The Main Definitions and Theorems.

The URT for S-wffs, (or the S-URT, or the URT for $\bar{S}$).

Let α be an S-wff, i.e. α belong to $\bar{S}$ (see p. 65 in the Notes). Then exactly one of the six possibilities happens:

(1) There exists a unique (i.e. exactly one) natural number n such that α is A_n, and A_n belongs to S.

(2) There exists a unique (i.e. exactly one) S-wff β, i.e. β belonging to $\bar{S}$, such that α is $(\neg\, \beta)$. Moreover β is also the unique wff (ordinary wff.) such that α is $(\neg\, \beta)$.

(3) There exists a unique (i.e. exactly one) S-wff β, i.e. β belonging to $\bar{S}$, and a unique (i.e. exactly one) S-wff γ, i.e. γ belonging to $\bar{S}$, such that α is $(\beta \wedge \gamma)$. Moreover, β and γ are also the unique (ordinary) wffs such that α is $(\beta \wedge \gamma)$.

(4)-(6) are similar to (3), with α being $(\beta \vee \gamma)$, $(\beta \to \gamma)$, $(\beta \leftrightarrow \gamma)$ respectively.

Comment. The "ordinary/basic" URT is the particular case of the URT for S-wffs when $S = \{A_1, A_2, \cdots A_n, \cdots\}$, i.e., when S is the set of all sentence symbols.

p. 54.1, 55 in the handwritten notes, or p. 34, 35 in typed Notes numbered pages 19–37.

So to get the "basic" URT from the typed up statement at the upper part of this page, we need to set $S = \{A_1, A_2, \cdots, A_n, \cdots\}$, and change S-wff (and/or α belongs to $\bar{S}$) to just "ordinary" wff.

The S_1, S_2-UR Agreement Theorem.

Suppose that α is both an S_1-wff, as well as an S_2-wff. Then α falls under exactly the same unique case, of the six cases of the S_1-URT as for the S_2-URT. Moreover it is the same unique case, that α falls under, for the "basic" URT. Furthermore, for each of the cases (1)-(6), the following holds:

Case (1). If α falls under this unique case for all three URTs (S_1, S_2, basic), then there is a unique natural number n such that the sentence symbol A_n belongs to S_1 as well as to S_2, and α is A_n.

Case (2). If α falls under this unique Case for all three URTs, then there is a unique wff β such that β is also an S_1-wff, as well as an S_2-wff, and α is $(\neg\,\beta)$. Thus α is expressed as $(\neg\,\beta)$ when we view α as an S_1-wff, as well as when we view α as S_2-wff, as well as when we view α as an ordinary wff.

Case (3). When α falls under this unique case for all three URTs, then there exists exactly one (unique) wff β, and exactly one (unique) wff γ such that α is $(\beta \wedge \gamma)$. Moreover, β and γ are also the unique S_1-wffs, as well as the unique S_2-wffs such that α is $(\beta \wedge \gamma)$.

Cases (4)-(6), dealing with "$\vee$", "$\rightarrow$", "$\leftrightarrow$" are similar to Case (3).

Another key Theorem/Result is what we have been calling the FACT which is typed up in various handouts (also emailed) and is originally stated on p. 66 in handwritten notes:

The FACT.

Let S' be a set of sentence symbols, and α be an expression of LSL. Then the following are equivalent:

(a) α is an S'-wff, i.e. α belongs to $\overline{S'}$;

(b) α is a wff (i.e. an ordinary wff) and all sentence symbols that occur in α belong to S'.

(See the last page of this attachment)

The Truth Assignment Theorem (Theorem 12A in the Book)

For any truth assignment ν for a set S (of sentence symbols) there is a UNIQUE function $\bar{\nu} : \bar{S} \to \{F, T\}$ satisfying the conditions 0. – 5.:

0. For every sentence symbol A_i in S, the value $\bar{\nu}(A_i)$ is equal to the given value $\nu(A_i)$, i.e. $\bar{\nu}(A_i) = \nu(A_i)$.

1. $\bar{\nu}((\neg\, \alpha)) = \begin{cases} T & \text{if } \bar{\nu}(\alpha) = F \\ F & \text{if } \bar{\nu}(\alpha) = T, \end{cases}$

which has to hold for every wff α belonging to $\bar{S}$, i.e. for every S-wff.

