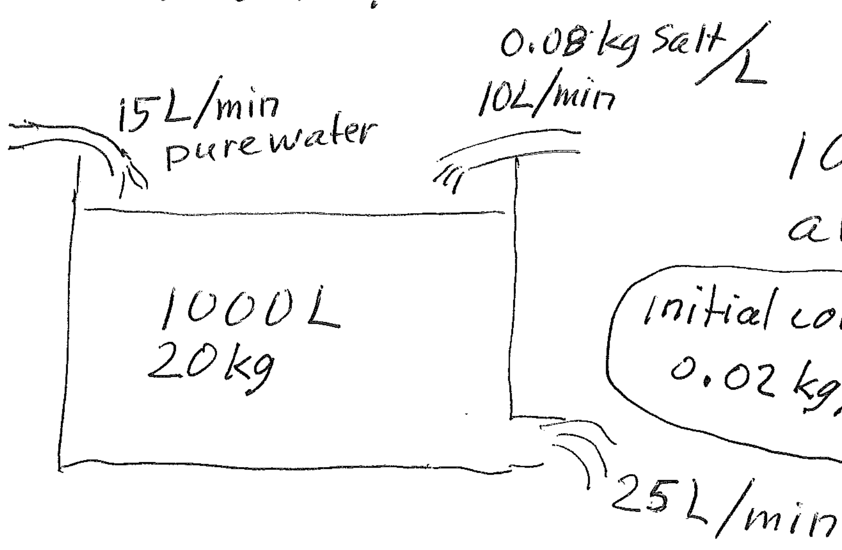
Mixing Problems.

HW due Tue. 3/4: 9.2, 9.3, 10.1 as on the syllabus

A tank contains 1000 L of brine with 20 kg of dissolved salt. Pure water enters the tank at the rate of 15 L/min. Brine that contains 0.08 kg of salt per liter enters the tank at the rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 25 L/min. How much salt is in the tank after t minutes; after 1 hour?



1000 L at all times

initial concentr. 0.02 kg/L

Concentration at time t is

$$\frac{y(t)}{1000}, \qquad y(0) = 20\,\text{kg}$$

Let $y(t) =$ the amount of salt in the tank at time t;

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$(\text{rate in}) = \left(0.08 \frac{kg}{L}\right)(10\,L/min) = \boxed{0.8\,kg/min}$$

The 15 L/min of pure water does not add in any salt.

$$(\text{rate out}) = \left(\frac{y(t)}{1000} \frac{kg}{L}\right)(25L/min) = \frac{y(t)}{40}\,kg/min$$

$$\frac{dy}{dt} = 0.8 - \frac{y}{40} = \frac{32-y}{40}, \quad \text{init. cond. } y(0) = 20$$

$$\frac{dy}{32-y} = \frac{1}{40}\,dt$$

$$\int \frac{dy}{32-y} = \int \frac{1}{40}\,dt$$

$$-\ln|32-y| = \frac{1}{40}t + C$$

$$\ln|32-y| = -\frac{1}{40}t + C$$

$$|32-y| = e^{-\frac{1}{40}t+C} = e^{-\frac{1}{40}t} \cdot e^{C}$$

$$= Ke^{-\frac{1}{40}t}, \quad K>0$$

$$32-y = Ke^{-\frac{1}{40}t}$$

$$y = 32-Ke^{-\frac{1}{40}t} = y(t) \quad \text{Find } K:$$

$$y(0) = 32-Ke^{0} = 32-K$$

$$\underset{20}{\overset{\parallel}{}} = 32-K$$

$$K=12$$

$$y = y(t) = 32-12e^{-\frac{1}{40}t}$$

$$\underbrace{y(60)} = 32-12e^{-\frac{1}{40}\cdot 60} \approx 29.32 \text{ kg}$$

the amount of salt after 1 hour

*

$$\lim_{t\to\infty} y(t) = 32 \text{ (kg)}$$

Similar to #19, p. 600.

Find an equation of the curve that passes through the point $(0,1)$ and whose slope at (x,y) is xe^{-y}.

Answer. If for every (x,y), the slope at the point (x,y) is xe^{-y}, that means that $\frac{dy}{dx} = xe^{-y}$

To pass through the point $(0,1)$ means $y(0)=1$. Hence we have to solve the initial value problem

$$\frac{dy}{dx} = xe^{-y}, \quad y(0)=1.$$

The general solution of the diff. eq.

$\frac{dy}{dx} = xe^{-y}$ is found at the bottom of p. 285, $y = \ln\left(\frac{1}{2}x^2 + C\right)$

Hence $y(0) = \ln\left(\frac{1}{2}0^2 + C\right) = \ln C$

Hence $y(0)=1 \Rightarrow \ln C = 1$, i.e. $C=e$.

