Rate of convergence to a stable law

Rachel Kuske [1]
Department of Mathematics
Stanford University
and
Joseph B. Keller[2]
Departments of Mathematics and Mechanical Engineering
Stanford University

Abbreviated title: Rate of convergence to a stable law.

ABSTRACT

The rate of convergence to a stable law is determined for the probability density of the normalized sum of n independent identically distributed random variables, as $n \to \infty$. Methods are given for using these results to fit data to such a law.

Keywords: Levy distributions, stable laws, limit laws, convergence rate
AMS subject classifications 60XX, 60E07, 60E10

1 Introduction

We consider the normalized sum of n independent identically distributed random variables X_i, $i = 1, \ldots, n$ with the common probability distribution function $F(x)$ and probability density $q(y) = F'(y)$. We suppose that the distribution function $F_n(y)$ of the normalized sum converges as $n \to \infty$ to a stable distribution or stable law. The only stable distributions are the normal distributions and the Levy distributions. We shall study the rate at which the density $q_n(y) = F_n'(y)$ converges to the density $q(y)$ of a stable law, which has been studied extensively for general $F(x)$. The book of Christoph and Wolf [1] and the recent paper by Juozulynas and Paulauskas [2] contain results most closely related to those in this paper, and give relevant references. See also Petrov [3]. By assuming that $F(x)$ has a probability density $p(x)$ with a very specific tail behavior, we study how the rate of convergence depends on the parameters describing the tail. We compare and contrast our results with those of [1] and [2] in sections 2 and 3.

Levy distributions have algebraically decaying densities. Therefore they often arise in nature describing the effects due to randomly placed sources producing slowly decaying fields. For example, Chandrasekhar [4] obtained the Holtsmark distribution, with tails decaying as the $-5/2$ power, for the magnitude of the gravitational force due to a random

---

[1] Supported in part by a NSF Mathematical Sciences Postdoctoral Fellowship.
Present address: School of Mathematics, University of Minnesota, 127 Vincent Hall,
206 Church St. SE, Minneapolis, MN 55455
[2] Supported in part by AFOSR.

distribution of stars. Similarly, Vlad [5] found algebraic decay for the probability density of the concentration of a rare mineral due to random sources. Jimenez [6] simulated two dimensional turbulent flow by representing it as the flow due to a random collection of n vortices. He showed that as n increased, the density of the components of velocity gradient converged at the rate $1/n$ to a Cauchy density, i.e. a Levy density with exponent $\alpha = 1$. The density of the velocity components converged much more slowly to a Gaussian, at the rate $1/\ln n$.

Levy distributions also describe many other types of data. For example, Peng et al. [7] considered the interval between successive beats for diseased human hearts, and also the increments between successive intervals. The increments of interbeat intervals had a Levy density with $\alpha = 1.7$ and the interbeat intervals had a Gaussian density. Mantegna [8], [9] considered the difference in price index from one day to the next on the Milan stock exchange and on the New York stock exchange. Both densities could be fit with Levy laws, with $\alpha = 1.16$ and $\alpha = 1.5$ respectively.

Before starting our analysis, we shall recall some basic results about the convergence of the normalized sums

$$Y_n = \frac{1}{B_n} \sum_{i=1}^{n} (X_i - M_n) . \tag{1.1}$$

When F has first and second moments m_1 and m_2, we set $B_n = n^{1/2}$ and $M_n = m_1$ in (1.1). Then the Central Limit Theorem states that as $n \to \infty$, the distribution of Y_n converges to a normal distribution with mean zero and second moment m_2. (Feller [10]). If F has a third moment m_3, then $q_n(y) - q(y) = O(n^{-1/2})$ uniformly in y, and if $m_3 = 0$ then the rate of convergence is $o(n^{-1/2})$. However, if F does not have a third moment, but if $F(x) = O(|x|^{-\alpha})$ as $x \to -\infty$ and $F(x) = 1 - O(x^{-\alpha})$ as $x \to +\infty$ with $2 < \alpha \leq 3$, then $q_n(y) - q(y) = O\left[n^{-(\alpha-2)/2}\right]$. In this case we see that the rate of convergence is slower than when there is a third moment.