2. $\bar{\nu}((\alpha \wedge \beta)) = \begin{cases} T & \text{if } \bar{\nu}(\alpha) = T \quad \text{and } \bar{\nu}(\beta) = T \\ F & \text{if } \bar{\nu}(\alpha) = T \quad \text{and } \bar{\nu}(\beta) = F \\ & \text{or } \bar{\nu}(\alpha) = F \quad \text{and } \bar{\nu}(\beta) = T \\ & \text{or } \bar{\nu}(\alpha) = F \quad \text{and } \bar{\nu}(\beta) = F, \end{cases}$

i.e. $\bar{\nu}((\alpha \wedge \beta)) = T$ if both $\bar{\nu}(\alpha) = T$ and $\bar{\nu}(\beta) = T$, and $\bar{\nu}((\alpha \wedge \beta)) = F$ if either one of $\bar{\nu}(\alpha)$, $\bar{\nu}(\beta) = F$, or both $\bar{\nu}(\alpha)$, $\bar{\nu}(\beta) = F$.

The condition 2., as well as the conditions 3.,4.,5. below, have to hold for every pair of wffs α, β in $\bar{S}$, i.e. for every pair α, β of S-wffs.

al 3. $\bar{\nu}((\alpha \vee \beta)) = \begin{cases} T & \text{if } \bar{\nu}(\alpha) = T \quad \text{or } \bar{\nu}(\beta) = T \\ F & \text{if } \bar{\nu}(\alpha) = F \quad \text{and } \bar{\nu}(\beta) = F \end{cases}$

4. $\bar{\nu}((\alpha \to \beta)) = \begin{cases} T & \text{if } \bar{\nu}(\alpha) = F \quad \text{or } \bar{\nu}(\beta) = T \\ F & \text{if } \bar{\nu}(\alpha) = T \quad \text{and } \bar{\nu}(\beta) = F \end{cases}$

5. $\bar{\nu}((\alpha \leftrightarrow \beta)) = \begin{cases} T & \text{if } \bar{\nu}(\alpha) = \bar{\nu}(\beta) = T, \quad \text{or } \bar{\nu}(\alpha) = \bar{\nu}(\beta) = F \\ F & \quad \text{if } \bar{\nu}(\alpha) = T \quad \text{and } \bar{\nu}(\beta) = F, \\ & \quad \text{or } \bar{\nu}(\alpha) = F \quad \text{and } \bar{\nu}(\beta) = T. \end{cases}$

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HW#13 due Fri., 3/6, is at the bottom of Next page

The following theorem is the Exercise 6 at the bottom of p.27 and top of p.28 in the Book.

The Truth Assignments Agreement Theorem

Let S_1 and S_2 be sets of sentence symbols and $\nu_1 : S_1 \to \{T, F\}$, $v_2 : S_2 \to \{T, F\}$ be truth assignments. Let

$$S = \{A_i \in S_1 \cap S_2 : \nu_1(A_i) = \nu_2(A_i)\}$$

Then for every S-wff α (i.e. α belonging to $\bar{S}$), we have

$$\bar{\nu}_1(\alpha) = \bar{\nu}_2(\alpha).$$

Proof.

First some comments: $\bar{\nu}_1 : \bar{S}_1 \to \{T, F\}$, $\bar{\nu}_2 : \bar{S}_2 \to \{T, F\}$ are as obtained in Theorem 12A.

(*) If α is in $\bar{S}$, i.e. if α is an S-wff, then $S(\alpha) \subset S \subset S_1 \cap S_2 \begin{cases} \subset S_1 \\ \subset S_2 \end{cases}$

Thus α is both an S_1-wff, as well as an S_2-wff, i.e. $\alpha \in \bar{S}_1$, and $\alpha \in \bar{S}_2$. Thus $\bar{\nu}_1(\alpha)$, $\bar{\nu}_2(\alpha)$ are both defined. If thus remains to be proved that $\bar{\nu}_1(\alpha) = \bar{\nu}_2(\alpha)$ for every such α in $\bar{S}$. We will prove this by induction on $\ell(\alpha)$, as we have done with other theorems, notably the FACT.

Base Case.

$\ell(\alpha) = 1$

α is an S-wff, so α is a wff, so $\ell(\alpha) = 1$ means α is a sentence symbol, i.e. α is A_i for a unique natural number i. But α is an S-wff, so A_i belongs to S . Thus by the Definition of S, we obtain $\nu_1(A_i) = \nu_2(A_i)$.

By the condition 0., of theorem 12A, we obtain $\bar{\nu}_1(A_i) = \nu_1(A_i)$, $\bar{\nu}_2(A_i) = \nu_2(A_i)$, hence $\bar{\nu}(A_i) = \bar{\nu}_2(A_i)$. But A_i is α, hence $\bar{\nu}_1(\alpha) = \bar{\nu}_2(\alpha)$.

Recall that $S(\alpha)$ is the set of those sentence symbols that occur in α. Thus if $S(\alpha) \subseteq S$, which in turn is contained in both S_1, S_2, then all sentence symbols occurring in α belong to both S_1 as well as S_2. Thus α is both an S_1-wff, as well as an S_2-wff. I.e., also $\alpha \in \bar{S}_1$, $\alpha \in \bar{S}_2$.

