## Section 11.3: The Integral Test.

The Integral Test is stated in the Box on page 716 in the Book. There is also a Note following the statement in the Box which makes the Test more applicable. So I will state it in the stronger form, incorporating the Note.

HW due the week starting Mon. 3/31: sect. 11.2, 11.3

**The Integral Test.**

Suppose $f$ is a continuous, positive function on $[n_0, \infty)$ where $n_0$ is some positive integer, and suppose that $f$ is decreasing on $[N, \infty)$ for some $N \geq n_0$.

(i) If $\int_{n_0}^{\infty} f(x)dx$ is convergent, then the series $\sum_{n=n_0}^{\infty} f(n)$ is convergent.

(ii) If $\int_{n_0}^{\infty} f(x)\,dx$ is divergent, then the series $\sum_{n=n_0}^{\infty} f(n)$ is divergent.

An easy application is the p-test for series, which is stated in the Box on p. 717:

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p \leq 1$.

This fact follows immediately from the Integral Test: We consider the function $\frac{1}{x^p}$ defined, continuous, $>0$, and decreasing on $[1, \infty)$.

Since $\int_1^{\infty} \frac{1}{x^p}\,dx$ is convergent for $p>1$, it follows that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p>1$ (using (i) on p. 443). Similarly, for $p \leq 1$, we conclude that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent (using (ii)).

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Example. Determine whether the series $\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n^2}$ converges or diverges.

Solution. We consider the function $\frac{(\ln x)^2}{x^2}$ which is defined and continuous on $[2,\infty)$, and also $>0$ on $[2,\infty)$.

To check if it is decreasing we calculate its derivative

$$\left(\frac{(\ln x)^2}{x^2}\right)' = \frac{2(\ln x)\cdot\frac{1}{x}\cdot x^2 - 2x(\ln x)^2}{x^4}$$

$$= \frac{2(\ln x)x - 2x(\ln x)^2}{x^4} =$$

$$= \frac{2x\ln x\,(1-\ln x)}{x^4} = \frac{2(\ln x)(1-\ln x)}{x^3}$$

For a function to be decreasing, its derivative has to be $<0$, which will happen if $1-\ln x<0$, i.e. $\ln x>1$, i.e. $x>e$.

Since $3 > e$, it follows that $f(x) = \frac{(\ln x)^2}{x^2}$ is decreasing on $[3, \infty)$.

Thus $f(x)$ satisfies all the assumptions of the Integral Test stated in the part above (i), (ii).

Thus if we determine whether $\int_2^{\infty} \frac{(\ln x)^2}{x^2}\,dx$ is convergent or divergent, we can conclude the same about the series $\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n^2}$.

To determine whether $\int_2^{\infty} \frac{(\ln x)^2}{x^2}\,dx$ is convergent or divergent, we examine $\lim_{t \to \infty} \int_2^{t} \frac{(\ln x)^2}{x^2}\,dx$.

Hence we calculate the antiderivative

$$\int \frac{(\ln x)^2}{x^2}\,dx = -\frac{1}{x}(\ln x)^2 + \int \frac{1}{x}(2\ln x)\cdot\frac{1}{x}\,dx$$

$$= -\frac{1}{x}(\ln x)^2 + 2\int \frac{1}{x^2}\ln x\,dx =$$

$$= -\frac{1}{x}(\ln x)^2 + 2\left[-\frac{1}{x}\ln x + \int \frac{1}{x}\cdot\frac{1}{x}\,dx\right]$$

$$= -\frac{1}{x}(\ln x)^2 - \frac{2}{x}\ln x - \frac{2}{x} + C$$

Hence $\displaystyle\int_2^t \frac{(\ln x)^2}{x^2}\,dx$

$$= -\frac{1}{t}(\ln t)^2 - \frac{2}{t}\ln t - \frac{2}{t} + \left(\frac{1}{2}(\ln 2)^2 + \frac{2}{2}\ln 2 + \frac{2}{2}\right)$$

$$= \frac{1}{2}(\ln 2)^2 + \ln 2 + 1 - \frac{(\ln t)^2 + 2\ln t}{t} - \frac{2}{t}$$

Finally we have to calculate

$$\lim_{t\to\infty} \int_2^t \frac{(\ln x)^2}{x^2}\,dx =$$

$$\lim_{t\to\infty}\left\{\frac{1}{2}(\ln 2)^2 + \ln 2 + 1 - \frac{(\ln t)^2 + 2\ln t}{t} - \underbrace{\frac{2}{t}}_{\to 0}\right\}$$

