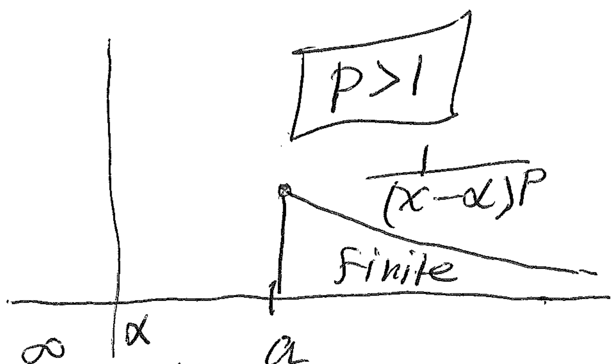
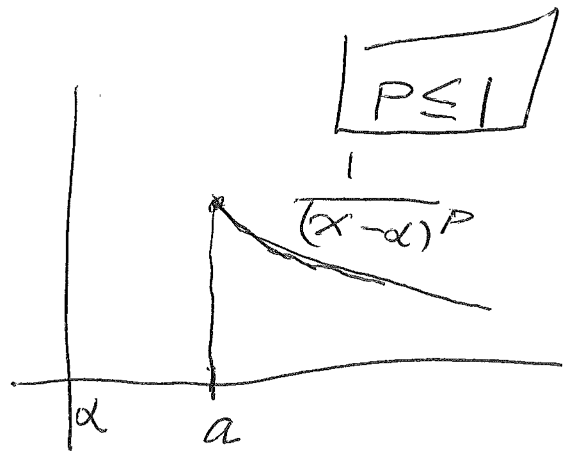
The p-Test(s) Associated with the Graphs on p. 207, stated using Improper Integrals:

Let α be any number, and a any number $> \alpha$.

(a) Then the improper integral $\int_a^\infty \frac{1}{(x-\alpha)^p}\,dx$ is convergent if $p>1$ and divergent if $p \le 1$.



$\int_a^\infty \frac{1}{(x-\alpha)^p}\,dx$ CONVERGENT



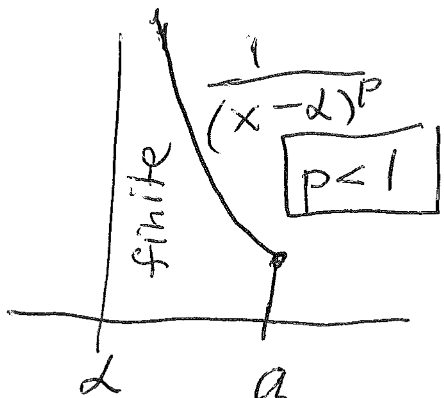
$\int_a^\infty \frac{1}{(x-\alpha)^p}\,dx$ DIVERGENT

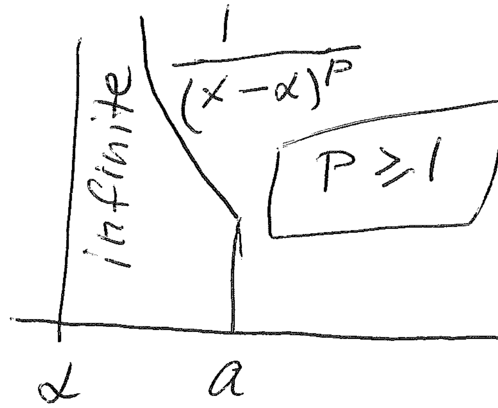
Continued from p. 208.

(b) The Improper Integral $\int_{\alpha}^{a} \frac{1}{(x-\alpha)^p}\,dx$

is convergent when $p<1$,

and divergent when $p \geq 1$.





$\int_{\alpha}^{a} \frac{1}{(x-\alpha)^p}\,dx$

CONVERGENT

$\int_{\alpha}^{a} \frac{1}{(x-\alpha)^p}\,dx$

DIVERGENT

Some Comments on p-Tests Discussed in preceding pages.

