# Asymptotic analysis of noise-amplified oscillations for subcritical delays

R. Kuske *†

We develop a multi-scale analysis for stochastic differential delay equations with additive noise. Such models are particularly sensitive to noise when the system is near a critical point, such as a Hopf bifurcation, which marks a transition to oscillatory behavior. In particular, we are interested in the case when the combined effects of the noise and the delay amplify oscillations which would decay in the deterministic system. The derivation of reduced equations for the envelope of the oscillations provides an efficient analysis of the dynamics by separating the influence of the noise from the intrinsic oscillations over long time scales.

## 1 Introduction

In this paper we give an asymptotic analysis of the effect of noise on bifurcations due to delays. In particular, we are interested in the case where the sensitivity to the noise is demonstrated through a resonance. In the presence of noise, oscillations are sustained in the subcritical region, where oscillations die out in the absence of noise. In the absence of external periodic forcing this phenomenon has been called autonomous stochastic resonance [1], where the noise excites the oscillations intrinsic to the deterministic dynamics. It has been observed in many systems with delays, including models of neurons, lasers, and a variety of oscillators with delayed feedback (see [2]-[6], and references therein). It has also been observed in systems without delays, where it has been studied in the context of stochastic bifurcations [7].

For concreteness, we focus on the particular system

$$dx = -\alpha x(t) + c\frac{\theta^n}{\theta^n + x^n(t-\tau)} + \delta dw(t), \tag{1.1}$$

with $w(t)$ a standard Brownian motion. This is a canonical model of delayed negative feedback which has been used in the modeling of the pupil light reflex [8] [9]. There the effects of several different types of noise were compared through numerical simulations. One essential feature of this system is that in the deterministic case there is a Hopf bifurcation at a certain critical value of the delay $\tau_c$. When there is no noise, oscillations about the steady state will decay in time when $\tau < \tau_c$, but they will not decay if $\tau > \tau_c$. This is a common behavior in models with delayed feedback based on delay differential equations [2]-[6], [10]-[13].

We focus on the subcritical case $\tau < \tau_c$ when the noise is small. In this case the dynamics are very sensitive to the noise, but the qualitative features of the stochastic dynamics are not

*Department of Mathematics, University of Minnesota, Minneapolis, MN 55455, email: rachel@math.umn.edu
†Supported in part by NSF grant DMS-0072311

dominated by the noise. Numerical simulations below show the typical effect of small noise for values of the delay near a Hopf bifurcation [7]. In the presence of small noise, oscillations with a frequency corresponding to the Hopf bifurcation of the deterministic system persist even for subcritical delays $\tau < \tau_c$. Without noise ($\delta = 0$), the amplitude of the oscillations would decay over time, and the system would approach the steady constant state. In the presence of noise, these oscillations are more prominent as the delay is increased, even though $\tau < \tau_c$ and the magnitude of the noise remains constant. Oscillations with the deterministic (natural) frequency are supported by the combination of the delay and the effect of the noise. While the deterministic frequency is the predominant feature in the oscillations, there are also small variations in the amplitude and the phase, due to the noise. These simulations show that there is an interaction of the deterministic and stochastic elements in the model.

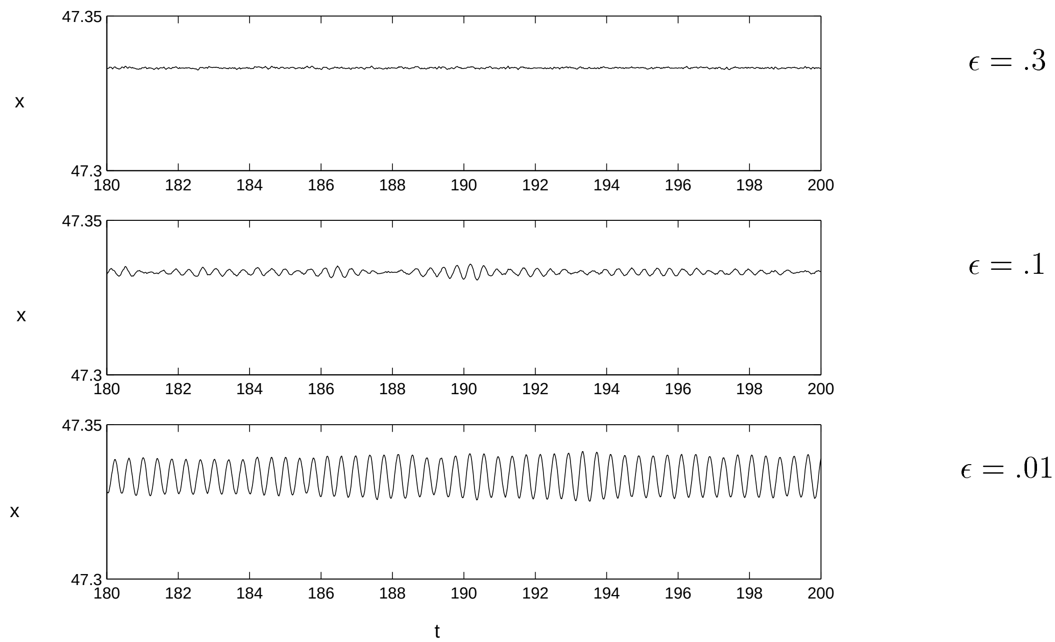



Figure 1: Simulation of equation (1.1) with $\delta = .001$, and $\tau = \tau_c - \epsilon^2$, with $\tau_c$ corresponding to Hopf bifurcation. Note that decreasing $\epsilon$ corresponds to approaching the critical delay. In the top graph the system is sufficiently far from the Hopf bifurcation point, so that the oscillations decay and the dynamics are essentially small random perturbations from the steady state. As $\epsilon$ decreases, the natural frequency of the system appears in the oscillations. For values of the delay close to the Hopf point, oscillations with the natural frequency of the deterministic system dominate the dynamics. There is also a small variation in amplitude and frequency due to the noise.

Because of the large variability over time and over a range of parameters, a systematic numerical investigation of the system can be expensive. Therefore we look for an efficient method for analyzing the model, which isolates the effects of the noise from the natural oscillations in the system.

The difficulty that arises in analyzing these effects in stochastic delay differential equations is primarily related to the presence of the delay. Using an approach based on either a forward or backward Kolmogorov equation or even weak solutions of (1.1) [14], one is inevitably faced with the memory in the system due to the delay. For example, the Fokker-Planck (forward Kolmogorov) equation for the probability density is an integro-differential equation [15]. Below we outline an approach which avoids this complication.

In this paper we use information about the deterministic dynamics to obtain a reduced model in which the influence of the noise can be separated from the oscillations related to the bifurcation. This reduced model is a stochastic equation for the envelope of the oscillations, which varies on a long time scale. This approach has several advantages. The stochastic envelope equation allows a separation of the effect of the noise on the system over long times from the oscillations with deterministic frequency. The model is efficient for simulating long time behavior, since it describes the dynamics on large time scales. The reduced models have certain features which are useful for analytical solutions, based on small delay approximations [15] or linearized approximations, which we discuss in Section 3. The phenomenon demonstrated here occurs in many systems with similar features, so this analysis can be applied to a wide variety of models. It has already been applied to a linear model and a logistic model with delays [17]. In this paper we demonstrate the method in a more general nonlinear setting.