When F does not have both first and second moments, the distribution of the Y_n may still converge. A necessary and sufficient condition for this is (Gnedenko and Kolmogorov [11])

$$\begin{aligned} F(x) &= [c_1 + r_1(x)]|x|^{-\alpha}, \quad x < 0 \\ &= 1 - [c_2 + r_2(x)]x^{-\alpha}, \quad x > 0, \end{aligned} \tag{1.2}$$

with $0 < \alpha \leq 2$, c_1 and c_2 positive constants, $r_1(x) \to 0$ as $x \to -\infty$ and $r_2(x) \to 0$ as $x \to +\infty$. When this condition holds and $0 < \alpha < 2$ we can set $B_n = n^{1/\alpha}$ in (1.1). Then the limit is a Levy distribution with density $q(y; \alpha, c, \gamma)$ characterized by α, $c > 0$ and γ, and $\gamma = 0$ when q is even. When $\alpha = 2$ we can set $B_n = h(n)$ where $h(n)$ satisfies $h^2 = n \ln h$, and the limit is a normal distribution. If F has an even density, we can set $M_n = 0$, and if F is not even but $\alpha > 1$ we can set $M_n = m_1$. The choice of M_n when the density is not even and $\alpha \leq 1$ will be explained in section 4.

In section 2, we will determine the rate of convergence of the density q_n to q when $F(x)$ has an even density with tail behavior (2.1). We contrast this result with that of

[2]. In section 3, we will show how this rate is modified when the tail behavior is changed to (3.1). In section 4, we consider non-symmetric densities with tail behavior (4.1). In section 5, we discuss methods for determining the exponent α from data, assuming that the densities have the appropriate moments.

2 Convergence rates for symmetric densities

To determine the probability density $q_n(y)$ of the normalized sum Y_n defined by (1.1), we assume that $F(x)$ has an even probability density $p(x)$. Furthermore, we assume that for some positive constants α, A and a, $p(x)$ has the form

$$p(x) = \frac{A}{2|x|^{1+\alpha}} \quad , \quad |x| > a. \tag{2.1}$$

We do not specify $p(x)$ for $|x| < a$, but we require that $p(x)$ and $p'(x)$ exist for $|x| < a$. We allow A and α to be arbitrary, and consider the specific sums Y_n given in (2.2). We now contrast this with the assumptions of [1] and [2]. There the tail of the density is assumed to be exactly the leading term in the expansion of the stable density, that is, they choose A to be a particular value. This is the obvious choice of A for assuring the existence of the pseudo-moments $\int_{-\infty}^{\infty} |x|^s \, dH$ and some related integrals, as required in [1] and [2] for certain values of $s \geq \alpha$. The function H is the difference between $F(x)$ and the corresponding attracting stable law. Therefore, for $s \geq \alpha$, the pseudo-moment will exist if the expression (2.1) is exactly the leading term in the expansion of the density $q(y; \alpha, c, 0)$ for $y \to \infty$. The particular choice of A to assure this is given in [2], using Theorem 1.4 of [1], with B_n chosen as below. Changing the value of A is generally equivalent to changing the scale of x. However for certain values of A and α, cancellation of terms results in higher order convergence. We shall note when the value of A can make a difference in the rate of convergence.

Now, because p is even, we can set $M_n = 0$ in (1.1). Then we choose $B_n = n^{1/\alpha}$ for $0 < \alpha < 2$, $B_n = h(n)$ for $\alpha = 2$ and $B_n = n^{1/2}$ for $\alpha > 2$ as was described in the preceding section. As a consequence, Y_n is given by

$$Y_n = n^{-1/\alpha} \sum_{i=1}^{n} X_i \quad , \qquad 0 < \alpha < 2 \tag{2.2a}$$

$$Y_n = \frac{1}{h(n)} \sum_{i=1}^{n} X_i \quad , \qquad \alpha = 2, \quad h^2(n) = n \ln h(n) \tag{2.2b}$$

$$Y_n = n^{-1/2} \sum_{i=1}^{n} X_i \quad , \qquad \alpha > 2. \tag{2.2c}$$

We shall prove the following theorem:

Theorem 1. Let X_i, $i = 1, \ldots, n$ be n i.i.d. random variables with the common even probability density $p(x)$ satisfying (2.1) and differentiable for $|x| \neq a$ and let Y_n be defined by (2.2a) -(2.2c) . Then Y_n has an even density $q_n(y) = F_n'(y)$ which converges, as $n \to \infty$, to the density $q(y)$ of a stable law given by

$$
\begin{aligned}
q(y) &= q(y;\alpha,c,0), \quad c = A\alpha^{-1}\Gamma(1-\alpha)\cos(\pi\alpha/2) \text{ for } \alpha \neq 1; \quad c = A\pi/2 \text{ for } \alpha = 1 && \text{(2.3a)}\\
q(y) &= (2\pi A)^{-1/2}\, e^{-y^2/2A} && \text{(2.3b)}\\
q(y) &= (2\pi m_2)^{-1/2}\, e^{-y^2/2m_2}, \quad m_2 = \text{ second moment of } X_1. && \text{(2.3c)}
\end{aligned}
$$

The rate of convergence is given uniformly in y by

$$
\begin{aligned}
q_n(y) - q(y) &= O\left[n^{-\nu(\alpha)}\right], && \text{(2.4a)}\\
q_n(y) - q(y) &= O\left[\frac{1}{\ln n}\right], && \text{(2.4b)}\\
q_n(y) - q(y) &= O\left[n^{-\nu(\alpha)}\right], && \text{(2.4c)}
\end{aligned}
$$

where

$$
\begin{aligned}
\nu(\alpha) &= 1, \qquad 0 < \alpha \leq 1 && \text{(2.5a)}\\
&= \frac{2}{\alpha} - 1, \quad 1 \leq \alpha < 2 && \text{(2.5b)}\\
\nu(\alpha) &= \frac{\alpha}{2} - 1, \quad 2 < \alpha \leq 4, && \text{(2.5c)}\\
&= 1, \qquad 4 \leq \alpha. && \text{(2.5d)}
\end{aligned}
$$

Proof From the relation $p(x) = F'(x)$ and (2.1), it follows that F satisfies (1.2) with $r_1(x) = 0$ for $x < -a$ and $r_2(x) = 0$ for $x > a$. Therefore the distribution F_n of Y_n given by (2.2a) converges to a Levy distribution with characteristic exponent α. The distribution F_n of Y_n given by (2.2c) converges to a normal distribution because F has first and second moments for $\alpha > 2$. In case (2.2b), F satisfies the condition [11]

$$\lim_{X\to\infty} \frac{X^2 \int_{|x|>X} dF(x)}{\int_{|x|<X} x^2 dF(x)} = 0, \tag{2.6}$$

which is necessary and sufficient for F_n to converge to a normal distribution.

We have assumed that $F(x)$ has a density $p(x)$ so each $F_n(y)$ has a density $q_n(y)$, and each limit distribution also has a density $q(y)$. We have to show that $q_n(y)$ converges to that particular $q(y)$ given by (2.3a)-(2.3c) and that the rate of convergence is given by (2.4a)-(2.4b). In case c, it follows from the Central Limit Theorem that q_n converges to q in (2.3c).

We begin by introducing $S(k)$, the characteristic function of X_i, which is the Fourier transform of $p(x)$. From (2.2a) the characteristic function of Y_n is $S^n(n^{-1/\alpha}k)$. Therefore

$$q_n(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky} S^n(n^{-1/\alpha}k)dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky+n\ln S(n^{-1/\alpha}k)}dk\ . \tag{2.7}$$

The corresponding representation of $q(y;\alpha,c,\gamma)$ is [9]

$$q(y;\alpha,c,\gamma)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iky-c|k|^{\alpha}\ e^{i\pi\gamma/2(k/|k|)}}\ dk. \tag{2.8}$$

When $0<\alpha<2$, to find $S(k)$ which occurs in (2.7), we use the evenness of $p(x)$ and (2.1) to write

$$S(k)=\int_{-\infty}^{\infty}e^{ikx}p(x)dx=2\int_{0}^{a}p(x)\cos kx dx+A\int_{a}^{\infty}\frac{\cos kx}{x^{1+\alpha}}dx\ . \tag{2.9}$$

We rewrite the last integral in (2.9) as follows:

$$\int_{a}^{\infty}\frac{\cos kx}{x^{1+\alpha}}dx=\int_{0}^{\infty}\frac{\cos kx-1}{x^{1+\alpha}}dx-\int_{0}^{a}\frac{\cos kx-1}{x^{1+\alpha}}dx+\int_{a}^{\infty}\frac{1}{x^{1+\alpha}}dx\ . \tag{2.10}$$