The BASIC p-Test(s) are Those for $\alpha=0$, i.e. $\int_a^\infty \frac{1}{x^p}\,dx$, $\int_0^a \frac{1}{x^p}\,dx$

Where a is any number > 0.

The Case $\alpha = 0$ is Described on p. 176 of these Notes and Example 4, p. 522 in the Book for the Integral when $a=1$

$$\int_a^\infty \frac{1}{x^p}\,dx \quad (\text{where } a > 0).$$

The Integral $\int_0^a \frac{1}{x^p}\,dx$ (again $a>0$)

is stated in the Exercise #57, p. 528 in the Book (when $a=1$);

The Exercise #57 is solved on pages 202–206 in these Notes.

*

SOME CORRECTIONS.

p. 104: The term "$2+x$" in two places near the bottom of the page should be replaced by "$2x+1$". The same correction should be made in two places near the top of p. 105.

p. 164: Line 9 which is

"is a bit larger than"

should be "is quite a bit larger than"

p. 191: In the statement of the Comparison Test, a very important part is missing: Namely, on line 9, which is

"and $f(x) \geq g(x)$ for all $x \geq a$"

should be

"and $f(x) \geq g(x) \underline{\underline{\geq 0}}$ for all $x \geq a$".

NEXT I REWRITE THE PAGE 193

TO make it better ORGANIZED and make another explanatory comment.

page 193 rewritten:

We choose the function $\frac{1}{x^2}$ since it is reasonable to guess that it contains the essential feature of $\frac{1}{x^2(3+\cos x)}$.

Since the integral $\int_5^\infty \frac{1}{x^2}\,dx$ is convergent, we should try to show that $\int_5^\infty \frac{1}{x^2(3+\cos x)}\,dx$ is likewise convergent. Thus we would like to show that $\frac{1}{x^2(3+\cos x)} \leq \frac{1}{x^2}$ for $x \geq 5$.

We should now carefully verify all assumptions of the Comparison Test:

First we show that both functions above are continuous for $x \geq 5$.

We of course know (have known) that $\frac{1}{x^2}$ is continuous:

That is if $f(x)$ is cont. on a certain domain, and $f(x) \neq 0$ on this domain, then $\frac{1}{f(x)}$ is also cont. on this domain. Since x^2 is cont (it is a polynomial), $x^2 \neq 0$ for $x > 0$, page 193 rewriting continued:

Next we show that both

$\frac{1}{x^2}$, $\frac{1}{x^2(3+\cos x)}$ are ≥ 0

for all $x \geq 5$: Clear for $\frac{1}{x^2} = \left(\frac{1}{x}\right)^2$.

For $\frac{1}{x^2(3+\cos x)}$: It suffices

to show that $x^2(3+\cos x) \geq 0$.

But we showed on bottom half of preceding page that $3+\cos x \geq 2$ for all x, hence $3+\cos x \geq 0$ since $2 \geq 0$. Then product of numbers ≥ 0 is ≥ 0.

Hence $x^2(3+\cos x) \geq 0$.

Finally we show that $\underbrace{\frac{1}{x^2(3+\cos x)}}$ $\leq \frac{1}{x^2}$

Since the quantities in the denominator are > 0, (they are $\neq 0$, and ≥ 0), the inequality above when $x \geq 5$ (in fact $x > 0$)

Hence we obtain $\frac{1}{x^2}$ cont. on $(0,\infty)$, hence on $[5,\infty)$.

Similarly $x^2(3+\cos x)$ is cont., since x^2 is cont., and $(3+\cos x)$ is cont. (trig. fncts. $\sin x, \cos x$ are cont. for all x). And product of cont. fncts. is continuous.

We further prove that $x^2(3+\cos x) \neq 0$ for any $x>0$:

We know $\cos x \geq -1$, for all x, hence

$3+\cos x \geq 3+(-1)$,

i.e. $3+\cos x \geq 2$ for all x.

Hence $3+\cos x \neq 0$ for all x.