Hence the eq. of the desired curve is $y = \ln\left(\frac{1}{2}x^2 + e\right)$ *

Exercise #48, p. 602 in the Book is a Mixing Problem: (picture next page)

A tank contains 1000 L of pure water. Brine that contains 0.05 kg of salt per liter enters the tank at a rate of 5 L/min. Brine that contains 0.04 kg of salt per liter enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. How much salt is in the tank (a) after t minutes; (b) after one hour?

As on p. 290 of the Notes, or p. 599 in the Book, we have

$$\boxed{\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})}$$

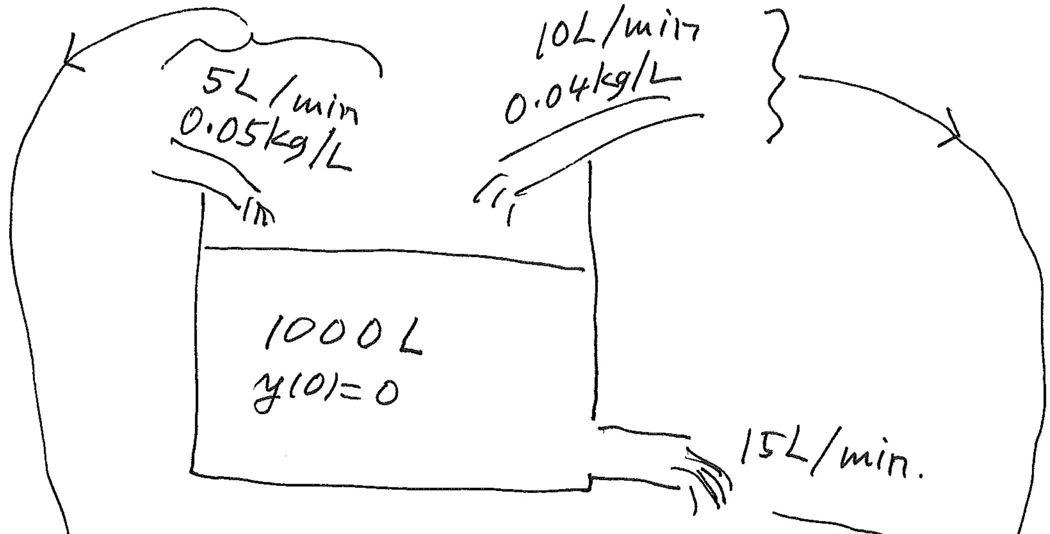
$$(\text{rate in}) = \left(0.05\,\frac{kg}{L}\right)\left(5\,\frac{L}{min}\right) + \left(0.04\,\frac{kg}{L}\right)\left(10\,\frac{L}{min}\right)$$

$$= 0.25\,\frac{kg}{min} + 0.40\,\frac{kg}{min} = 0.65\,\frac{kg}{min}$$

$$(\text{rate out}) = \left(\frac{y(t)}{1000}\,\frac{kg}{L}\right)\left(15\,\frac{L}{min}\right) = \frac{15}{1000}\,y(t)\,\frac{kg}{min}$$

$$= \frac{3}{200}\,y(t)\,\frac{kg}{min}$$

For #48, p. 602 in the Book.



$$\mathrm{I}\begin{cases}\text{part of rate in}\\ (0.05)\cdot(5) = 0.25\,(\text{kg/min})\end{cases}$$

$$\mathrm{II}\begin{cases}\text{part of rate in}\\ (0.04)\cdot(10) = 0.40\,(\text{kg/min})\end{cases}$$

$$\boxed{\text{Rate out}: \quad \frac{y(t)}{1000}\cdot 15 = \frac{y}{1000}\cdot 15}$$

$$\boxed{\text{Rate in} = \mathrm{I}+\mathrm{II} = 0.25+0.40 = 0.65}$$

Diff. Eq.: $\dfrac{dy}{dt} = 0.65 - \dfrac{y}{1000}\cdot 15$

Initial Condition: $y(0) = 0$

Note that the amount that enters the tank per minute equals $5L + 10L = 15L$, and the amount that drains from the tank per minute likewise equals $15L$. Hence the volume of brine in the tank equals $1000\,L$ at all times. Thus the concentration of the brine in the tank at time t is

$\frac{y(t)}{1000} \frac{kg}{L}$. Thus we take

$$\left(\frac{y(t)}{1000} \frac{kg}{L}\right) \cdot \left(15 \frac{L}{min}\right) = \frac{15y(t)}{1000} \frac{kg}{min}$$

to be the rate out at time t (where $y(t)$ is the amount of salt in the tank at time t).