We evaluate the first integral on the right side of (2.10) in terms of the gamma function. In the second integral, we expand $\cos kx$ in a Taylor series and integrate term by term. The third integral we evaluate explicitly. Then we use these results in (2.10), and substitute (2.10) into (2.9) to get

$$\begin{aligned}S(k)&=2\int_{0}^{a}p(x)\cos kx dx+A\left[\frac{a^{-\alpha}}{\alpha}-|k|^{\alpha}\frac{\Gamma(1-\alpha)}{\alpha}\cos\frac{\pi\alpha}{2}+\sum_{j=1}^{\infty}\frac{|k|^{2j}}{(2j)!}(-1)^{j}\frac{a^{2j-\alpha}}{2j-\alpha}\right]\\&\equiv 1-c|k|^{\alpha}+\sum_{j=1}^{\infty}B_{2j}|k|^{2j}\ . \end{aligned}\tag{2.11}$$

The constant 1 in (2.11) is the integral of $p(x)$. The coefficient of $-|k|^{\alpha}$ in the first expression is denoted c, and it has exactly the value given in (2.3a) for $\alpha\neq 1$. The value at $\alpha=1$ is just the limit of this expression as $\alpha\to 1$. It can be found by writing c as $A\alpha^{-1}\Gamma(2-\alpha)(1-\alpha)^{-1}\cos(\pi\alpha/2)$, which has the limit $A\pi/2$ at $\alpha=1$.

In evaluating (2.7) for n large we need the behavior of $\ln S(k)$ for small $|k|$. The lowest power of k in (2.11) is $|k|^{\alpha}$ and the next lowest is $k^{2\alpha}$ for $\alpha\leq 1$ and k^{2} for $\alpha\geq 1$. Thus for small $|k|$

$$\ln S(k)=-c|k|^{\alpha}-\frac{c^{2}}{2}|k|^{2\alpha}+B_{2}k^{2}+O(|k|^{2+\alpha})+O(|k|^{3\alpha})\ . \tag{2.12}$$

From (2.12) we get for $n^{-1/\alpha}|k|$ small

$$n\ln S(k/n^{1/\alpha})=-c|k|^{\alpha}-\frac{c^{2}}{2n}|k|^{2\alpha}+B_{2}\frac{k^{2}}{n^{2/\alpha-1}}+O\left(\frac{|k|^{2+\alpha}}{n^{2/\alpha}}\right)+O\left(\frac{|k|^{3\alpha}}{n^{2}}\right)\ . \tag{2.13}$$

Now we write (2.7) in the form

$$q_{n}(y)=\frac{1}{2\pi}\int\limits_{-n^{1/\alpha-\delta}}^{n^{1/\alpha-\delta}}e^{-iky+n\log S(n^{-1/\alpha}k)}\ dk+R_{n}(y)\ . \tag{2.14}$$

for $0 < \delta < \min(1/\alpha, 2/\alpha - 1)$. In the Appendix, we show that

$$|R_n(y)| < C_1 \left[1 - \frac{c}{n^{\delta\alpha}} + O(n^{-2\delta})\right]^n + C_2 n^{-\mu n} \tag{2.15}$$

for C_1, C_2 constants, $\mu > 0$. Thus the remainder R_n is exponentially small for large n. Within the range of integration in (2.14), $|n^{-1/\alpha}k| \le n^{-\delta}$. Therefore, the small argument expansion (2.13) is valid throughout the range. Upon using it in the integral, and expanding the exponential, we can write (2.14) as

$$q_n(y) = \frac{1}{2\pi} \int\limits_{-n^{1/\alpha-\delta}}^{n^{1/\alpha-\delta}} e^{-iky-c|k|^\alpha} \left[1 - \frac{c^2}{2n}|k|^{2\alpha} + B_2 \frac{k^2}{n^{2/\alpha-1}} + O\left(\frac{|k|^{2+\alpha}}{n^{2/\alpha}}\right) + O\left(\frac{|k|^{3\alpha}}{n^2}\right)\right] dk + R_n(y)\,. \tag{2.16}$$

As $n \to \infty$, the limits of integration in (2.16) tend to $\pm\infty$. The integral of the exponential times the first term, 1, in the integrand tends to exactly (2.8), with $\gamma = 0$, which proves (2.3a).

