Section 11.1.

Sequences are functions whose values are defined only for positive integers. Thus for any functions $f(x)$ defined for all $x \geq 0$, or for all $x \geq 1$, we can define the corresponding sequence

$a_n = f(n)$, for $n \geq 1$,

or $a_n = f(n)$, for $n \geq 0$,

or $a_n = f(n)$, for $n \geq 3$, etc.

Typically, in the book, unless otherwise stated, the sequence starts with $n=1$, although it can start with $n=0$, $n=2$, or simply any positive integer.

Thus if $f(x) = \dfrac{1}{x^2+2x+3}$, the corresponding sequence, for $n \geq 1$, is

$a_n = \dfrac{1}{n^2+2n+3}$. Of course we can also have the sequence $\dfrac{1}{n^2+2n+3}$, $n \geq 0$,

$$\frac{1}{n^2+2n+3}, \quad n \geq 5.$$

For $f(x) = \dfrac{2x^3 - x + 5}{5x^3 + x^2 - 1}$, we obtain

the sequence $a_n = \dfrac{2n^3 - n + 5}{5n^3 + n^2 - 1}$.

For $f(x) = \dfrac{e^{2x} + 3e^{x} - e^{-x}}{5e^{2x} - 2e^{-x}}$,

we obtain the sequence

$$a_n = \frac{e^{2n} + 3e^{n} - e^{-n}}{5e^{2n} - 2e^{-n}}$$

etc.

There are also sequences which do not correspond to easily defined functions, for example

$$a_n = \frac{(-1)^n n^2}{3n^2+5}$$

or $b_n = \dfrac{n!}{e^n}$;

One can use letters a_n, b_n, c_n etc.

For more introductory comments about sequences read the section 11.1 in the Book.

Limits of Sequences.

this is similar to limits of functions at infinity as studied in Section 2.6 in first semester Calculus:

Example. Find (or Evaluate) $\lim_{n\to\infty} \frac{3n^2-n-2}{5n^2+4n+1}$

Similar to Example 3, section 2.6, p. 133, which is

Find (Evaluate) $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$:

In section 2.6 it is done as follows:

$$\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}-\frac{x}{x^2}+\frac{2}{x^2}}{\frac{5x^2}{x^2}+\frac{4x}{x^2}+\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3 - \overbrace{\frac{1}{x}}^{\to 0} + \overbrace{\frac{2}{x^2}}^{\to 0}}{5 + \underbrace{\frac{4}{x}}_{\to 0} + \underbrace{\frac{1}{x^2}}_{\to 0}} = \frac{3}{5}$$

We proceed similarly with

$$\lim_{n\to\infty} \frac{3n^2-n-2}{5n^2+4n+1} = \lim_{n\to\infty} \frac{\frac{3n^2}{n^2}-\frac{n}{n^2}-\frac{2}{n^2}}{\frac{5n^2}{n^2}+\frac{4n}{n^2}+\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{3-\overbrace{\frac{1}{n}}^{\to 0}-\overbrace{\frac{2}{n^2}}^{\to 0}}{5+\underbrace{\frac{4}{n}}_{\to 0}+\underbrace{\frac{1}{n^2}}_{\to 0}} = \underline{\frac{3}{5}}$$

So one divides, in both the numerator and the denominator, by the highest power of n, which increases most rapidly as $n \to \infty$. *

<u>Limit equals ∞, or $-\infty$.</u>

Similar to functions.