We thus substitute into the equation in the box on p. 293, obtaining

$$\frac{dy}{dt} = 0.65 - \frac{3}{200} y(t)$$

The differential equation that we solve (for y) is then

$$\frac{dy}{dt} = 0.65 - \frac{3}{200} y$$

(i.e. we omit the t in $y(t)$).

Also, the problem is an initial value problem, the initial condition being determined from the amount of the salt in the tank initially — for #48 (p. 293 in the Notes) we have $y(0) = 0$.

Thus the initial value problem is stated as

$$\frac{dy}{dt} = 0.65 - \frac{3}{200} y, \quad y(0) = 0$$

You can finish solving this init. value problem for yourselves (see pages 289 – 291 for a similar problem).

Answer. $y(t) = \frac{130}{3}\left(1 - e^{-3t/200}\right)$

After 1 hour: $y(60) \approx 25.7$ kg *

Parametric Equations of Curves.

Parametric Equations allow us to represent motion of a particle along a curve. For example:

$x = t, \; y = t^2, \; -\infty < t < \infty$

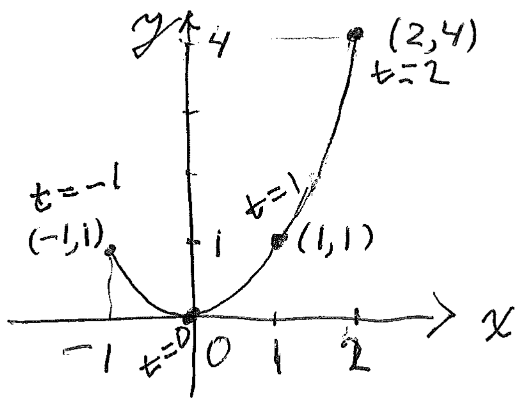
We can think of t as time.

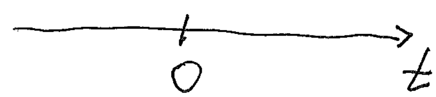
Then at time t (e.g. seconds),

the particle is at the point

$(x, y) = (t, t^2)$; e.g.

t	-1	0	1	2
x	-1	0	1	2
y	1	0	1	4





The equations $x = t$, $y = t^2$ imply $y = x^2$ (substituting x for t in the 2nd equation). thus the particle moves along the parabola $y = x^2$.

Now consider the parametric equations

$$x = 3t, \ y = 9t^2, \quad -\infty < t < \infty$$

Again we see $x^2 = (3t)^2 = 9t^2 = y$.
Thus this 2nd set of param. equations also represents the motion of a particle along the parabola $y = x^2$. However the particles move in a different manner. They are both at $(0,0)$ at time $t = 0$. But at time $t = 1$, the particle from p. 296 is at the point $(1, 1)$, whereas for the particle motion described on this page, the particle is at $(3, 9)$ at time $t = 1$.

Another way to describe this situation is to say that an x,y - equation of a curve describes a set of points in the plane — e.g. the parabola $y = x^2$ or

$x^2+y^2=4$ (the circle of radius 2 centered at the origin), whereas a set of two parametric equations describes a manner in which a curve is traced out (by pen or the motion of a particle along such a curve).

The param. equations

$$x = 2\cos t\ ,\ y = 2\sin t\ ,\ -\infty < t < \infty$$

describes the circle $x^2+y^2=4$ being repeatedly traced out:

If at time t the particle finds itself at the point $(x,y) = (2\cos t, 2\sin t)$, then

$$x^2+y^2 = 4\cos^2 t + 4\sin^2 t = 4,$$

hence the particle is at a point (x,y) on the circle $x^2+y^2=4$.

At time $t=0$ the particle is at $(2\cos 0, 2\sin 0) = (2,0)$.

Example. Find parametric equations for the path of a particle which travels clockwise along the circle $x^2+y^2=4$ and at time $t=0$ finds itself at $(\sqrt{2}, \sqrt{2})$.