We focus on the phenomenon of autonomous stochastic resonance, considering parameter ranges in the vicinity of the Hopf bifurcation where small noise has a pronounced effect. A reduced model for the envelope of the oscillations is obtained via a multi-scale analysis, which is a well-known technique used near bifurcations in deterministic models [18], [19]. The method is combined with a projection onto the oscillatory modes of the linearized, deterministic system, which has similarities to Hale's reduction method [13]. Similar projection ideas have been used to reduce the stochastic Allen-Cahn equation to a system of stochastic differential equations for the coupled interfaces [16]. The analysis of this paper has similarities with multi-scale analysis of modulation equations in coupled oscillators [18] or patterns dynamics [19], but there are certain adaptations necessary for stochastic models. A discussion of this method for the supercritical case ($\tau > \tau_c$) in the context of the logistic equation with delay and its the relationship of the method to a Fourier series approximation is covered in [17].

The paper is organized as follows. In Section 2 we outline the multi-scale approach for deriving stochastic envelope equations for the oscillations in the subcritical case $\tau < \tau_c$. For clarity the technical details of this derivation are moved to the Appendix. In Section 3 we compare the stationary density of the process obtained from the approximation with that of the original process described by (1.1). We discuss these results and further applications of the method in Section 4.

## 2 Derivation of the envelope equations

In this section we outline the derivation of the stochastic envelope equations which gives an asymptotic approximation to the long time behavior of the noisy system (1.1) in the vicinity of the critical delay. We focus on the case where small noise $0 < \delta \ll 1$ amplifies the oscillations even though the delay is in the subcritical range $\tau < \tau_c$. In this model the combination of parameters $c$, $\theta$, $n$ and $\tau$ all play an important role in the dynamics. We focus in this paper on varying the length of the delay $\tau$ rather than the other parameters. Then we view the Hopf point as a value $\tau_c$ which of course depends on the values of the other parameters. Similar effects can be achieved by fixing the delay and varying the other parameters [9].

## 2.1 Deterministic problem

The derivation via the multi-scale analysis uses information about the Hopf bifurcation in the deterministic problem,

$$\frac{dx}{dt} = -\alpha x(t) + c\frac{\theta^n}{\theta^n + x^n(t-\tau)}. \tag{2.1}$$

From this equation we determine the relationships between the critical delay $\tau_c$, the natural frequency of the oscillations $r$, and the other parameters, $\alpha$, $c$, $\theta$, $n$. The oscillations are oscillations about a steady state $x_0$, which satisfies

$$\alpha x_0 = c\frac{\theta^n}{\theta^n + x_0^n}. \tag{2.2}$$

It is possible to determine $\alpha$, $c$, $\theta$, and $n$ such that the steady state $x = x_0$ is stable for a given $\tau$. In order to focus on oscillations about $x = x_0$, we rewrite the equation (2.1) in terms of $y = x - x_0$. Then (2.1) becomes

$$\frac{dy}{dt} = -\alpha y(t) + c\frac{\theta^n\left(x_0^n - [x_0 + y(t-\tau)]^n\right)}{(\theta^n + [x_0 + y(t-\tau)]^n)(\theta^n + x_0^n)}. \tag{2.3}$$

Next we determine an expression for the critical delay in the deterministic problem, corresponding to the Hopf bifurcation point, which will be useful in the calculations below. We linearize (2.3) about $y = 0$,

$$\frac{dy}{dt} \approx -\alpha y(t) - \frac{cn\theta^n x_0^{n-1}}{(\theta^n + x_0^n)^2} y(t-\tau), \tag{2.4}$$

and look for oscillatory solutions,

$$y(t) = e^{(\sigma + ir)t}. \tag{2.5}$$

That is, we look for a Hopf bifurcation from the $y = 0$ solution, which gives oscillations about $x = x_0$. Then, substituting (2.5) into (2.4) and using (2.2) yields equations for $\tau = \tau_c$ which must be satisfied in order to have purely oscillatory solutions to (2.4) ($\sigma = 0$),

$$-r + \frac{\alpha n x_0^n}{\theta^n + x_0^n}\sin r\tau_c = 0 \tag{2.6}$$

$$-\alpha - \frac{\alpha n x_0^n}{\theta^n + x_0^n}\cos r\tau_c = 0. \tag{2.7}$$

These equations relate the frequency $r$ and the critical delay $\tau_c$ to the other parameters. It is a standard calculation to show that for $\tau > (<)\tau_c$, $\sigma > (<)0$.

## 2.2 Multiscale analysis of the noisy nonlinear problem

Here we use the method of multiple scales to derive stochastic equations which describe the envelope of the oscillations in (1.1). Since we are interested in oscillations about the deterministic steady state $x_0$, we rewrite (1.1) in terms of $y = x - x_0$,

$$dy = \left(-\alpha y(t) + c\frac{\theta^n\left(x_0^n - [x_0 + y(t-\tau)]^n\right)}{(\theta^n + [x_0 + y(t-\tau)]^n)(\theta^n + x_0^n)}\right)dt + \delta dw(t). \tag{2.8}$$

Now we apply the method of multi-scale analysis for delays near $\tau_c$, that is, $\tau = \tau_c + \epsilon^2\tau_2$ for $\epsilon^2 \ll 1$ in the vicinity of the Hopf bifurcation point. Here we confine our attention to the case when $\tau < \tau_c$ ($\tau_2 < 0$), so that in the absence of noise, $x \to x_0$, $y \to 0$ as $t \to \infty$. The approach is very similar to the standard method of multiple scales used to analyze Hopf bifurcations in deterministic systems [18], [19]. We make appropriate adjustments for the stochastic terms.

We begin with the expansion for $y$,

$$\begin{aligned} y &= \epsilon y_1 + \epsilon^2 y_2 + \dots \\ &= \epsilon[A(T)\cos rt + B(T)\sin rt] + \epsilon^2[C(T)\cos 2rt + D(T)\sin 2rt + E(T)] + \mathcal{O}(\epsilon^3), \quad (2.9) \\ \tau &= \tau_c + \epsilon^2\tau_2, \quad \tau_2 < 0. \quad (2.10) \end{aligned}$$

Here $T = \epsilon^2 t$ is a slow time, with the scale $\epsilon$ related to the proximity to the bifurcation point $\tau = \tau_c$. Thus the leading term in the expansion is simply an oscillatory function with a slowly varying amplitude or envelope, described by the functions $A(T)$ and $B(T)$. In order to capture the influence of the noise on the $T$ time scale, we look for a solution (2.9) which is periodic on the fast time scale $t$, with an amplitude that varies stochastically over the long time scale; that is, $A$ and $B$ are stochastic quantities on the $T$ time scale. The frequency of the oscillation is $r$, given by (2.7) with $\tau = \tau_c$. The functions $C(T)$, $D(T)$, $E(T)$ depend on $A(T)$ and $B(T)$ (A.5)-(A.6), and are determined in the analysis. In the following, we omit the argument $T$ from $A(T)$, $B(T)$, $C(T)$, $D(T)$, and $E(T)$, except when the argument is at the delayed time $T - \epsilon^2\tau$.