<u>Example.</u> Find $\lim\limits_{n\to\infty} \dfrac{3n^2+5}{2n+1}$

$$\lim_{n\to\infty} \frac{3n^2+5}{2n+1} = \lim_{n\to\infty} \frac{\frac{3n^2}{n}+\frac{5}{n}}{\frac{2n}{n}+\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\overbrace{3n}^{\to\infty}+\overbrace{\frac{5}{n}}^{\to 0}}{2+\underbrace{\frac{1}{n}}_{\to 0}} = \infty$$ *

Example. Find $\displaystyle\lim_{n\to\infty} \frac{-5e^{2n}+e^{-n}}{e^{n}-e^{-n}}$

Solution.

$$\lim_{n\to\infty} \frac{-5e^{2n}+e^{-n}}{e^{n}-e^{-n}} = \lim_{n\to\infty} \frac{-5\frac{e^{2n}}{e^{n}}+\frac{e^{-n}}{e^{n}}}{\frac{e^{n}}{e^{n}}-\frac{e^{-n}}{e^{n}}} =$$

$$= \lim_{n\to\infty} \frac{-5\overbrace{e^{n}}^{\to\infty} + \overbrace{\frac{1}{e^{2n}}}^{\to 0}}{1-\underbrace{\frac{1}{e^{2n}}}_{\to 0}} = -\infty,$$

since $\displaystyle\lim_{n\to\infty}(-5e^{n}) = -\infty.$

Example. Find $\displaystyle\lim_{n\to\infty} \frac{n!}{5^{n}}$

Solution.
We can write

$$\frac{n!}{5^{n}} = \frac{1\cdot 2\cdot 3\cdot\ldots\cdot n}{5\cdot 5\cdot 5\cdot\ldots\cdot 5} =$$

$$=\left(\frac{1}{5}\cdot\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{4}{5}\cdot\frac{5}{5}\right)\cdot\left(\frac{6}{5}\cdot\frac{7}{5}\cdot\ldots\cdot\frac{n-1}{5}\right)\cdot\frac{n}{5}$$

$$=\frac{24}{125}\cdot\left(\frac{6}{5}\cdot\frac{7}{5}\cdot\ldots\cdot\frac{n-1}{5}\right)\cdot\frac{n}{5}$$

We note that

$$\left(\frac{6}{5}\cdot\frac{7}{5}\cdots\cdot\frac{n-1}{5}\right) > 1$$

for all $n > 5$, and $\frac{n}{5} \to \infty$ as $n \to \infty$.

Thus the entire expression

$$\frac{24}{125}\cdot\left(\frac{6}{5}\cdot\frac{7}{5}\cdots\cdot\frac{n-1}{5}\right)\cdot\frac{n}{5}, \text{ when } n > 5,$$

equals an expression $> \frac{24}{125}$ times an expression $\left(\frac{n}{5}\right)$ which approaches ∞. Hence the entire expression $\frac{n!}{5^n}$ approaches ∞, i.e. has limit ∞. $\divideontimes$

More Similarities with Limits of Functions:

Limit Laws: The Box at the bottom of p. 693 is similar to the Box at the top of p. 99:

For example

$$\lim (f(x)+g(x)) = \lim f(x) + \lim g(x)$$

$$\lim_{n\to\infty} (a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)} \quad \text{if } \lim g(x) \neq 0$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

etc.

Direct Relations Between Limits of Functions and Limits of Sequences.

The following Theorem is in the Box at the center of page 695 in the Book:

Theorem. Suppose that the function $f(x)$ is continuous at L (i.e. $\lim_{x \to L} f(x) = f(L)$), and

$\lim_{n \to \infty} a_n = L$.

Then $\lim_{n \to \infty} f(a_n) = f(L)$.

Example. Find $\lim_{n \to \infty} e^{\frac{1}{n}}$.

Solution. $\lim_{n \to \infty} \frac{1}{n} = 0 = L$ and e^x is cont. at $x = 0$,

hence $\lim_{n \to \infty} e^{\frac{1}{n}} = e^L = e^0 = 1$

$\ast$

Example. Find $\lim_{n\to\infty} \cos\left(\frac{3\pi n^2+2n+1}{n^2+4}\right)$.