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(Induction Step) α is an S-wff, and

Induction Hyp

Suppose $\bar{\nu}_1(\beta) = \bar{\nu}_2(\beta)$, $\bar{\nu}_1(\gamma) = \bar{\nu}_2(\gamma)$, etc., for all β, γ, etc. belonging to $\bar{S}$ such that $\ell(\beta), \ell(\gamma) \leq n$.

Suppose $\ell(\alpha) = n + 1 \geq 2$ (since $\ell(\alpha) = 1$ is done in the base case). α is an S-wff, hence α is both an S_1-wff, as well as S_2-wff (see (*) preceding page). Thus by the S_1, S_2-URT (pages 91,92), one of the Cases (2)-(6) must occur. Suppose Case (2) occurs. Thus (see near top of p.92) there exists a unique β which is both an S_1-wff, as well as an S_2-wff, such that α is $(\neg\, \beta)$. But α is also an S-wff, hence β is an S-wff, and $\ell(\beta) < n$. Thus by the Induction Hyp, we obtain $\bar{\nu}_1(\beta) = \bar{\nu}_2(\beta)$. But then $\bar{\nu}_1((\neg\, \beta) = \bar{\nu}_2((\neg\, \beta))$ since $\bar{\nu}_1((\neg\, \beta)) = \begin{cases} T & \text{if } \bar{\nu}_1(\beta) = F \\ F & \text{if } \bar{\nu}_1(\beta) = T \end{cases}$; likewise $\bar{\nu}_2((\neg\beta)) = \begin{cases} T \text{ if } \bar{\nu}_2(\beta) = F \\ F \text{ if } \bar{\nu}_2(\beta) = T \end{cases}$

S_1, S_2 - UR Agreement Theorem

But $\bar{\nu}_1(\beta) = \bar{\nu}_2(\beta)$ by the Induction Hyp.

Thus if $\bar{\nu}_1(\beta) = \bar{\nu}_2(\beta) = F$, then $\bar{\nu}_1((\neg\beta)) = \bar{\nu}_2((\neg\beta)) = T$, i.e. $\bar{\nu}_1(\alpha) = \bar{\nu}_2(\alpha) = T$.

And if $\bar{\nu}_1(\beta) = \bar{\nu}_2(\beta) = T$, then $\bar{\nu}_1((\neg\beta)) = \bar{\nu}_2((\neg\beta)) = F$, i.e. $\bar{\nu}_1(\alpha) = \bar{\nu}_2(\alpha) = F$.

But either $\bar{\nu}_1(\beta) = \bar{\nu}_2(\beta) = T$, or $\bar{\nu}_1(\beta) = \bar{\nu}_2(\beta) = F$.

Thus since in each case, $\bar{\nu}_1(\alpha) = \bar{\nu}_2(\alpha)$,

the Method of Proof by Cases (we obtain that)

allows us to conclude that $\bar{\nu}_1(\alpha) = \bar{\nu}_2(\alpha)$.

Thus the Case (2) of URT (i.e. negation) is completed.

Case (3) (i.e. "$\wedge$") of the S_1, S_2 - UR Agreement Thm.

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Case (4) (i.e. "$\vee$")

Case (5) (i.e. "$\rightarrow$")

Case (6) (i.e. "$\leftrightarrow$")

ANY OF THESE CASES, including (1), (2), (3), can be on the Exam, March 20,

Part of HW# 13 also

We now define a set of wffs $\overline{\mathcal{S}'}$, i.e., $\mathcal{S}'$-wffs as at the bottom third of p. 20 in the Book, to consist of those wffs that can be obtained in the manner (a), (b), (c) at the bottom of p. 16 in the Book, but with the parts (a) (b) and (c) altered as follows;

(a$\mathcal{S}'$) Every sentence symbol in $\mathcal{S}'$ is an $\mathcal{S}'$-wff;

(b$\mathcal{S}'$) If α and β are $\mathcal{S}'$-wffs, then $(\neg\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ are $\mathcal{S}'$-wffs;

(c$\mathcal{S}'$) No expression is an $\mathcal{S}'$-wff unless it is compelled to be one by (a$\mathcal{S}'$) and (b$\mathcal{S}'$).

We then define $\overline{\mathcal{S}'}$ to consist of all $\mathcal{S}'$-wffs.

The bottom-up process of building up $\mathcal{S}'$-wffs, i.e. wffs in $\overline{\mathcal{S}'}$, from sentence symbols in $\mathcal{S}'$

So this is the same as (a), (b), (c) bottom of p. 16 in Book, but adjusted so only sentence symbols in $\mathcal{S}'$ are allowed.