Hence $x^2(3+\cos x) \neq 0$.

Hence $\dfrac{1}{x^2(3+\cos x)}$ is cont. on $(0,\infty)$.

page 193 rewriting continued:

is equivalent $x^2(3+\cos x) \geq x^2$

This is however true since $3+\cos x \geq 2$.

Thus we have verified all assumptions of the Comparison Test, part (a), with $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x^2(3+\cos x)}$, on the interval $[5,\infty)$. Thus since $\int_5^{\infty} \frac{1}{x^2}\,dx$ is convergent, it follows that $\int_5^{\infty} \frac{1}{x^2(3+\cos x)}\,dx$ is convergent. (as on p. 194).

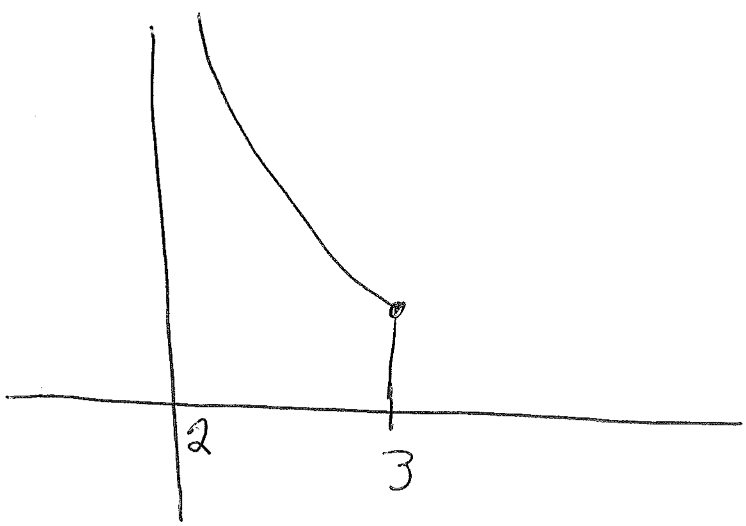
$\ast$

Use the Comparison Test to Determine whether the Integral

$$\int_2^3 \frac{2x+1}{\sqrt{x^5-32}}\,dx$$

is convergent or divergent.

This is an Improper integral of Type 2 since the function $\frac{2x+1}{\sqrt{x^5-32}}$ is continuous on $(2,3]$ and has a vertical asymptote at $x=2$ (hence discont. on $[2,3]$).



We would like to compare $\dfrac{2x+1}{\sqrt{x^5-32}}$

to $c \cdot \dfrac{1}{(x-2)^p}$, for a suitable p,

on some interval $(2, a)$, where $a \leq 3$.

Since $x^5 - 32$ is a polynomial which $= 0$ for $x = 2$, it follows that $x^5 - 32$ is divisible by $(x-2)^k$ for some positive integer $k \geq 1$.

We want to find largest such k.

As already stated, $k \geq 1$, i.e. $x^5 - 32$ is divisible by $(x-2)$.

To check if $x^5 - 32$ is divisible by $(x-2)^2$, we can differentiate (x^5-32):

$(x^5 - 32)' = 5x^4$, which $\neq 0$ for $x = 2$,

It thus follows that $(x^5 - 32)$ is not divisible by $(x-2)^2$.

(Or one can just carry out long division of $(x-2)$ into $(x^5 - 32)$, then check if the resulting polynomial $= 0$ for $x = 2$; also by College algebra

$$x^n - b^n = (x-b)(x^{n-1} + x^{n-2}b + x^{n-3}b^2 + \cdots + x^2 b^{n-3} + x b^{n-2} + b^{n-1})$$

when n odd)

Thus we will compare $\dfrac{2x+1}{\sqrt{x^5-32}}$

with $c \cdot \dfrac{1}{\sqrt{x-2}}$; SEE COMMENT ON PAGE 218

(if $x^5 - 32$ were divisible by $(x-2)^2$, we would compare with $c \cdot \dfrac{1}{\sqrt{(x-2)^2}}$,

if divisible by $(x-2)^3$, we would compare with $c \cdot \dfrac{1}{\sqrt{(x-2)^3}}$, etc.)