Solution. We observe that the simplest param. representation of the circle (p. 298) traces out the circle counterclockwise:

t	0	$\pi/4$	$\pi/2$	etc.
x	2	$\sqrt{2}$	0	
y	0	$\sqrt{2}$	2	

We can reverse this if in the equations $x = 2\cos t$, $y = 2\sin t$ we substitute $(-t)$ for t, obtaining

$$x = 2\cos(-t), \quad y = 2\sin(-t)$$

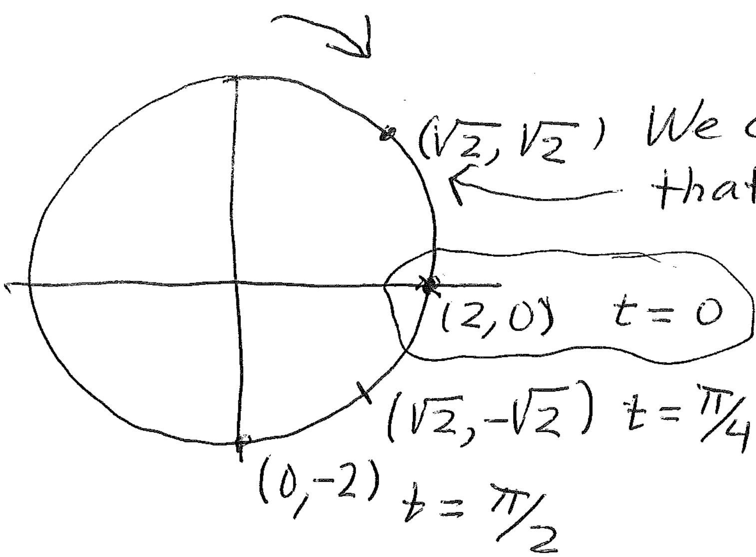
But $\cos(-t) = \cos t$, $\sin(-t) = -\sin t$, hence we obtain

$$x = 2\cos t, \quad y = -2\sin t$$

We can check that the circle is being traced out clockwise, but still starting at $(2,0)$ when $t=0$.

t	0	$\pi/4$	$\pi/2$	etc.
x	2	$\sqrt{2}$	0	
y	0	$-\sqrt{2}$	-2	

$$x = 2\cos t, \quad y = -2\sin t$$



So we can make the particle be at $(\sqrt{2}, \sqrt{2})$ at time $t=0$ if we substitute $(t - \frac{\pi}{4})$ for t in the equations above:

$$x = 2\cos\left(t - \frac{\pi}{4}\right), \quad y = -2\sin\left(t - \frac{\pi}{4}\right)$$

$*$

We can also obtain param. equations for circles centered at an arbitrary point (a,b). Namely,

$$x = a + r\cos t \ , \ y = b + r\sin t$$

is a set of param. equations for the circle of rad. r centered at (a,b) since $(x-a)^2 + (y-b)^2 =$

$$= (r\cos t)^2 + (r\sin t)^2 = r^2$$

But $(x-a)^2 + (y-b)^2 = r^2$ is an eq. of the circle of rad $= r$ centered at (a,b).

Again, the above set of equations are for a particle tracing the circle counterclockwise. Substituting $(-t)$ for t makes the circle traced out clockwise, etc. $*$

#2, p. 641. Sketch the curve defined by the parametric equations

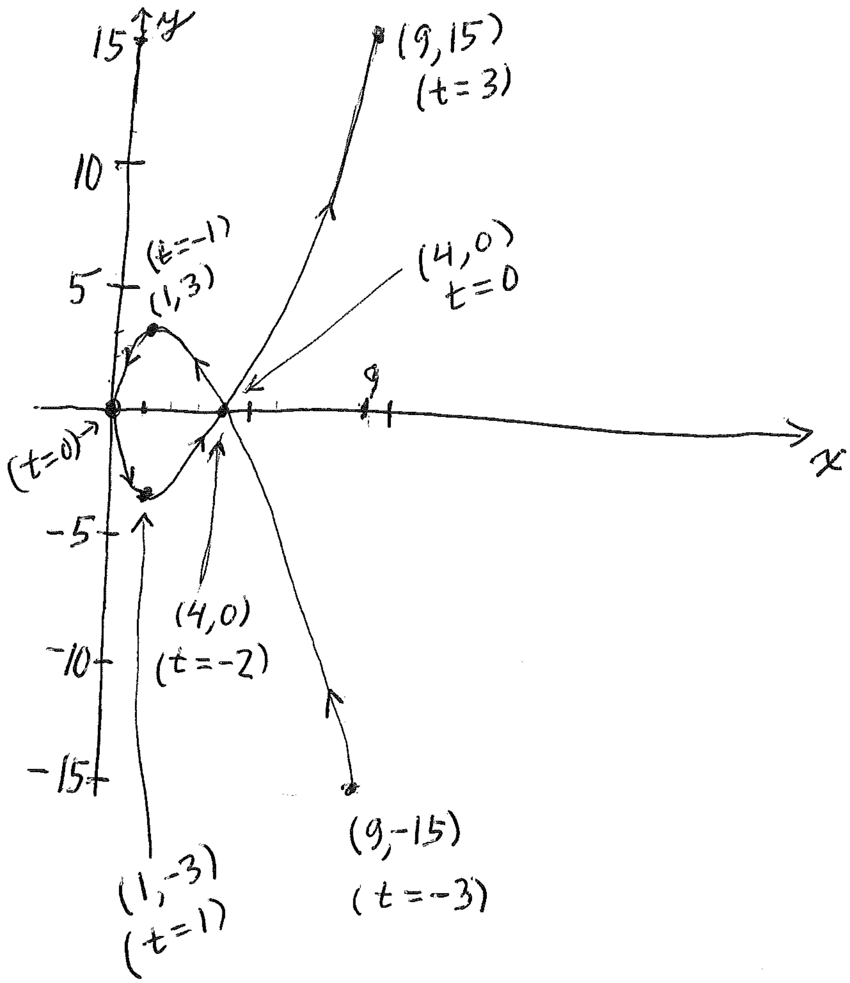
$$x = t^2, \quad y = t^3 - 4t, \quad -3 \le t \le 3$$