Solution.

$$\lim_{n\to\infty} \frac{3\pi n^2+2n+1}{n^2+4} = \lim_{n\to\infty} \frac{\frac{3\pi n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}+\frac{4}{n^2}}$$

$$= \lim_{n\to\infty} \frac{3\pi+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{4}{n^2}} = 3\pi$$

The function $\cos x$ is continuous at $\underset{=L}{3\pi}$.

Hence

$$\lim_{n\to\infty} \cos\left(\frac{3\pi n^2+2n+1}{n^2+4}\right) = \cos L = \cos 3\pi = -1$$

Another connection between limits of functions and limits of sequences goes back to the Example on pages 385, 386 where we showed that the limit

$$\lim_{n\to\infty} \frac{3n^2-n-2}{5n^2+4n+1}$$

can be evaluated in a similar manner as

$$\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$$

in first semester calculus.

In fact the connection is even stronger, and it is expressed in the Theorem in the Box near the top of page 693 in the Book:

> If $\lim_{x\to\infty} f(x) = L$ and we define the sequence a_n by $a_n = f(n)$, then $\lim_{x\to\infty} a_n = L$, i.e. $\lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x)$.

This Theorem means that we did not need to calculate

$$\lim_{n\to\infty} \frac{3n^2-n-2}{5n^2+4n+1}$$

all over again, once we calculated $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1} = \frac{3}{5}$. Once we know the theorem at the bottom of the preceding page, we can just say that since $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1} = \frac{3}{5}$, it follows that $\lim_{n\to\infty} \frac{3n^2-n-2}{5n^2+4n+1} = \frac{3}{5}$ also.

However, the Theorem is quite useful in more complicated situations:

<u>Example</u>. Find $\lim_{n\to\infty} \frac{\ln(3n^2+2n+5)}{\ln(n^2+10)}$

Solution. Clearly, $\lim_{n\to\infty} \ln(3n^2+2n+5)=\infty$ and likewise $\lim_{n\to\infty} \ln(n^2+10)=\infty$

So lets consider

$$\lim_{x\to\infty} \frac{\ln(3x^2+2x+5)}{\ln(x^2+10)}$$

Both the numerator and denominator have limit ∞, as $x \to \infty$. Hence we can use L'Hospital's Rule:

$$= \lim_{x\to\infty} \frac{\frac{6x+2}{3x^2+2x+5}}{\frac{2x}{x^2+10}} = \lim_{x\to\infty} \frac{(6x+2)(x^2+10)}{2x(3x^2+2x+5)}$$

$$= \lim_{x\to\infty} \frac{6x+2}{2x} \cdot \lim_{x\to\infty} \frac{x^2+10}{3x^2+2x+5}$$

since we (should) know that if the numer. & the denom. are polynomials of the same degree, then the quotient has a finite limit

$$= \lim_{x\to\infty}\left(\frac{6x}{2x}+\frac{2}{2x}\right) \cdot \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{10}{x^2}}{\frac{3x^2}{x^2}+\frac{2x}{x^2}+\frac{5}{x^2}}$$

$$= \lim_{x\to\infty}\left(3+\frac{1}{x}\right) \cdot \lim_{x\to\infty} \frac{1+\frac{10}{x^2}}{3+\frac{2}{x}+\frac{5}{x^2}} = 3 \cdot \frac{1}{3} = 1$$

It then follows that

$$\lim_{n\to\infty} \frac{\ln(3n^2+2n+5)}{\ln(n^2+10)} = 1,$$ likewise

using the Theorem stated in the Box at the bottom of page 392

*

When you use the L'Hospital's Rule to calculate the limit of a sequence, always first change n to x, as done in the preceding Example (started at the bottom of p. 393), or in the Example 6 at the bottom of p. 694 in the Book.

*

L'Hospital's Rule does not always work. For Example, the Examples on p. 387 can not be done using L'Hospital's Rule, since after applying L'Hospital's Rule, the expression does not simplify. Similarly,

$$\lim_{n\to\infty} \frac{\sqrt{\ln(3n^2+2n+5)}}{\sqrt{\ln(n^2+10)}}$$

can not be evaluated by directly applying L'Hospital's Rule.