We derive equations for the stochastic amplitudes $A$ and $B$, assuming the following forms:

$$\begin{aligned} dA &= \psi_A dT + \sigma_A d\beta_A(T), \quad (2.11) \\ dB &= \psi_B dT + \sigma_B d\beta_B(T). \end{aligned}$$

Here $\beta_A$ and $\beta_B$ are two independent standard Brownian motions, and the coefficients $\psi_A$, $\psi_B$, $\sigma_A$, and $\sigma_B$ are unknown and may depend on $A$ and $B$. The assumed form of these equations will be shown to be consistent in the following analysis. We obtain these coefficients by deriving two equations for $dy$, first using Ito's formula, and then using the equation (1.1).

First, using Ito's formula we have

$$dy = \frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial A}dA + \frac{\partial y}{\partial B}dB + \frac{\sigma_A^2}{2}\frac{\partial^2 y}{\partial A^2}dT + \frac{\sigma_B^2}{2}\frac{\partial^2 y}{\partial B^2}dT. \quad (2.12)$$

Combining (2.9) and (2.12) gives

$$\begin{aligned} dy &= \left(\epsilon\frac{\partial y_1}{dt} + \epsilon^2\frac{\partial y_2}{dt}\right)dt + \left[\epsilon\cos rt + \epsilon^2\frac{\partial y_2}{\partial A}\right](\psi_A dT + \sigma_A d\beta_A(T)) \quad (2.13) \\ &+ \left[\epsilon\sin rt + \epsilon^2\frac{\partial y_2}{\partial B}\right](\psi_B dT + \sigma_B d\beta_B(T)) + \epsilon^2\left(\frac{\sigma_A^2}{2}\frac{\partial^2 y_2}{\partial A^2} + \frac{\sigma_B^2}{2}\frac{\partial^2 y_2}{\partial B^2}\right)dT + \mathcal{O}(\epsilon^3). \end{aligned}$$

Note that in the spirit of multi-scale analysis, we treat functions of the slow time $T = \epsilon^2 t$ as independent of $t$.

Now we compare (2.13) to a second equation for $dy$, obtained by substitution of (2.9) into the right hand side of (2.8),

$$dy = -\Big[\alpha\left(\epsilon[A\cos rt + B\sin rt] + \mathcal{O}(\epsilon^2)\right) \quad (2.14)$$

$$+ \quad \left. \frac{c\theta^n \left(x_0^n - (x_0 + \epsilon[A(T-\epsilon^2\tau)\cos r(t-\tau_c) + B(T-\epsilon^2\tau)\sin r(t-\tau_c)] + \mathcal{O}(\epsilon^2))^n\right)}{(\theta^n + (x_0 + \epsilon[A(T-\epsilon^2\tau)\cos r(t-\tau_c) + B(T-\epsilon^2\tau)\sin r(t-\tau_c)] + \mathcal{O}(\epsilon^2))^n)(\theta^n + x_0^n)} \right] dt$$
$$+\delta dw.$$

In the absence of noise ($\delta = 0$), a multi-scale analysis would proceed as follows. Using a perturbation expansion of $y$ for $\epsilon \ll 1$ in (2.14), and setting the coefficients of like powers of $\epsilon$ equal to zero, yields a sequence of equations,

$$\mathcal{O}(\epsilon): \qquad y_1'(t) + \alpha y_1(t) + \frac{cn\theta^n x_0^{n-1}}{(\theta^n + x_0^n)^2} y_1(t-\tau_c) = 0, \tag{2.15}$$

$$\mathcal{O}(\epsilon^j): \qquad y_j'(t) + \alpha y_j(t) + \frac{cn\theta^n x_0^{n-1}}{(\theta^n + x_0^n)^2} y_j(t-\tau_c) = R(y_1, y_2, \dots y_{j-1}) \qquad \text{for } j > 1. \tag{2.16}$$

Equation (2.15) is simply the linear equation (2.4) which has the periodic solution $A\cos rt$ and $B\sin rt$. Allowing for a slowly varying modulation or envelope, we take $A(T)$ and $B(T)$ as functions of the slow time $T$. These functions are undetermined from (2.15), since $T$ is treated as independent of $t$. The left hand side of (2.16) is the same linear operator as in (2.15) acting on $y_j$, and the right hand side depends on lower order terms $y_k$ for $k < j$ in the expansion for $y$. In particular, it includes terms which involve $A(T)$ and $B(T)$, as well as $\cos mrt$ and $\sin mrt$ for $m$ an integer. In order for (2.16) to have a solution, the right hand side must be orthogonal to the homogeneous solutions $\cos rt$ and $\sin rt$. Imposing these solvability conditions while treating the functions of the slow time $A(T)$ and $B(T)$ as independent of $t$, we obtain equations for $A(T)$ and $B(T)$. That is, $A(T)$ and $B(T)$ satisfy the equation which results from projecting the right hand side of (2.16) onto $\cos rt$ and $\sin rt$, while treating $T$ as independent of $t$,

$$\int_0^{2\pi/r} R(y_1, y_2, \dots y_{j-1}) \left\{ \begin{array}{c} \cos rt \\ \sin rt \end{array} \right\} dt = 0. \tag{2.17}$$

The difference in the analysis of this paper is that we must also treat the noise. Then the procedure is as follows. We use a perturbation expansion for $\epsilon \ll 1$ for the drift in (2.14). Then we equate the two equations for $dy$, (2.13) and the expansion of (2.14). We equate separately the drift and diffusion coefficients in order to obtain the drift and diffusion coefficients in (2.11). As in the deterministic analysis, the projection onto $\cos rt$ and $\sin rt$ is used to determine these terms in the equations for $A$ and $B$.

To treat the diffusion terms, we first rewrite the noise in (2.14),

$$\begin{aligned} \delta dw(t) &= \cos rt dw_1(t) + \sin rt dw_2(t) \\ &= \frac{\delta}{\epsilon}(\sin rt\, dw_1(\epsilon^2 t) + \cos rt\, dw_2(\epsilon^2 t)) \\ &= \frac{\delta}{\epsilon}\cos rt\, dw_1(T) + \frac{\delta}{\epsilon}\sin rt\, dw_2(T), \end{aligned} \tag{2.18}$$

using well-known properties of white noise [20], with $w_1$ and $w_2$ two independent standard Brownian motions. The diffusion terms in (2.13) are

$$\left[\epsilon\cos rt + \mathcal{O}(\epsilon^2)\right]\sigma_A d\beta_A(T) + \left[\epsilon\sin rt + \mathcal{O}(\epsilon^2)\right]\sigma_B d\beta_B(T). \tag{2.19}$$

Then we neglect the $\mathcal{O}(\epsilon^2)$ terms and equate the noise terms from both equations (2.14) and (2.13) for $dy$, as given in (2.18) and (2.19), to get the leading order expressions for $\sigma_A$ and $\sigma_B$,

$$\epsilon \cos rt \sigma_A d\beta_A(T) + \epsilon \sin rt \sigma_B d\beta_B(T) \sim \frac{\delta}{\epsilon}(\cos rt \, dw_1(T) + \sin rt \, dw_2(T)). \tag{2.20}$$