by using the equations to plot points.

t	-3	-2	-1	0	1	2	3
x	9	4	1	0	1	4	9
y	-15	0	3	0	-3	0	15

We also observe that the curve is symmetric about the -axis, since for every positive x, a point (x, y) is on the curve if and only if the point $(x, -y)$ is on the curve. Also, only points with positive (≥ 0) x-coordinate are on the curve, and the curve passes through the origin $(0,0)$.

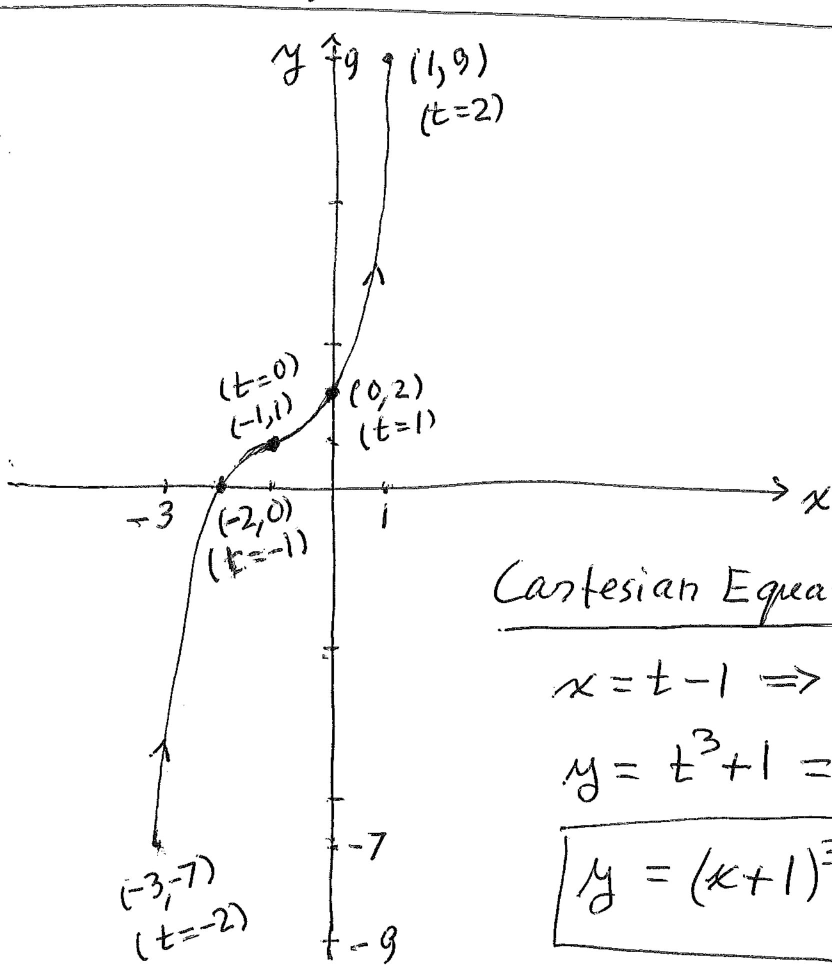
#2, p. 641 continued



#8, p. 641 Sketch the curve by plotting points and also find a Cartesian equation of the curve (i.e. an x, y-equation):

$$x = t-1, \quad y = t^3+1, \quad -2 \le t \le 2$$

t	-2	-1	0	1	2
x	-3	-2	-1	0	1
y	-7	0	1	2	9



Cartesian Equation:

$x = t-1 \Rightarrow t = x+1$

$y = t^3+1 = (x+1)^3+1$

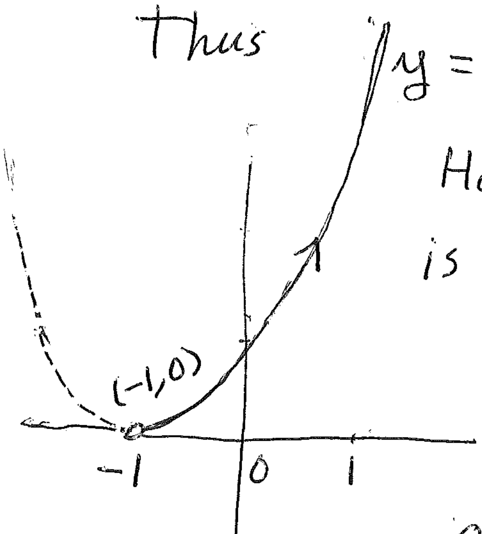
$$\boxed{y = (x+1)^3+1}$$

#14, p. 641. Eliminate the parameter to find a Cartesian Equation of the curve. Also sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter t increases.

$$x = e^t - 1, \quad y = e^{2t}$$

We see that $y = (e^t)^2$, while $e^t = x + 1$ from the first equation.

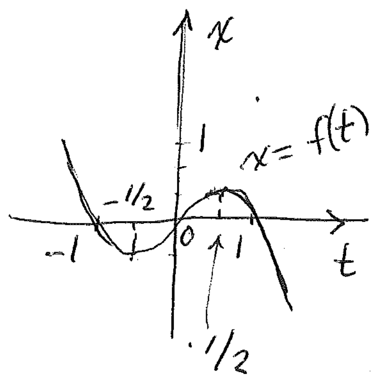
Thus $y = (x+1)^2$ is the Cartesian Equation.