However,

$$= \lim_{n\to\infty} \sqrt{\frac{\ln(3n^2+2n+5)}{\ln(n^2+10)}}$$

$$= \sqrt{1} = 1$$

since $\lim_{n\to\infty} \frac{\ln(3n^2+2n+5)}{\ln(n^2+10)} = 1$

(See Example at the bottom of p. 393)

and the function $\sqrt{x}$ is cont. at $x=1$, hence we can use the Theorem on p. 390.

*

<u>Example.</u> Find
$$\lim_{n\to\infty}\left(2+\frac{1}{n^2}\right)^{\frac{1}{n}}$$

<u>Solution.</u> Here the simplest method is to write
$$\left(2+\frac{1}{n^2}\right)^{\frac{1}{n}} = e^{\frac{1}{n}\ln\left(2+\frac{1}{n^2}\right)}$$

If we can find
$$\lim_{n\to\infty}\frac{1}{n}\ln\left(2+\frac{1}{n^2}\right)$$
then we can again use the Theorem on p. 390 with the function $f(x)=e^x$, exactly as in the example at the bottom of p. 390:

However, to calculate
$$\lim_{n\to\infty}\frac{1}{n}\ln\left(2+\frac{1}{n^2}\right),$$
we apply $\lim\limits_{n\to\infty} a_n b_n = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$

since $\lim\limits_{n\to\infty}\frac{1}{n}=0$

and $\lim\limits_{n\to\infty} \ln\left(2+\frac{1}{n^2}\right)$

can be calculated using the Theorem at the top of p. 390:

$$\lim_{n\to\infty}\left(2+\frac{1}{n^2}\right) = 2$$

and the function $\ln x$ is continuous at $x=2$, hence by the Thm. at the top of p. 390,

$$\lim_{n\to\infty} \ln\left(2+\frac{1}{n^2}\right) = \ln 2 .$$

Hence

$$\lim_{n\to\infty} \frac{1}{n}\ln\left(2+\frac{1}{n^2}\right) = \left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim \ln\left(2+\frac{1}{n}\right)\right)$$

$$= 0\,(\ln(2)) = 0.$$

Hence $\lim\limits_{n\to\infty}\left(2+\frac{1}{n^2}\right)^{\frac{1}{n}} = \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln\left(2+\frac{1}{n^2}\right)}$

$$= e^0 = 1.$$

A sequence is called CONVERGENT if $\lim_{n\to\infty} a_n = L$ for some real number L, i.e. L has to be a finite real number.

Otherwise a_n is called DIVERGENT. In particular, if $\lim_{n\to\infty} a_n = \infty$, or $\lim_{n\to\infty} a_n = -\infty$, then a_n is DIVERGENT.

An Important Special Kind of Sequence.

Example 11, p. 696 in the Book:

The sequence r^n, where r is real number:

$r = 1$: $\lim_{n\to\infty} r^n = \lim_{n\to\infty} 1^n = 1$

$r = -1$: $(-1)^n$ alternates between

values $-1, 1$, namely
$(-1)^n = 1$ if n is even, $(-1)^n = -1$
when n is odd.

Hence the sequence $(-1)^n$ is
DIVERGENT — its terms can not become arbitrarily close to any finite number.

When $|r| < 1$, i.e. $-1 < r < 1$,
then $\lim\limits_{n\to\infty} r^n = 0$ —
e.g. $\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$.

If $r > 1$, then $\lim\limits_{n\to\infty} r^n = \infty$
e.g. $\lim\limits_{n\to\infty} (1.1)^n = \infty$, hence r^n DIVERG.

If $r < -1$, then, as n becomes large, r^n alternates between large positive and large negative values, hence r^n is DIVERGENT.