Now we treat this equation from the viewpoint of the method of multiple scales, where the two time scales $t$ and $T$ are viewed as independent, and we use the the orthogonality of the modes $\sin jrt$ and $\cos jrt$. Here we use the inner products

$$\int_0^{2\pi/r} \left[\cos rt \left(\epsilon\sigma_A d\beta_A(T) - \frac{\delta}{\epsilon} dw_1(T)\right) + \sin rt \left(\epsilon\sigma_B d\beta_B(T) - \frac{\delta}{\epsilon} dw_2(T)\right)\right] \left\{ \begin{array}{c} \cos rt \\ \sin rt \end{array} \right\} dt = 0, \tag{2.21}$$

and integrate with respect to the fast time scale, treating the terms which depend on the slow time scale $T$ as independent of $t$ in this integration. Thus we identify $\beta_A(T) = dw_1(T)$ and $\beta_B(T) = dw_2(T)$, which then yields

$$\sigma_A = \sigma_B = \frac{\delta}{\epsilon^2}. \tag{2.22}$$

For the drift, essentially we follow the procedure outlined for the deterministic case in (2.15)-(2.17) after equating the drift terms in (2.14) and (2.13). We collect terms of like powers of $\epsilon$ and set them equal to zero. In the Appendix we show that there is cancellation up to $\mathcal{O}(\epsilon^3)$ in the drift for (2.14), using the definitions of $r$ and $\tau_c$, and the coefficients $C$, $D$, and $E$ (A.5)-(A.6). Then we take the inner product of the $\mathcal{O}(\epsilon^3)$ terms with $\cos rt$ and $\sin rt$, which yields two equations which involve $\psi_A$, $\psi_B$ in terms of $A$ and $B$ and the other parameters in the problem. The details of the analysis appear in the Appendix. Then the equations are

$$\begin{aligned}
\psi_A &= b_1 \left[ -\frac{B(T-\epsilon^2\tau) - B(T)}{\epsilon^2} \sin r\tau_c + \frac{A(T-\epsilon^2\tau) - A(T)}{\epsilon^2} \cos r\tau_c \right. \\
&\quad \left. - r\tau_2 \Big( B(T)\cos r\tau_c + A(T) \sin r\tau_c \Big) \right] \\
&\quad + \frac{b_2}{2} \Big[ (AC+DB)\cos r\tau_c + (BC-AD)\sin r\tau_c + EA\cos r\tau_c - EB \sin r\tau_c \Big] \\
&\quad + \frac{b_3(A^2+B^2)}{8} \Big[ A\cos r\tau_c - B \sin r\tau_c \Big] + \mathcal{O}(\epsilon) \qquad (2.23) \\
\psi_B &= b_1 \left[ \frac{B(T-\epsilon^2\tau) - B(T)}{\epsilon^2} \cos r\tau_c + \frac{A(T-\epsilon^2\tau) - A(T)}{\epsilon^2} \sin r\tau_c \right. \\
&\quad \left. - r\tau_2 \Big( B(T)\sin r\tau_c - A(T) \cos r\tau_c \Big) \right] \\
&\quad + \frac{b_2}{2} \Big[ (AC+DB)\sin r\tau_c - (BC-AD)\cos r\tau_c + EA\sin r\tau_c + EB \cos r\tau_c \Big] \\
&\quad + \frac{b_3(A^2+B^2)}{8} \Big[ A\sin r\tau_c + B \cos r\tau_c \Big] + \mathcal{O}(\epsilon) \qquad (2.24)
\end{aligned}$$

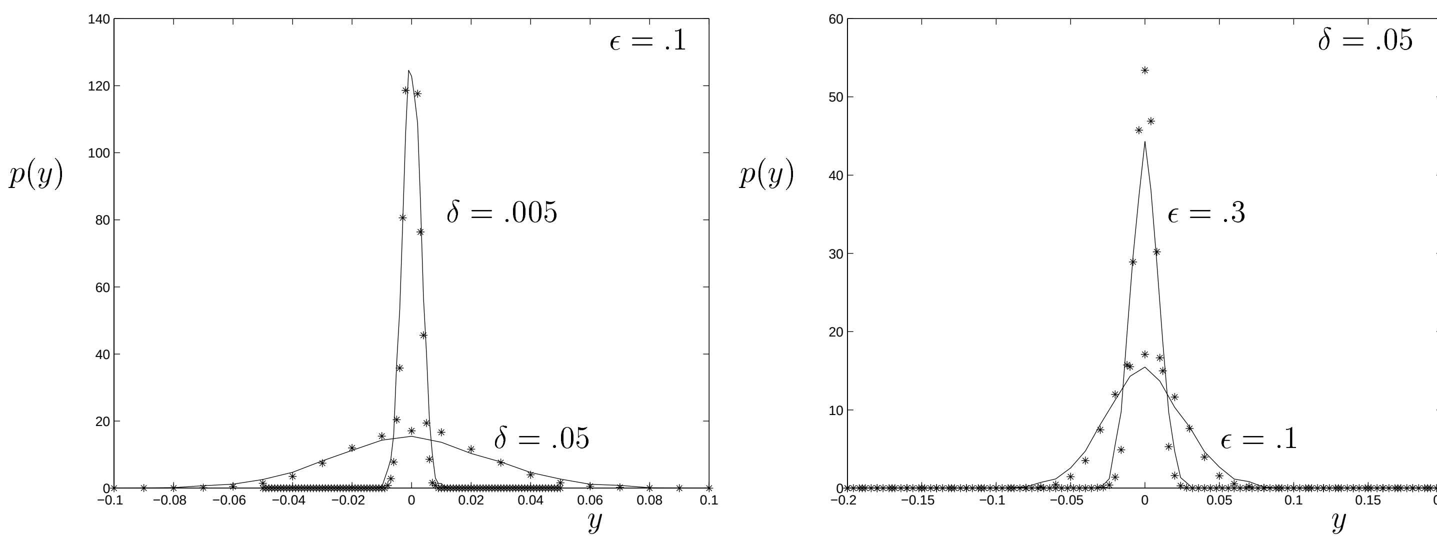
Neglecting the $\mathcal{O}(\epsilon)$ terms in these expressions, we combine them with (2.22) to get the coefficients in (2.11).

# 3 Comparison of the original model with the multi-scale approximation

In the following figures we illustrate the performance of the multi-scale approximation (2.9) for $y = x - x_0$ by comparison with the solution to the original model (1.1). The choice of parameter values is $n = 21$, $\theta = 50$, $c = 200$, $\alpha = 3.21$. With these parameter values, $x_0 \approx 47.332$, $r \approx 15.8775$ and $\tau_c \approx .1115$. This choice of parameter values is similar to that used in [9]. While it may appear that the critical delay $\tau_c$ is small in this case, in fact, $\tau_c$ is not small relative to the period of the oscillations. That is, using a rescaled time $\tilde{t} = t/\tau_c$, we get a critical delay of $\tilde{\tau_c} = 1$, and a natural frequency of $\tilde{r} \approx 1.7703$.