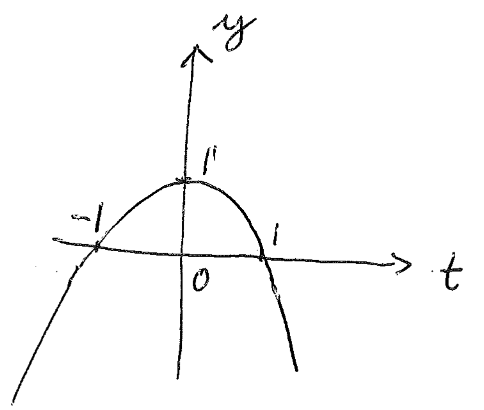


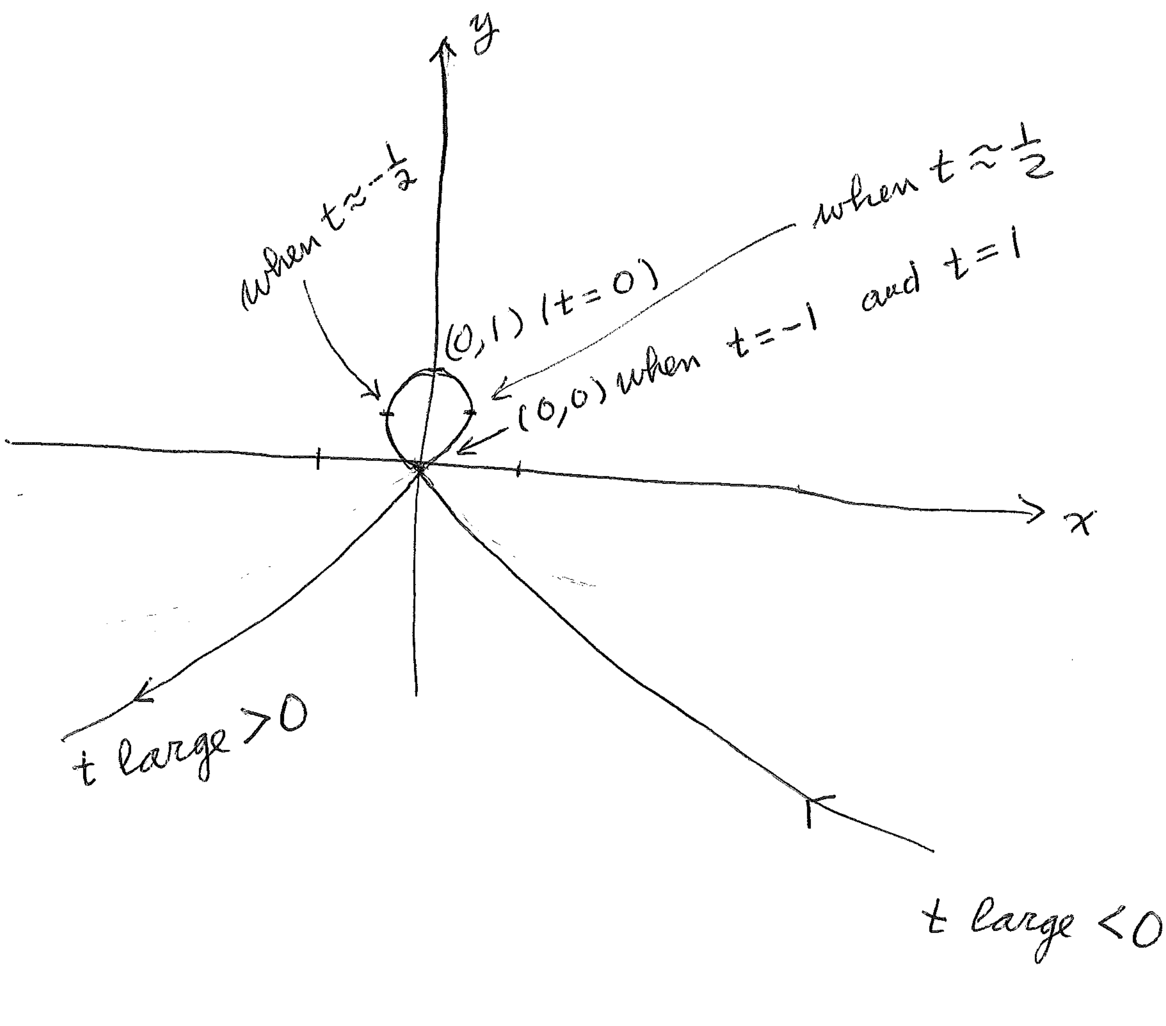
However, while the Cartesian equation is the equation of the entire parabola $y = (x+1)^2$, the Param. Equations define only a part of this parabola, namely those points for which $x = e^t - 1$ for some t. Since $e^t > 0$ for all t, only those x can be obtained which are > -1. Hence only the right half of the parabola is obtained, vertex not included.

*

#26, p. 542. The graphs of $x = f(t)$, $y = g(t)$ are shown. Sketch the curve $x = f(t)$, $y = g(t)$







Midterm 1 Solutions. 2/13

307

1.(5 pts) The substitution required when evaluating

$$\int \sin^7 x \, dx$$

is $\boxed{u = \cos x}$

No justification is required.

2.(5 pts) The substitution required when evaluating

$$\int \tan^5 x \sec^3 x \, dx$$

is $\boxed{u = \sec x}$

No justification is required.

COMMENT. The substitutions above may be done after some "preparatory" work. You are not required to show this work.

Work for #1 (Does not have to be shown.)

$$\sin^7 x = \sin^6 x \sin x = (1-\cos^2 x)^3 \sin x$$

$$\cos x = u$$

Work For #2 (Does not have to be shown.)

$$\tan^5 x \sec^3 x = \tan^4 x \sec^2 x (\sec x \tan x)$$

$$= (\sec^2 x - 1)^2 \sec^2 x (\sec x \tan x)$$

$$u = \sec x$$

308

3.(10 pts) Evaluate

$$\int \frac{e^x}{e^{2x}-4}\,dx$$

$e^x = u$, $du = e^x dx$

$$= \int \frac{1}{u^2-4}\,du$$

$$\frac{1}{u^2-4} = \frac{A}{u+2} + \frac{B}{u-2}$$

$u^2-4=(u+2)(u-2)$

$$= \frac{(A+B)u + (2B-2A)}{u^2-4}$$

$$\boxed{\begin{aligned} A+B&=0 \\ 2B-2A&=1 \end{aligned}}$$

$A=-B$

$2B-2(-B)=1$

$4B=1$

$\boxed{B=\frac{1}{4}}$

$$\frac{1}{u^2-4} = \frac{1}{4}\cdot\frac{1}{u-2} - \frac{1}{4}\cdot\frac{1}{u+2}$$

$$= \frac{1}{4}\int\left(\frac{1}{u-2} - \frac{1}{u+2}\right)du =$$

$$\frac{1}{4}\left(\ln|u-2| - \ln|u+2|\right) + C$$

$$= \frac{1}{4}\left(\ln|e^x-2| - \ln|e^x+2|\right) + C$$

4.(10 pts) Write out the form of the partial fraction decomposition. Do not evaluate the coefficients:

$$\frac{x}{(x^2-2x+1)(x^4-1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1} + \frac{Ex+F}{x^2+1}$$