The fact that $\gamma = 0$ is a consequence of the symmetry of $p(x)$. The difference between the integral with limits $\pm n^{1/\alpha-\delta}$ and that with limits $\pm\infty$ is bounded by a multiple of the right side of (2.15), so it is exponentially small for large n. The next term in the integrand of (2.16) is $O(n^{-1})$ and it is the leading correction if $0 < \alpha < 1$, while if $1 < \alpha < 2$, the $O(n^{1-2/\alpha})$ term is the leading correction. For $\alpha = 1$ they are both of the same order, $O(n^{-1})$. This proves (2.4a) and (2.5a).

For $\alpha > 2$, we calculate $S(k)$ in the same way, with (2.10) modified to

$$\begin{aligned} \int_a^\infty \frac{\cos kx}{x^{1+\alpha}} dx &= \int_0^\infty \frac{\cos kx - 1 + \frac{1}{2}k^2x^2}{x^{1+\alpha}} dx - \int_0^a \frac{\cos kx - 1 + \frac{1}{2}k^2x^2}{x^{1+\alpha}} dx \\ &+ \int_a^\infty \frac{1 - \frac{1}{2}k^2x^2}{x^{1+\alpha}} dx\,. \end{aligned} \tag{2.17}$$

Since $S''(k)|_{k=0} = m_2/2$, where m_2 is the second moment of X_i, we conclude that

$$S(k) \sim 1 - \frac{m_2}{2}k^2 - c|k|^\alpha + B_4 k^4 + O(k^5). \tag{2.18}$$

Here c is the coefficient of $|k|^\alpha$ in (2.11), where it is given explicitly. From (2.18) we get

$$n \ln S(k/n^{1/2}) = -\frac{m_2 k^2}{2} - c\frac{|k|^\alpha}{n^{\alpha/2-1}} - \frac{c^2}{2n}|k|^{2\alpha} + O\left(\frac{|k|^4}{n}\right)\,. \tag{2.19}$$

Thus

$$\begin{aligned} q_n(y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iky-m_2k^2/2} \left[1 - \frac{c}{n^{\alpha/2-1}}|k|^\alpha |B_3| + O\left(\frac{|k|^{2\alpha}}{n}\right) + O\left(\frac{|k|^4}{n}\right)\right] dk \\ &= \frac{1}{\sqrt{2\pi m_2}}\, e^{-y^2/2m_2} + O(n^{-\alpha/2+1}) + O(n^{-1})\,. \end{aligned} \tag{2.20}$$

The first term in (2.20) proves (2.3c) and the next two terms prove that (2.4c) holds with $\nu(\alpha)$ given by (2.5c).

For $\alpha = 2$ we calculate $S(k)$ as before with (2.10) replaced by

$$\int_a^\infty \frac{\cos kx}{x^3}dx = \int_0^\infty \frac{\cos kx - 1 + \frac{1}{2}k^2 x \sin x}{x^3}dx \\ - \int_0^a \frac{\cos kx - 1 + \frac{1}{2}k^2 x \sin x}{x^3}dx + \int_a^\infty \frac{1 - \frac{1}{2}k^2 x \sin x}{x^3}dx \ . \tag{2.21}$$

We find

$$S(k) \ = \ 1 + \frac{A}{2}k^2 \ln|k| + C_2 k^2 + \sum_{j=3}^{\infty} C_j k^j \ . \tag{2.22}$$

The coefficients C_j are determined by evaluating the integrals in (2.21). Then

$$n \ln S[k/h(n)] \ \sim \ -\frac{A}{2}\frac{n\ln(h(n))}{h^2(n)}k^2 + \frac{A}{2}k^2 \ln(k)\frac{n}{h^2(n)} + \\ C_2 k^2 \frac{n}{h^2(n)} + O\left(\frac{n \ln^2 h(n)}{h^4(n)}\right) \ . \tag{2.23}$$

By using (2.23) in (2.7) we get

$$\begin{aligned} q_n(y) \ &= \ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky} e^{n \ln S[k/h(n)]} dk \\ &= \ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky - \frac{A}{2}k^2}\left[1 + \frac{1}{\ln h(n)}[C_2 k^2 + \frac{A}{2}k^2 \ln k] + O\left(\frac{1}{n}\right)\right] dk \ . \end{aligned} \tag{2.24}$$

The leading order term in (2.24) is just that given in (2.3b) which proves (2.3b). The next order term is $O\left[1/\ln h(n)\right] \sim O\left[1/\ln n\right]$ since $h(n) \sim n^{1/2}(\ln n)^{1/2}$, which proves (2.4b). This completes the proof of Theorem 1.