We compare results from realizations for times sufficiently long so that the process follows the stationary density. From 5000 realizations of (1.1) we approximate the stationary density of $y = x - x_0$, and from 5000 realizations of (2.11) we approximate the stationary density using the multiscale approximation $y \sim A\cos rt + B\sin rt$, for large $t$. These approximations of the stationary density of $y$ are compared in the following graphs. Initial conditions are such that the process starts near $x = x_0$, or $A = B = 0$.

In Figure 2a, we compare the behavior for $\epsilon = .1$ with $\delta = .005$ and $\delta = .05$. As expected, as the noise increases, the variance of the process increases and the concentration of the density decreases. In Figure 2b we compare the behavior for $\epsilon = .1$ and $\epsilon = .3$ for $\delta = .05$. As $\tau$ approaches $\tau_c$, $\epsilon$ decreases, and the oscillations increase as the noise causes a resonance. Then the variance of the process increases.




(a) (b)

Figure 2: Comparison of the densities obtained from (2.9) ($*$'s) and (1.1) (solid lines) for (a) $\epsilon = .1$, and $\delta = .005, .05$ ($*$'s) and (b) $\epsilon = .1, .3$, and $\delta = .05$ ($*$'s). Note that the variance of the process increases as the noise increases, and as the proximity to the critical delay decreases.

In both cases in Figure 2b), $\delta/\epsilon^2$, the diffusion coefficient indicating the effective magnitude of the noise in the envelope equations (2.11), is not small. Figure 3 provides a numerical exploration to test the range of validity of the multi-scale approximation. It gives a good approximation if $\delta/\epsilon \leq 1$, but it gives a variance which is too large if $\delta/\epsilon$ exceeds unity. In order to get reasonable results from these extreme values of $\epsilon$ and $\delta$, we rewrite $\tau = \tau_c - \epsilon^2\tau_2$, as $\tau_c - \hat{\epsilon}^2\hat{\tau}_2$, choosing $\hat{\epsilon}$ so that $\delta/\hat{\epsilon} < 1$. Under such a rescaling, there is a balance between drift and diffusion in the envelope equations, so that the multi-scale approximation gives a good approximation.

The fact that the asymptotic analysis provides a good approximation for a large range of parameters is due to the dissipative behavior of the drift terms in the equations for $A$ and $B$. Then, for values of $\delta$ and $\epsilon$ for which the effective magnitude noise in (2.11) is not too large, variations away from $A = B = 0$ do not dominate the dynamics since the damping in the drift drives the system towards zero amplitude variation. In Figure 4 we demonstrate that for $\tau < \tau_c$, the linear approximation to (2.11) gives a good approximation to the behavior of $A$ and $B$, and we discuss the asymptotic validity in the context of that approximation.

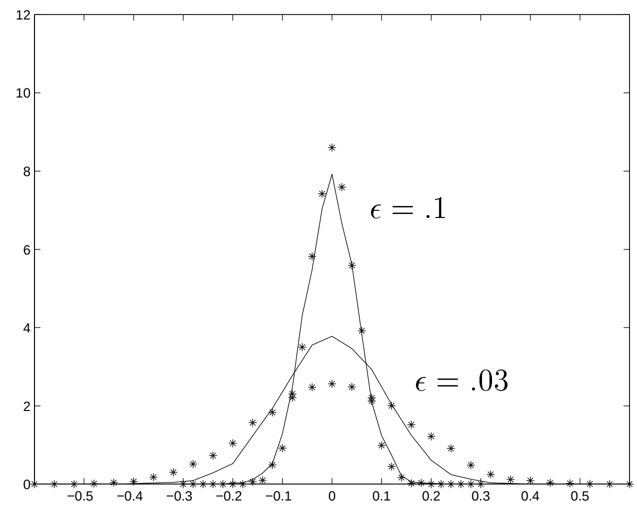



Figure 3: Comparison of the densities obtained from (2.9) ($*$'s) and (1.1) (solid lines) for $\delta = .1$ and $\epsilon = .1$ and $\epsilon = .03$, values for which one might expect that the asymptotic approximation would break down. For $\epsilon = .1$, the multi-scale approximation gives a good approximation, but as $\epsilon$ decreases the multi-scale approximation breaks down, as it describes a process with a variance which is too large.

Neglecting the nonlinear terms in the equation for $A$ and $B$,

$$dA \approx b_1 \left[ -\frac{B(T-\epsilon^2\tau) - B(T)}{\epsilon^2} \sin r\tau_c + \frac{A(T-\epsilon^2\tau) - A(T)}{\epsilon^2} \cos r\tau_c - r\tau_2 \Big( B(T)\cos r\tau_c + A(T)\sin r\tau_c \Big) \right] dT + \frac{\delta}{\epsilon^2} d\beta_1(T), \tag{3.1}$$

$$dB \approx b_1 \left[ \frac{B(T-\epsilon^2\tau) - B(T)}{\epsilon^2} \cos r\tau_c + \frac{A(T-\epsilon^2\tau) - A(T)}{\epsilon^2} \sin r\tau_c - r\tau_2 \Big( B(T)\sin r\tau_c - A(T)\cos r\tau_c \Big) \right] dT + \frac{\delta}{\epsilon^2} d\beta_2(T). \tag{3.2}$$

In Figure 4 simulations using the linearized approximations to the envelope equations (3.1)-(3.2) are compared with simulations using the original system (1.1). The good agreement demonstrates that the linear terms in $\psi_A$ and $\psi_B$ dominate the dynamics of the slow envelope described by $A$ and $B$.

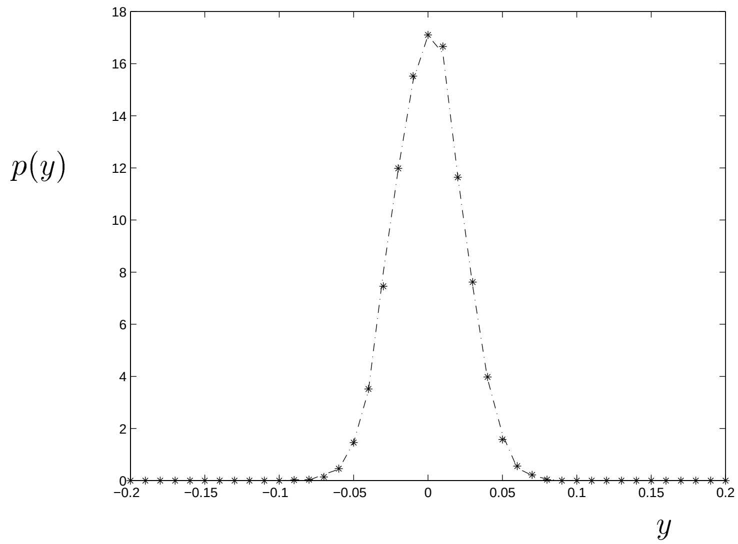



Figure 4: Comparison of the densities obtained for $y = A\cos rt + B\sin rt$, with $A$ and $B$ given by the linear approximation (3.1)-(3.2) (dashdotted), and the nonlinear equations (2.11) (*'s), for $\delta = .05$, $\epsilon = .1$