So we need to calculate

$$\lim_{t\to\infty} \frac{(\ln t)^2 + 2\ln t}{t} \underset{\text{L'H.Rule}}{=}$$

$$= \lim_{t\to\infty} \frac{2(\ln t)\cdot\frac{1}{t} + \frac{2}{t}}{1} = \lim_{t\to\infty} \frac{2\ln t}{t} + \underbrace{\lim_{t\to\infty}\frac{2}{t}}_{=0}$$

$$\underset{\text{L'H.Rule}}{=} \lim_{t\to\infty} \frac{2\cdot\frac{1}{t}}{1} = 0$$

We thus conclude that

$$\lim_{t\to\infty} \int_2^t \frac{(\ln x)^2}{x^2}\,dx = \frac{1}{2}(\ln 2)^2 + \ln 2 + 1$$

Hence $\lim_{t\to\infty} \int_2^t \frac{(\ln x)^2}{x^2}\,dx$ exists and is finite. Thus $\int_2^\infty \frac{(\ln x)^2}{x^2}\,dx$ is convergent. Hence by part (i) on p. 443, the series

$$\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n^2} \text{ is convergent.}$$

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Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ is convergent or divergent.

Solution. The corresponding Improper Integral is $\int_1^\infty \frac{1}{x\sqrt{x^2+1}}\,dx$. Thus the function $f(x)$ is $f(x) = \frac{1}{x\sqrt{x^2+1}}$.

In order to use the Integral Test which <u>is stated in full</u> on p. 443, we have to verify the assumptions:

<u>$f(x)$ is continuous:</u> Yes: $x$, $\sqrt{x^2+1}$ are cont., product of cont. functs is continuous, and $x\sqrt{x^2+1} \neq 0$ on $[1,\infty)$, hence $\dfrac{1}{x\sqrt{x^2+1}}$ is cont. on $[1,\infty)$

Moreover $f(x) \geq 0$.

Also $f(x)$ is decreasing on $[1,\infty)$. We can see this as follows:

$x\sqrt{x^2+1}$ is increasing on $[1,\infty)$ since both $x$, $\sqrt{x^2+1}$ are increasing and product of increasing positive functions is increasing. Furthermore if $g(x)>0$ and increasing, then $\dfrac{1}{g(x)}$ is decreasing. Thus $\dfrac{1}{x\sqrt{x^2+1}}$ is decreasing on $[1,\infty)$.

Thus The Integral Test can be applied.

Thus we have to determine whether the improper integral $\int_1^{\infty} \frac{1}{x\sqrt{x^2+1}}\,dx$ is convergent or divergent.

To make an intelligent guess (which we then have to substantiate) we count the "total" powers of x in the denominator:

The power of x in $\sqrt{x^2+1} = (x^2+1)^{1/2}$ is counted as $(x^2)^{1/2} = x^1$,

hence the "total" power of x in the denominator is $x^1 \cdot x^1 = x^2$, i.e. $p=2$. Thus since the improper integral $\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent (by the p-test) we "guess" that the improper integral $\int_1^{\infty} \frac{1}{x\sqrt{x^2+1}}\,dx$ is convergent. However this is <u>not a proof</u> since the p-test applies "directly" only to integrals of the type $\int_a^\infty \frac{1}{x^p}dx$ (in our case, $\int_1^\infty \frac{1}{x^2}dx$).

So we need to use the Comparison Test. It suffices to use it in the simplest form (pages 191, 192 in the Notes).

Thus we will apply the Comparison Test to the functions $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x\sqrt{x^2+1}}$, both of which are $\geq 0$ on $[1, \infty)$ and both are contin. on $[1, \infty)$. Since we wish to establish that $\int \frac{1}{x\sqrt{x^2+1}}dx$ is convergent, we want to show that $g(x) \leq f(x)$ on $[1, \infty)$, i.e. $\frac{1}{x\sqrt{x^2+1}} \leq \frac{1}{x^2}$ on $[1, \infty)$.

Since $x\sqrt{x^2+1}$, $x^2$ are $>0$ on $[1,\infty)$, it is equivalent to showing that $x\sqrt{x^2+1} \geq x^2$, i.e. $\sqrt{x^2+1} \geq x$, i.e. $x^2+1 \geq x^2$ (squaring both sides), which is clearly true.

Thus all assumptions of the Comparison Test have been verified.

Thus since $\int_1^\infty \frac{1}{x^2}dx$ is convergent, we conclude from part (a) of the Comparison Test (on p. 191 of the Notes) that $\int_1^\infty \frac{1}{x\sqrt{x^2+1}}dx$ is convergent. Thus the Integral Test can be applied to the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ and the Improper Integral $\int_1^\infty \frac{1}{x\sqrt{x^2+1}}dx$, Concluding that the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

is convergent, using the part (i)

of the Integral Test on p. 443.

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