The exponent $\nu(\alpha)$ of the rate of convergence of q_n to q can be defined as $\nu(\alpha) = -\lim_{n\to\infty} \ln[q_n(y) - q(y)]/\ln n$. It is given by (2.5a)-(2.5d) for $\alpha \neq 2$, and from (2.4b) $\nu(2) = 0$. It is shown in Figure 1.

In [2] the density is considered in which $A = 2c$ and $a = c^{-1/\alpha}$ in (2.1), and the density is nonzero only for $x > a$. This case corresponds to one in which there is some cancellation in the terms in (2.13) so that a different rate of convergence is obtained for the isolated values of $\alpha = 1/2$ and $\alpha = 1$.

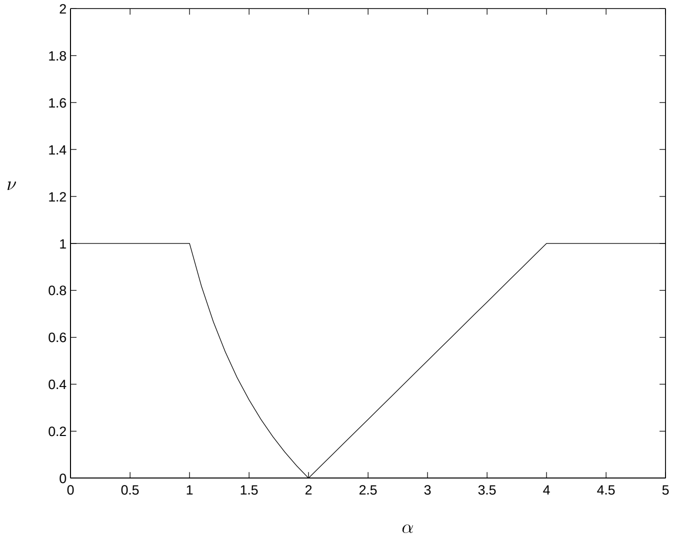


Figure 1: The exponent $\nu(\alpha)$ of the rate of convergence of q_n to q, given by (2.5a)-(2.5d), as a function of α for $0 < \alpha \le 4$. For $\alpha \ge 4$, $\nu(\alpha) = 1$. The density $p(x)$ satisfies (2.1).

3 Convergence rates for symmetric densities with modified tails

Next we shall determine the change in the rate of convergence to a Levy distribution when the tail behavior (2.1) is changed to

$$p(x) = \frac{A}{2|x|^{1+\alpha}} + \frac{B}{2|x|^{1+\beta}} \text{ for } |x| > a, \quad 0 < \alpha < 2, \quad \alpha < \beta, \quad A > 0, \quad B > 0. \tag{3.1}$$

The corresponding $F(x)$ satisfies (1.1) with exponent α, since we have assumed that $\alpha < \beta$, and with

$$r_1(x) = r_2(x) = \frac{B}{2\beta|x|^{\beta-\alpha}} \quad , \qquad c_1 = c_2 = \frac{A}{2\alpha}. \tag{3.2}$$

We still assume that $p(x)$ is even, so $q_n(y)$ converges to a Levy density $q(y; \alpha, c, 0)$.