From the linearized equations we can see how the ratio $\delta/\epsilon < 1$ naturally occurs as a criterion for the asymptotic validity of the multi-scale approximation. Replacing $\epsilon A = a$ and $\epsilon B = b$ in the linearized approximation (3.1)-(3.2) results in equations for $a$ and $b$

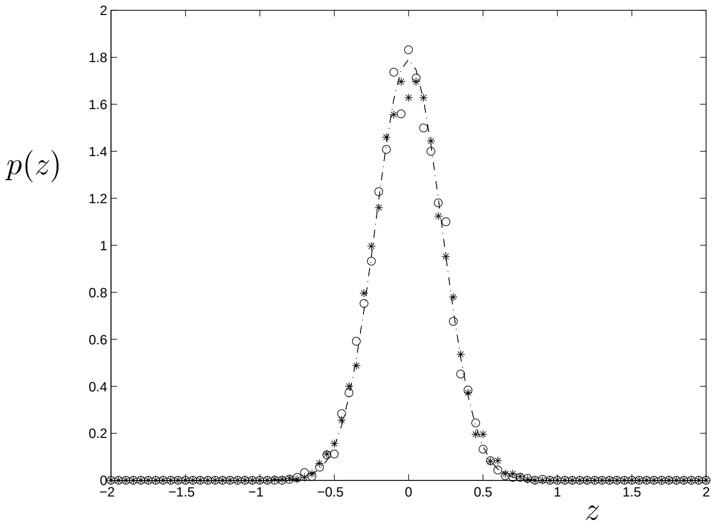
$$\begin{aligned} da &\approx \psi_a dT + \frac{\delta}{\epsilon} d\beta_A(T) \\ db &\approx \psi_b dT + \frac{\delta}{\epsilon} d\beta_B(T). \end{aligned} \tag{3.3}$$

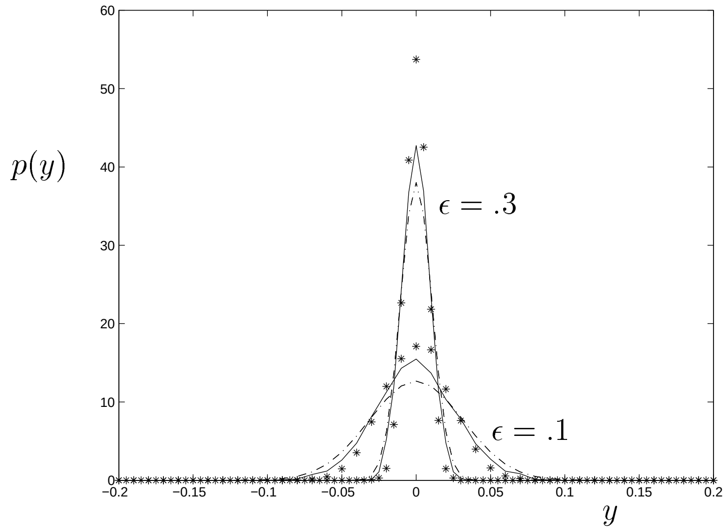
Here $\psi_a$ and $\psi_b$ are just the drift coefficients from (3.1)-(3.2) with $A$ and $B$ replaced by $a$ and $b$, respectively. When $\delta/\epsilon$ is large, the system (3.3) is dominated by the noise. This violates an implicit assumption in the multi-scale expansion: the noise does not dominate the dynamics. If the noise was the dominant feature in the dynamics, the dynamics in the original variable $x$ would also be dominated by the noise. Then it would be incorrect to assume an approximate form of solution as in (2.9) in which the behavior on the fast time scale $t$ was simply a periodic oscillation. In this way, the multi-scale approximation breaks down as $\delta/\epsilon$ increases.

As an additional approximation, we consider the terms in (3.1) and (3.2) which are at the delayed time $(T - \epsilon^2\tau)$. Since $\epsilon^2\tau$ is a small delay, well below any possible critical delay which would result in oscillations for $A$ and $B$ in the absence of noise, we consider the approximation which results from neglecting the delay $\epsilon^2\tau$ for $\epsilon^2 \ll 1$ in (3.1)-(3.2). Then the equations for the approximations $\hat{A}$ and $\hat{B}$ are

$$d\hat{A} = \left[-b_1 r\tau_2(\hat{B}(T)\cos r\tau_c + \hat{A}(T)\sin r\tau_c)\right] dT + \frac{\delta}{\epsilon^2} d\beta_A(T) \tag{3.4}$$

$$d\hat{B} = \left[-b_1 r\tau_2(\hat{B}(T)\sin r\tau_c - \hat{A}(T)\cos r\tau_c)\right] dT + \frac{\delta}{\epsilon^2} d\beta_B(T). \tag{3.5}$$




$(a)$ $(b)$

Figure 5: a) Comparison of the steady state density obtained from simulation of (2.11) for $A$ ($*$'s) and $B$ (o's) with the normal density $p(z) = (2\pi\sigma)^{-1/2} e^{-z^2/(2\sigma)}$ (dash-dotted line) with $\sigma$ given by (3.7). Here $\delta = .05$, $\epsilon = .1$. b) Comparison of the steady state density of $x - x_0$ obtained from (1.1) (solid line), $y$ obtained from the multi-scale approximation using (2.11) ($*$'s), and the normal density $p(y) = (4\pi\sigma)^{-1/2} e^{-y^2/(4\sigma)}$ (dash-dotted line) for $\delta = .05$, $\epsilon = .1$ and $\epsilon = .3$. These are the same parameter values as in Figure 2.

This coupled system describes a two-variable Ornstein-Uhlenbeck process, where the stationary density is normal with the following properties,

$$\mathbf{E}[\hat{A}] = \mathbf{E}[\hat{B}] = \mathbf{E}[\hat{A}\hat{B}] = 0, \tag{3.6}$$

$$\mathbf{E}[\epsilon^2 \hat{A}^2] = \mathbf{E}[\epsilon^2 \hat{B}^2] = \sigma \equiv \frac{\delta^2}{2\epsilon^2 b_1 r \tau_2 \sin r\tau_c}. \tag{3.7}$$

Then in the steady state $\hat{A}$ and $\hat{B}$ are two independent normal random variables which are identically distributed as $\mathcal{N}(0, \sigma)$. Furthermore, $y = \hat{A}\cos rt + \hat{B}\sin rt$ is a normal random variable with zero mean and variance $2\sigma$. In Figure 5 we compare the stationary density of $A$, $B$ and $y$ obtained from the full multi-scale approximation (2.23)-(2.24) with the density of the two variable Ornstein-Uhlenbeck process described by (3.4)-(3.5). The approximation does surprisingly well, even though the nonlinear terms and the delay in the envelope equations have been neglected.

# 4 Discussion

We apply a multiscale analysis to analyze the effect of noise on a model with delayed negative feedback. This model is particularly sensitive to noise when the delay is near a critical value, corresponding to a Hopf bifurcation. In the subcritical case, oscillations which would decay in the absence of noise are amplified by the interaction of the noise and the delay. We derive stochastic equations for the slow evolution of the envelope of these oscillations at the Hopf frequency. The analysis leads to a description which separates the deterministic and stochastic elements of the dynamics. Advantages of this approximation include efficiency of simulations over large times and relatively simple dynamics described by the envelope equations.

