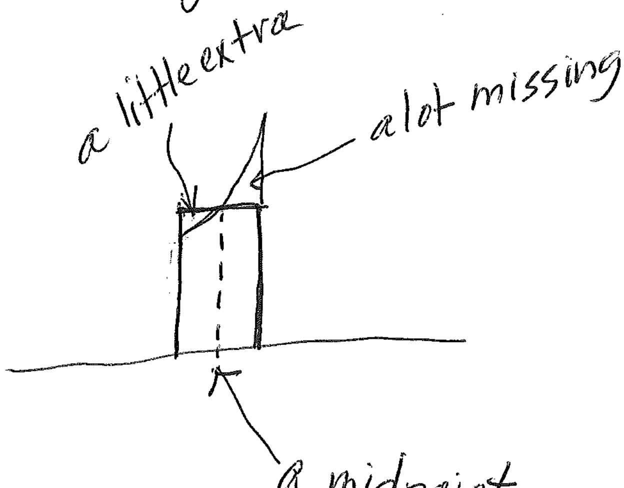
Let's take a look at the errors.
The errors E_M, E_T, E_S are defined on pages 153, 154: Let's calculate these errors for $\int_0^3 e^{x^2}dx$ when $n=6$:

$$E_M = \int_0^3 e^{x^2}dx - M_6 \approx 1{,}444.55 - 1{,}055.8 = 388.75$$

$$E_T = \int_0^3 e^{x^2}dx - T_6 \approx 1{,}444.55 - 2{,}319.1 = -874.55$$

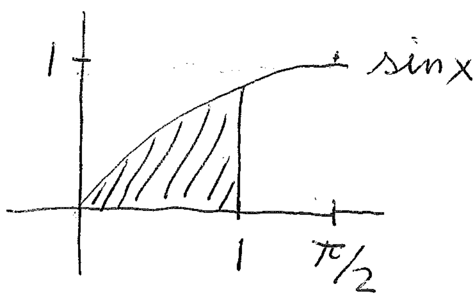
$$E_S = \int_0^3 e^{x^2}dx - S_6 \approx 1{,}444.55 - 1{,}722.3 = -277.75$$

Picture Explanation why the Midpoint Rule rectangles come up short:



Simpson's Rule is typically more accurate than the other two Rules.

Question. How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_0^1 \sin x\,dx$ is accurate to within 0.0001?



We need to choose n so that the error E_S, in absolute value, is < 0.0001

From the Box on p. 514 in the Book we obtain
$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$
where K is such that $|f^{(4)}(x)| \leq K$ on $[a,b]$; $f^{(4)}(x)$ means the fourth derivative of $f(x)$ ($= \sin x$ in our Example).

So we have $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$.

Now we use the important fact that $|\sin x| \leq 1$ for all x (and likewise for $\cos x$). Actually, on the interval $[0,1]$, $0 \leq \sin x \leq \sin 1 < 1$, but that is not a very large difference, so we will use $K = 1$.

Also $b-a = 1-0 = 1$, hence we obtain

$$|E_S| \leq \frac{1\,(1)^5}{180n^4} = \frac{1}{180n^4}$$

Thus if we choose n so that

$$\boxed{\frac{1}{180n^4} < 0.0001}$$

then we will obtain $|E_S| < 0.0001$

Furthermore Note that n must be even for using Simpson's Rule.

Multiply be 10^4:

$$\frac{10^4}{180n^4} < 1 \Rightarrow \frac{10^4}{180} < n^4$$

So $n > \sqrt[4]{\dfrac{10^4}{180}} \approx 2.73$

Thus n must be at least 3,

and since n must be even, we

can choose $n = 4$.

(We can not choose $n = 3$.) $\ast$

Let's answer the Question at the top of p. 166 with Simpson's Rule replaced by the Midpoint Rule:

The formula for the bound on $|E_M|$ is

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \quad \text{(Box, p. 510)}$$

where $|f''(x)| \leq K$ on $[a,b]$

As before $[a,b] = [0,1]$, hence

$b - a = 1 - 0 = 1$, and

$f''(x) = (\sin x)'' = -\sin x$,

hence $|$

Thus we can choose $K = 1$.

Thus $|E_M| \le \dfrac{1}{24n^2}$

Hence $|E_M| < 0.0001$ if we choose n so that $\dfrac{1}{24n^2} < 0.0001$;

Multiply by 10^4: $\dfrac{10^4}{24n^2} < 1$,

Multiply by n^2: $\dfrac{10^4}{24} < n^2$,

i.e. $n > \sqrt{\dfrac{10^4}{24}} \approx 20.41$

Hence we can choose $\underline{\underline{n = 21}}$.

Let us return to the discussions of the approximations

$M_6 \approx 1{,}055.8$, $T_6 \approx 2{,}319.1$, $S_6 \approx 1{,}722.3$

for the integral

$\int_0^3 e^{x^2} dx \approx 1{,}444.55$ which is an accurate value obtained by MATHEMATICA.

On p. 160 we obtained bounds for the errors E_M, E_T and on p. 161 a bound for E_S. We had

$|E_M| \leq \dfrac{171 e^9}{4n^2}$, $|E_T| \leq \dfrac{171 e^9}{2n^2}$,

$|E_S| \leq \dfrac{2{,}349 e^9}{n^4}$

Thus with $n=6$ we obtain $|E_M| \leq 9{,}622.42$, but in fact, the actual error is much less than that, namely $E_M \approx 1{,}444.55 - 1{,}055.8$

$E_M \approx 388.75$. Similarly for E_T and E_S.

A comment to this effect is on p. 510, above Example 2: The Actual Error is often substantially smaller than what we obtain from the formulas for the error bounds:

i.e. $|E_T| \leq \dfrac{K(b-a)^3}{12n^2}$, $|E_M| \leq \dfrac{K(b-a)^3}{24n^2}$

$|E_S| \leq \dfrac{K(b-a)^5}{180n^4}$ (K is different for E_S from that for E_T, E_M)

Another Comment. An important part of the information about the error bounds is that E_T and E_M approach zero at the rate $\frac{1}{n^2}$ (i.e. some constant times $\frac{1}{n^2}$), whereas E_S approaches zero at the rate $\frac{1}{n^4}$. Thus Simpson's Rule is more accurate.

*

A comment on the Homework Question #29, p. 517:

How to pick the values of the function in the graph: At integer points, pick the values to be integers — that's reasonable enough, looking at the graph.

At the midpoints pick the nearest multiple of 1/4. So for example, pick $f(5.5) = 3.25$.

By doing what is suggested here you will not get the same answers as listed in the Answers Section in the back of the Book.

HW: 7.5, 7.7, part of 7.8. through #37 (inclusive)

due Tue. Feb. 11