The following Theorem is near the top of p. 694 in the Book:

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

<u>Example.</u> Find $\lim_{n\to\infty} \frac{(-1)^n}{n}$.

<u>Solution.</u> $\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$ since if n is even $\frac{(-1)^n}{n} = \frac{1}{n} > 0$,

and if n is odd, then $\frac{(-1)^n}{n} = -\frac{1}{n}$

and $\left|-\frac{1}{n}\right| = \frac{1}{n}$ for n positive.

But $\lim_{n\to\infty} \frac{1}{n} = 0$. Hence by the Theorem quoted at the top of this page, $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ also. $\times$

Finally we have some important additional concepts:

A sequence a_n is <u>increasing</u> if $a_n < a_{n+1}$ for all $n \geq 1$

This is similar to a function $f(x)$ being increasing.

Likewise,

A sequence a_n is <u>decreasing</u> if $a_n > a_{n+1}$ for all $n \geq 1$

There is also the notion of monotonically increasing/decreasing which is obtained if we replace $<$ / $>$ in the above definitions by $\leq$ / $\geq$.

A sequence is <u>monotonic</u> if it is either increasing, or decreasing, or monotonically increasing or monot. decreasing.

A sequence a_n is <u>bounded above</u> if there is a number M such that $a_n \le M$ for all $n \geq 1$.

A sequence a_n is <u>bounded below</u> if there is a number m such that $a_n \geq m$ for all $n \geq 1$.

A sequence a_n is <u>bounded</u> if it is bounded both below and above.

Now there is an Important Fact, called the <u>Monotonic Sequence Theorem</u>, that involves these concepts:

Monotonic Sequence Theorem.

Every bounded monotonic sequence is convergent.

Example.

Let $a_n = \sum_{k=1}^{n} \frac{1}{2^k} \cdot \frac{k}{k+1}$

i.e. $a_1 = \frac{1}{2} \cdot \frac{1}{1+1} = \frac{1}{4}$

$$a_2 = \frac{1}{2} \cdot \frac{1}{1+1} + \frac{1}{2^2} \cdot \frac{2}{2+1} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$$a_3 = \frac{1}{2} \cdot \frac{1}{1+1} + \frac{1}{2^2} \cdot \frac{2}{2+1} + \frac{1}{2^3} \cdot \frac{3}{3+1},$$

$$\vdots$$

$$a_n = \frac{1}{2} \cdot \frac{1}{1+1} + \frac{1}{2^2} \cdot \frac{2}{2+1} + \frac{1}{2^3} \cdot \frac{3}{3+1} + \cdots + \frac{1}{2^n} \cdot \frac{n}{n+1}$$

So $a_n = \sum_{k=1}^{n} \frac{1}{2^k} \cdot \frac{k}{k+1}$

and also $a_{n+1} = a_n + \frac{1}{2^{n+1}} \cdot \frac{n+1}{n+2}$

Then, for all $n \geq 1$,

$$a_n < \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right)$$

$$= \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n} < 1$$

Thus a_n is an increasing sequence and thus $a_n \geq a_1$ for all $n \geq 1$, i.e. $a_n \geq \frac{1}{4}$ for all $n \geq 1$, hence a_n is bounded below.

But a_n is also bounded above since $a_n < 1$ for all $n \geq 1$.

Thus a_n is <u>bounded</u>, being bounded above, as well as below.

Since a_n is increasing, it <u>is monotonic</u>.

Thus by the Monotonic Sequence Theorem, <u>a_n is convergent</u>, i.e. $\lim_{n \to \infty} a_n = L$ for some (<u>finite</u>) real number L.

Moreover $L \leq 1$ since all

numbers a_n are < 1, and a_n get arbitrarily close to L as n becomes very large. Similarly $L \geq \frac{1}{4}$ — in fact $L > a_n$ for every $n \geq 1$ since the sequence a_n is increasing.

However calculating the limit L is not easy, or not clear how to do.