The asymptotic validity of the approximation is explored by comparison with the original model. Since the drift in the stochastic envelope equations is dissipative, certain approximations are valid in a significant range of parameters. One particularly simple approximation which performs well is the linearized system for the envelope equations. In addition, since the delay in the envelope equation is very small, approximations which eliminate the delay can also be used. This leads to an analytical approximation for the behavior of system.

The method has also been applied to a linear system and the logistic model, both with delay and small noise [17]. For the linear system the approximation is compared with a result obtained from the Fourier analysis. For the logistic model both the sub- and super- critical cases are considered; that is, the analysis is applied for delays above and below the critical delay. Due to the bifurcation structure and large amplitudes of the limit-cycle oscillations for super-critical delays in (1.1) [9], a considerable adaptation would be required for extending the multi-scale method to this case, so we leave it for future study.

The phenomenon demonstrated here occurs in many applications with similar features, including neuronal models, optics, and information systems [2]-[6]. The analysis exploits the interaction of the noise with the Hopf bifurcation, features which are common to these other models. Thus the analysis can be applied to a wide variety of stochastic models, including those without delays. The method is particularly useful for systems with delays since it avoids complications due to the memory in the system, which can make other analyses impractical.

# A Appendix

We start with the two equations for $dy$ obtained from Ito's formula (2.13) and (2.14) obtained by substitution of (2.9) into (2.8). We derive the nonlinear evolution equations for the subcritical $\tau < \tau_c$ ($\tau_2 < 0$) case. Since we are interested in the delay-related bifurcation from $y = 0$, a Taylor series expansion about $y = 0$ of the nonlinear term on the right hand side of the equation (2.14) is useful,

$$c\frac{\theta^n\left(x_0^n - (x_0 + y(t-\tau))^n\right)}{(\theta^n + (x_0 + y(t-\tau))^n)(\theta^n + x_0^n)} = b_1 y(t-\tau) + \frac{b_2}{2!}y(t-\tau)^2 + \frac{b_3}{3!}y(t-\tau)^3 + \dots \tag{A.1}$$

$$b_1 = -\frac{n\alpha x_0^n}{\theta^n + x_0^n} \tag{A.2}$$

$$b_2 = -\frac{n(n-1)\alpha x_0^{n-1}}{\theta^n + x_0^n} + \frac{2n^2\alpha x_0^{2n-1}}{(\theta^n + x_0^n)^2} \tag{A.3}$$

$$b_3 = -\frac{n(n-1)(n-2)\alpha x_0^{n-2}}{\theta^n + x_0^n} + \frac{6n^2(n-1)\alpha x_0^{2n-2}}{(\theta^n + x_0^n)^2} - \frac{6n^3\alpha x_0^{3n-2}}{(\theta^n + x_0^n)^3} \tag{A.4}$$

Here we have used the expression for $x_0$ (2.2). Furthermore, we take

$$C(T) = \frac{b_2}{2(4r^2+\alpha)}\Bigg( -2r[(A^2 - B^2)\sin r\tau_c \cos r\tau_c + AB(\cos^2 r\tau_c - \sin^2 r\tau_c)]$$

$$+\alpha[-2AB\sin r\tau_c\cos r\tau_c + \frac{A^2-B^2}{2}(\cos^2 r\tau_c - \sin^2 r\tau_c)]\Big)$$

$$D(T) = \frac{b_2}{2(4r^2+\alpha)}\Big(\alpha[(A^2-B^2)\sin r\tau_c\cos r\tau_c + AB(\cos^2 r\tau_c - \sin^2 r\tau_c)]$$
$$+2r[-2AB\sin r\tau_c\cos r\tau_c + \frac{A^2-B^2}{2}(\cos^2 r\tau_c - \sin^2 r\tau_c)]\Big) \quad (A.5)$$

$$E(T) = \frac{b_2}{4\alpha}(A^2+B^2) \quad (A.6)$$

This choice leads to some cancellation, which we describe below. From Ito's formula (2.12) we have:

$$\begin{aligned} dy &= \epsilon(-rA(T)\sin rt + rB(T)\cos rt - 2\epsilon rC(T)\sin 2rt + 2\epsilon rD(T)\cos 2rt + \mathcal{O}(\epsilon^2))dt + \\ &\left[\epsilon\cos rt + \epsilon^2\left(\frac{\partial C}{\partial A}\cos 2rt + \frac{\partial D}{\partial A}\sin 2rt + \frac{\partial E}{\partial A}\right) + \mathcal{O}(\epsilon^3)\right](\psi_A dT + \sigma_A d\beta_A(T)) \\ &+ \left[\epsilon\sin rt + \epsilon^2\left(\frac{\partial C}{\partial B}\cos 2rt + \frac{\partial D}{\partial B}\sin 2rt + \frac{\partial E}{\partial B}\right) + \mathcal{O}(\epsilon^3)\right](\psi_B dT + \sigma_B d\beta_B(T)) \quad (A.7) \\ &+\epsilon^2\left(\frac{\sigma_A^2}{2}\left[\frac{\partial^2 C}{\partial A^2}\cos 2rt + \frac{\partial^2 D}{\partial A^2}\sin 2rt + \frac{\partial^2 E}{\partial A^2} + \mathcal{O}(\epsilon)\right]\right. \\ &\left.+\frac{\sigma_B^2}{2}\left[\frac{\partial^2 C}{\partial B^2}\cos 2rt + \frac{\partial^2 D}{\partial B^2}\sin 2rt + \frac{\partial^2 E}{\partial B^2} + \mathcal{O}(\epsilon)\right]\right) dT \end{aligned}$$

and from substitution of (A.1) and (2.9) into (2.14) we have

$$\begin{aligned} dy &= -\epsilon\alpha\left[A\sin rt - B\cos rt\right] \\ &+ \epsilon b_1\left(A(T)\cos r(t-\tau_c) + B(T)\sin r(t-\tau_c)\right) \\ &- \epsilon^2\left(\alpha\left[C\cos 2rt + D\sin 2rt + E\right] - \frac{b_2}{2}(A\cos rt + B\sin rt)^2\right) \\ &+ \epsilon^3\Big[rb_1\tau_2(A\sin r(t-\tau_c) - B\cos r(t-\tau_c)) \\ &+ b_1\left(\frac{A(T-\epsilon^2\tau)-A(T)}{\epsilon^2}\cos r(t-\tau_c) + \frac{B(T-\epsilon^2\tau)-B(T)}{\epsilon^2}\sin r(t-\tau_c)\right) \\ &+ b_2\left(A\cos r(t-\tau_c) + B\cos r(t-\tau_c)\right)\left(C\cos 2r(t-\tau_c) + D\sin 2r(t-\tau_c) + E\right) \\ &+ \frac{b_3}{3!}\left(A\cos r(t-\tau_c) + B\sin r(t-\tau_c)\right)^3\Big] dt + \mathcal{O}(\epsilon^4) + \delta dw(t) \quad (A.8) \end{aligned}$$

Here we have used (A.1) for $\epsilon \ll 1$, and we have also rewritten

$$A(T-\epsilon^2\tau) = A(T) + \epsilon^2\frac{A(T-\epsilon^2\tau)-A(T)}{\epsilon^2} \quad (A.9)$$
$$B(T-\epsilon^2\tau) = B(T) + \epsilon^2\frac{B(T-\epsilon^2\tau)-B(T)}{\epsilon^2}$$

and have assumed that

$$\frac{A(T-\epsilon^2\tau)-A(T)}{\epsilon^2} = \mathcal{O}(1), \quad \frac{B(T-\epsilon^2\tau)-B(T)}{\epsilon^2} = \mathcal{O}(1) \quad (A.10)$$

Then we equate (A.7) and (A.8) and substitute (A.5)-(A.6) into (A.8). This results in the cancellation of the $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ terms, using the definitions of $r$ and $\tau_c$ (2.7), and $C$, $D$, and $E$ from (A.5)- (A.6).