To make the comparison, we first calculate

$$\lim_{x \to 2+} \frac{\dfrac{2x+1}{\sqrt{x^5-32}}}{\dfrac{1}{\sqrt{x-2}}} = \lim_{x \to 2+} (2x+1) \cdot \frac{\sqrt{x-2}}{\sqrt{x^5-32}} =$$

$$= \lim_{x\to 2+} (2x+1)\sqrt{\frac{x-2}{x^5-32}}$$

We first calculate $\underbrace{\lim_{x\to 2+} \frac{x-2}{x^5-32}}$

Using L'H. Rule, $= \lim_{x\to 2+} \frac{1}{5x^4} = \frac{1}{80}$

Hence $\lim_{x\to 2+} \underbrace{(2x+1)}_{\to 5}\sqrt{\frac{x-2}{x^5-32}}$

$$= 5\sqrt{\frac{1}{80}} = \frac{\sqrt{5}}{4}$$

COMMENT FOR PAGE 217:

BUT $\int_2^a \frac{1}{\sqrt{x-2}}\,dx$, for $a>2$, is convergent by the p-test.

Thus we want to show that the integral $\int_2^3 \frac{2x+1}{\sqrt{x^5-32}}\,dx$ is conv.

Hence we want to show that

$$\frac{2x+1}{\sqrt{x^5-32}} \le c \cdot \frac{1}{\sqrt{x-2}}$$

(at least for x such that $2<x<a$, some $a>2$)

Hence $\lim_{x \to 2+} \dfrac{\frac{2x+1}{\sqrt{x^5-32}}}{\frac{1}{\sqrt{x-2}}} = \dfrac{\sqrt{5}}{4}$

But then, if we choose any number $c > \frac{\sqrt{5}}{4}$, the quotient

$$\frac{\frac{2x+1}{\sqrt{x^5-32}}}{\frac{1}{\sqrt{x-2}}} \quad \text{must be} < c$$

for x sufficiently close to 2 and >2.

I.e. for some $a > 2$, it is true that

$$\frac{\frac{2x+1}{\sqrt{x^5-32}}}{\frac{1}{\sqrt{x-2}}} < c \quad \text{for all } x \text{ such that } 2 < x < a$$

I.e. $\dfrac{2x+1}{\sqrt{x^5-32}} < c \cdot \dfrac{1}{\sqrt{x-2}}$

for all x such that $2 < x < a$,
where c is any number $> \frac{\sqrt{5}}{4}$
and $a > 2$ (a is chosen after c is chosen)
We can choose $c = \frac{3}{4}$ or $c = 1$ etc.

Thus we will now summarize what was proved in order to show that we have verified all assumptions of the comparison Test and thus we can apply the Comparison Test to show that $\displaystyle\int_2^3 \frac{2x+1}{\sqrt{x^5-32}}\,dx$

is convergent.

(1) $\dfrac{2x+1}{\sqrt{x^5-32}}$, $\dfrac{1}{\sqrt{x-2}}$ both are continuous on $(2,3]$ and ≥ 0

(2) There is some $a > 2$, such that

$$\frac{2x+1}{\sqrt{x^5-32}} < 1 \cdot \frac{1}{\sqrt{x-2}} \quad \text{for all } x$$

such that $2 < x < a$

(3) $\displaystyle\int_2^3 \frac{1}{\sqrt{x-2}}\,dx$ is convergent.

Conclusion, by using the Compar. Test:
$\displaystyle\int_2^3 \frac{2x+1}{\sqrt{x^5-32}}\,dx$ is Conv.

$\ast$

COMMENT. Note the similarity of the INTEGRAL on p. 215 with the one on p. 185, and likewise the similarity of $\int_2^3 \frac{1}{\sqrt{x-2}}\,dx$ to the one on p. 185.