$(x^2-2x+1)(x^4-1)$

$= (x-1)^2(x^2+1)(x^2-1)$

$= (x-1)^2(x^2+1)(x+1)(x-1)$

$= (x-1)^3(x+1)(x^2+1)$

5.(10 pts) Evaluate

$$\int \sqrt{x}e^{\sqrt{x}}dx$$

$\sqrt{x} = u$, $\quad du = \frac{1}{2\sqrt{x}}dx = \frac{1}{2u}dx$

$$\Downarrow$$

$$\boxed{dx = 2u\,du}$$

$$= \int ue^{u}\cdot 2u\,du = 2\int u^2e^u du = 2\left(e^u u^2 - \int 2ue^u du\right) =$$

$$= 2e^u u^2 - 4\int ue^u du = 2e^u u^2 - 4\left(e^u u - \int e^u du\right) =$$

$$= 2u^2e^u - 4e^u u + 4e^u + C$$

$$= 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$$

6.(10 pts) Evaluate

$$\int x \arctan x \, dx$$

$$= \frac{1}{2}x^2\arctan x - \int \frac{1}{2}x^2 \cdot \frac{1}{x^2+1}dx =$$

$$= \boxed{\frac{1}{2}x^2\arctan x - \frac{1}{2}\int \frac{x^2}{x^2+1}dx}$$

$$\int \frac{x^2}{x^2+1}dx = \int \frac{(x^2+1)-1}{x^2+1}dx = \int\left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right)dx$$

$$= \int\left(1 - \frac{1}{x^2+1}\right)dx = x - \arctan x + C$$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}(x - \arctan x) + C$$

310

7.(10 pts) Evaluate

$$\int x \sin x \cos x \, dx$$

Using $\sin 2x = 2\sin x \cos x$

$$= \frac{1}{2}\int x \sin 2x\, dx =$$

$$= \frac{1}{2}\left(-\frac{\cos 2x}{2} x - \int \left(-\frac{\cos 2x}{2}\right) dx \right) =$$

$$= -\frac{1}{4} x \cos 2x + \frac{1}{4}\int \cos 2x \, dx =$$

$$= -\frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + C$$

OR: $$\int \underbrace{x}_{u} \underbrace{\sin x \cos x\, dx}_{dv} = \left(\frac{1}{2}\sin^2 x\right) x - \int \frac{1}{2}\sin^2 x\, dx$$

$$v = \frac{1}{2}\sin^2 x$$

$$= \frac{1}{2} x \sin^2 x - \frac{1}{2}\int \frac{1}{2}(1 - \cos 2x)\, dx =$$

$$= \frac{1}{2} x \sin^2 x - \frac{1}{4} x + \frac{1}{8}\sin 2x + C$$

(311)

8.(10 pts) (a) Find the exact value of the Simpson's Rule approximation S_2 for $\int_0^{\pi/2} \sin x \, dx$.
Your value should include quantities like π, $\sqrt{2}$, $\sqrt{3}$, etc., no decimal approximations.

S_2 : $n=2$, $\Delta x = \dfrac{\frac{\pi}{2} - 0}{2} = \dfrac{\pi}{4}$

$S_2 = \dfrac{\Delta x}{3}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right)\right]$

$= \dfrac{\pi}{12}\left(0 + 4\sin\frac{\pi}{4} + \sin\frac{\pi}{2}\right) =$

$= \dfrac{\pi}{12}\left(4\frac{\sqrt{2}}{2} + 1\right) = \underline{\underline{\dfrac{\pi}{12}\left(2\sqrt{2} + 1\right)}}$

(b) Use the fact that $\left(\frac{\pi}{2}\right)^5 < 10$ to decide if the approximation S_2 in part (a) is accurate to within $\frac{1}{200}$.

COMMENT. All required calculations can be readily performed without the use of a calculator.

$|E_S| \le \dfrac{K(b-a)^5}{180 n^4}$; $K = \max|f''''(x)|$

$f''''(x) = (\sin x)'''' = \sin x$, $\max|\sin x| = 1$;

Hence we can choose $K = 1$, and, again, $n = 2$;

$|E_S| \le \dfrac{\left(\frac{\pi}{2}\right)^5}{180(2^4)} < \dfrac{10}{180(16)} = \dfrac{1}{18(16)} = \dfrac{1}{288}$,

and $\dfrac{1}{288} < \dfrac{1}{200}$

Hence S_2 is accurate to within $\frac{1}{200}$

(Since it is accurate even to within $\frac{1}{288}$.)