The remaining terms are then

$$\begin{aligned}
& \epsilon \sin rt(\psi_B dT + \sigma_B d\beta_B(T)) + \epsilon \cos rt(\psi_A dT + \sigma_A d\beta_A(T)) \\
& +\epsilon^2 \left( \frac{\sigma_A^2}{2} \left[ \frac{\partial^2 C}{\partial A^2} \cos 2rt + \frac{\partial^2 D}{\partial A^2} \sin 2rt + \frac{\partial^2 E}{\partial A^2} \right] + \frac{\sigma_B^2}{2} \left[ \frac{\partial^2 C}{\partial B^2} \cos 2rt + \frac{\partial^2 D}{\partial B^2} \sin 2rt + \frac{\partial^2 E}{\partial B^2} \right] \right) dT \\
& + \text{other modes and higher order terms} \\
= & \left( \epsilon^3 \left[ r b_1 \tau_2 (A \sin r(t-\tau_c) - B \cos r(t-\tau_c)) \right.\right. \qquad \text{(A.11)} \\
+ & \; b_1 \left( \frac{A(T-\epsilon^2\tau) - A(T)}{\epsilon^2} \cos r(t-\tau_c) + \frac{B(T-\epsilon^2\tau) - B(T)}{\epsilon^2} \sin r(t-\tau_c) \right) \\
+ & \; b_2 \left( A \cos r(t-\tau_c) + B \cos r(t-\tau_c) \right) \left( C \cos 2r(t-\tau_c) + D \sin 2r(t-\tau_c) + E \right) \\
+ & \left.\left. \frac{b_3}{3!} \left( A \cos r(t-\tau_c) + B \sin r(t-\tau_c) \right)^3 \right] + \mathcal{O}(\epsilon^4) \right) dt + \delta dw(t)
\end{aligned}$$

The slowly varying functions $C(T-\epsilon^2\tau)$, $D(T-\epsilon^2\tau)$, and $E(T-\epsilon^2\tau)$ are rewritten as in (A.9), so that these functions with delayed arguments are included in the $\mathcal{O}(\epsilon^4)$ terms.

As described at the end of Section 2, we equate noise terms and drift terms separately, which leads to $\sigma_A$ and $\sigma_B$ is given in (2.18)-(2.22). For the drift terms we again use the projection as in (2.21). This is equivalent to equating the coefficients of both $\cos rt$ and $\sin rt$, and neglecting all other modes and higher order terms. For example, the terms

$$+\epsilon^2 \left( \frac{\sigma_A^2}{2} \left[ \frac{\partial^2 C}{\partial A^2} \cos 2rt + \frac{\partial^2 D}{\partial A^2} \sin 2rt + \frac{\partial^2 E}{\partial A^2} \right] + \frac{\sigma_B^2}{2} \left[ \frac{\partial^2 C}{\partial B^2} \cos 2rt + \frac{\partial^2 D}{\partial B^2} \sin 2rt + \frac{\partial^2 E}{\partial B^2} \right] \right) dT,$$

do not play a role since they contain terms which are orthogonal to the primary modes $\sin rt$ and $\cos rt$ on the $t$ time scale. Then we are left with equations for $\psi_A$ and $\psi_B$, which yield (2.23)-(2.24).

## References


[1] H. Gang, T. Ditzinger, C. Z. Ning, and H. Haken, *Phys. Rev. Lett.* **71** 807 (1993).

[2] J. Garcia-Ojalvo and R. Roy, "Noise amplification in a stochastic Ikeda model", *Phys. Lett. A* **224** (1996), 51-56.

[3] S. Kim, Seon Hee Park, and H.-B. Pyo, "Stochastic resonance in coupled oscillator systems with time delay", *Phys. Rev. Lett.* **82** (1999) 1620-1623.

[4] J. L. Cabrera, J. Gorronogoitia, and F. J. de la Rubia, "Noise-correlation-time-mediated localization in random nonlinear dynamical systems", *Phys. Rev. Lett.* **82** (1999), 2816-2819.

[5] T. Ohira and Y. Sato, "Resonance with Noise and Delay", *Phys. Rev. Lett.* **82** (1999) 2811-2815.

[6] A. Beuter, J. Belair, and C. Labrie, "Feedback and delays in neurological diseases: a modeling study using dynamical systems", *Bull. Math. Bio.* **55** (1993) 525-541.

[7] *Stochastic dynamics*, H. Crauel and M. Gundlach, eds. Springer-Verlag, New York, 1999.

[8] A. Longtin, J.G. Milton, J. D. Bos, M. C. Mackey, "Noise and critical behavior of the pupil light reflex at oscillation onset", Phys. Rev. A **41** 1990, 6992-7005.

[9] A. Longtin, "Noise-induced transitions at a Hopf bifurcation in a first-order delay-differential equation", Phys. Rev. A, **44**, 1991, 4801-4813.

[10] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations for Population Dynamics*

[11] D. Pieroux, T. Erneux, and K. Otsuka, "A minimal model of a class B laser with delayed feedback: Cascading branching of periodic solutions and period doubling bifurcations" *Phys. Rev. A* **50** (1994), 1822-29.

[12] S.A. Campbell, J. Belair, T. Ohira, and J. Milton, "Complex dynamics and multistability in a damped harmonic oscillator with delayed negative feedback" *Chaos* **5** (1995) 640-645.

[13] J.K. Hale, *Theory of Functional Differential Equations* Springer-Verlag, 1977.

[14] T. Shardlow, "Weak Approximation of Stochastic Delay Equations", *Stochastic Partial Differential Equations and Related Topics*, University of Warwick, March, 2001.

[15] S. Guillouzic, I. L'Heureux, and A. Longtin, "Small delay approximation of stochastic delay differential equations", *Phys. Rev. E.* **59** 1999, 3970-3982.

[16] T. Shardlow, "Stochastic perturbations of the Allen-Cahn equation", *Electron. J. Diff. Eqns.* **2000** 1-19, (2000).

[17] R. Kuske and M.Klosek, "Multi-scale analysis of stochastic delay differential equations", preprint.

[18] J. Kevorkian and J.D. Cole, *Multiple scale and singular perturbation methods.* Springer-Verlag, New York, 1996.

[19] P. Manneville, *Dissipative Structures and Weak Turbulence*, (1990), Academic Press, San Diego.

[20] Z. Schuss, *Introduction to stochastic differential equations*