Furthermore the similarity of to the one on bottom half of p. 187, the only difference being $p = \frac{1}{2}$ and $p = \frac{2}{3}$ on p. 187, so on p. 187 we can consider the function both to the right as well as to the left of the point 2 ($\sqrt{x-2}$ is defined only for $x \geq 2$, whereas $(x-2)^{2/3}$ is defined both for $x \geq 2$ as well as $x \leq 2$).

Also, compare $\int_2^3 \frac{1}{\sqrt{x-2}}\,dx$ with the two in the box on p. 188.

Furthermore, compare with the integral(s) $\int_1^6 \frac{1}{x^2-4x+4}\,dx = \int_1^6 \frac{1}{(x-2)^2}\,dx$,

and $\int_1^2 \frac{1}{(x-2)^2}\,dx$, $\int_2^6 \frac{1}{(x-2)^2}\,dx$,

(hence $\int_1^2 \frac{1}{(x-2)^p}\,dx$, $\int_2^6 \frac{1}{(x-2)^p}\,dx$,

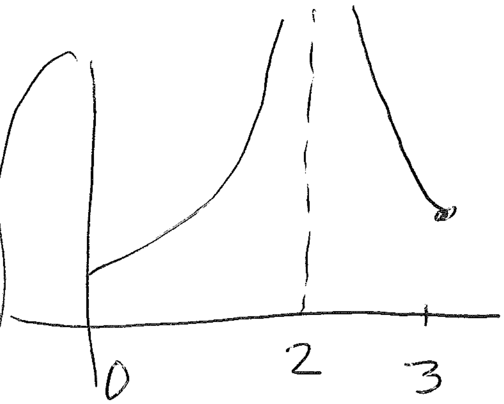
with p now $= 2$) which are the integrals in Question #3 of the posted sample exam.

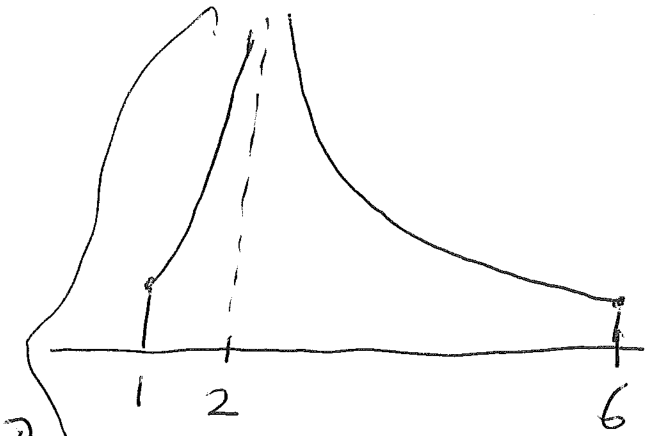
<u>FURTHERMORE</u>, COMPARE DIRECTLY THE EXAMPLE WORKED ON pages 187 – 190 in the NOTES, of the INTEGRAL

$$\int_0^3 \frac{1}{(x-2)^{2/3}}\,dx,$$

with $\int_1^6 \frac{1}{(x-2)^2}\,dx$ from Question #3 SAMPLE EXAM.

CONTINUING THE COMMENT "FURTHERMORE" from p. 222:





$$\int_0^3 \frac{1}{(x-2)^{2/3}}\,dx$$

CONVERGENT

p. 187–190 NOTES

$$\int_1^6 \frac{1}{(x-2)^2}\,dx$$

DIVERGENT

SAMPLE EXAM, QUESTION #3

THE MAIN DIFFERENCE IS

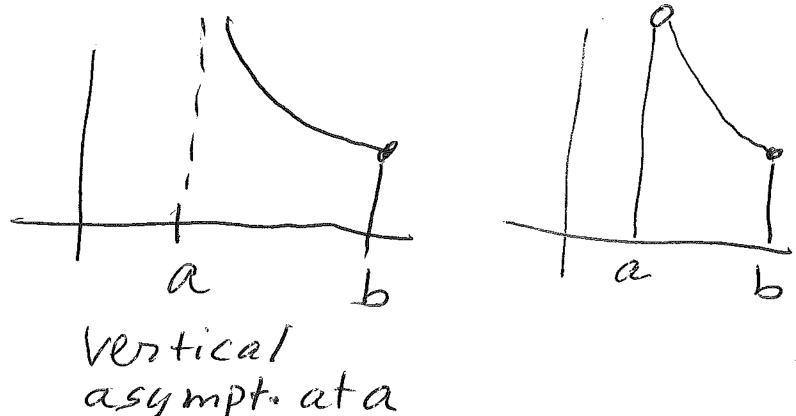
$\begin{cases} p = 2/3 \text{ (p. 187–190, NOTES)} \\ \text{hence } p<1, \text{ hence CONVERGENT} \end{cases}$

$\begin{cases} p = 2. \text{ (SAMPLE EXAM, Quest. \#3)} \\ \text{hence } p>1, \text{ hence DIVERGENT.} \end{cases}$

COMPARISON TEST For IMPROPER INTEGRAS OF TYPE 2:

We will state it for INTEGRALS of the Form

$\int_a^b f(x)\,dx$ where $f(x)$ is cont. on $(a, b]$ and has a discontinuity at a:



BASIC FORM: Let $f(x), g(x)$ be both cont. on $(a, b]$, $f(x) \geq 0$, $g(x) \geq 0$, and $f(x) \leq g(x)$ on $(a, b]$.

(a) If $\int_a^b g(x)\,dx$ is convergent, then $\int_a^b f(x)\,dx$ is convergent.

(b) If $\int_a^b f(x)\,dx$ is divergent, then $\int_a^b g(x)\,dx$ is divergent.

AMPLIFIED FORM.

Let $f(x)$, $g(x)$ both be cont. on $(a,b]$, $f(x) \geq 0$, $g(x) \geq 0$ on $(a,b]$.

and suppose that there is a constant $c > 0$ and some α in $(a,b]$ such that $f(x) \leq c\,g(x)$ for all x in (a,α).

If $\int_a^b g(x)\,dx$ is conv.,

then $\int_a^b f(x)\,dx$ is conv.

If $\int_a^b f(x)\,dx$ is div.,

then $\int_a^b g(x)\,dx$ is div.

Comment. We can state the part for DIVERGENCE also in the form where the condition "$f(x) \leq c\,g(x)$"

is replaced by "$f(x) \geq Cg(x)$":

Then the conclusion becomes:

If $\int_a^b g(x)\,dx$ is div., then $\int_a^b f(x)\,dx$ is div. $\ast$

We can have improper integrals which are ~~a~~ a combination of Improper Integrals of type 1 and type 2. Homework Exercise #51, p. 528 is of this type.

Example. Determine if

$$\int_2^{\infty} \frac{2x+1}{\sqrt{x^5-32}}\,dx$$

is convergent or divergent.

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To solve this problem, we break up the integral into two improper integrals by choosing any $a > 2$ and considering

$$\int_2^a \frac{2x+1}{\sqrt{x^5-32}}\,dx, \text{ and } \int_a^\infty \frac{2x+1}{\sqrt{x^5-32}}\,dx$$

The original integral $\int_2^\infty \frac{2x+1}{\sqrt{x^5-32}}\,dx$ is then convergent if both integrals above are convergent, and it is divergent if one or both integrals above are divergent.

We can choose $a = 3$ (but it makes no difference what we choose; we could choose $a = 5$ or $a = 100$ etc.)

We have already shown that both integrals $\int_3^\infty \frac{2x+1}{\sqrt{x^5-32}}\,dx$ and $\int_2^3 \frac{2x+1}{\sqrt{x^5-32}}\,dx$ are convergent, namely pages 195 – 201 for the first, and pages 215 – 220 for the second.

It follows that the integral

$$\int_2^\infty \frac{2x+1}{\sqrt{x^5-32}}\,dx \text{ is convergent.}$$

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