We calculate $S(k)$ as we did before and find

$$S(k) = 1 - c|k|^\alpha - c'|k|^\beta + \sum_{j=1}^{\infty} B_{2j}|k|^{2j}. \tag{3.3}$$

The constant c' is obtained by changing α to β in the definition of c given in (2.3a). From (3.3) we get

$$n \ln S(k/n^{1/\alpha}) \quad \sim \quad -c|k|^\alpha - c' \frac{|k|^\beta}{n^{\beta/\alpha-1}} - \frac{c^2}{2n}|k|^{2\alpha} - \frac{(c')^2}{2n^{2\beta/\alpha-1}}|k|^{2\beta}$$

$$-\frac{c\,c'}{n^{\beta/\alpha}}|k|^{\beta+\alpha}+B_2\frac{k^2}{n^{2/\alpha-1}}+O\left(\frac{|k|^{2+\alpha}}{n^{2/\alpha}}\right)\ . \tag{3.4}$$

Now we use (3.4) in (2.7) to obtain, when $\beta < 2$,

$$\begin{aligned} q_n(y) \ &= \ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky-ck^\alpha}\left[1-c'\frac{|k|^\beta}{n^{\beta/\alpha-1}}-\frac{c^2}{2n}|k|^{2\alpha}-\frac{(c')^2}{2n^{2\beta/\alpha-1}}|k|^{2\beta}\right. \\ &\left. -\frac{c\,c'}{n^{\beta/\alpha}}|k|^{\beta+\alpha}+B_2\frac{k^2}{n^{2/\alpha-1}}+O\left(\frac{|k|^{2+\alpha}}{n^{2/\alpha}}\right)\right]\ dk\ . \end{aligned} \tag{3.5}$$

The first term in (3.5) is just (2.8) with $\gamma = 0$, so q_n converges to q given in (2.3a). The leading order correction is $O(n^{-1})$ when $\beta > 2\alpha$ and $O(n^{1-\beta/\alpha})$ when $\beta < 2\alpha$. Thus the exponent of the rate of convergence is

$$\begin{aligned} \nu \ &= 1, && 0<\alpha\le 1, && \beta > 2\alpha \\ &= 1-\tfrac{2}{\alpha}, && 1\le\alpha<2, && \beta>2\alpha \\ &= \tfrac{\beta}{\alpha}-1, && 0<\alpha<\beta<2\alpha. \end{aligned} \tag{3.6}$$

In the first two cases, ν is the same as $\nu(\alpha)$ given by (2.5a) and in the third case ν is less than $\nu(\alpha)$ in (2.5a) for $1 \le \alpha < 2$. Thus when $\beta > 2\alpha$ the rate of convergence is unaffected by the $O\left[1/|x|^{1+\beta}\right]$ term in $p(x)$, while when $\beta < 2\alpha$ the rate of convergence is reduced when $\alpha < 1$.

When $0 < \alpha < 1$ and $\beta = 2\alpha$, both leading order correction terms in (3.5) are $O(n^{-1})$, so $\nu = 1$ unless the two terms cancel. Then the leading order correction is $O(n^{-2})$ for $O < \alpha < 2/3$ and $O(n^{1-2/\alpha})$ for $2/3 < \alpha < 1$, which is faster convergence than that given by (2.4a) and (2.5a). Cancellation occurs if $c^2 + 2c' = 0$, which occurs when A, B and α satisfy the equation

$$\frac{A^2}{2B}=-\frac{\alpha}{2^{2\alpha}\sqrt{\pi}}\frac{\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\tan\frac{\alpha\pi}{2}=\frac{d_1^2}{d_2}\ . \tag{3.7}$$

The ratio (3.7) is equal to d_1^2/d_2 where d_1 and d_2 are the first two coefficients in the expansion of $q(y;\alpha,c,0)$ in the form

$$q(y)=\sum_{j=1}^{\infty} d_j\frac{1}{y^{\alpha j+1}}\ . \tag{3.8}$$

By proceeding in the same way, we can calculate the rate of convergence with $p(x)$ symmetric and satisfying (3.1), but with $\beta \ge 2$. The results are shown in Table 1, along with the results in (3.6), and those for $\beta = 2\alpha$, with $c^2 \ne -2c'$ and $c^2 = -2c'$.

The results can be summarized in the following theorem.

Theorem 2. Let $X_i, i = 1, \ldots, n$ be n i.i.d. random variables with a common even probability density satisfying (3.1). Then $Y_n = n^{-1/\alpha}\sum_{i=1}^{n} X_i$ for $0 < \alpha < 2$, and $Y_n =$

β α	$\alpha < \beta < 2\alpha$	$\beta > 2\alpha$	$\beta = 2\alpha$	$\alpha \leq \beta = 2$
$0 < \alpha < 1$	$n^{1-\beta/\alpha}$	n^{-1}	n^{-1} for $c^2 \neq -2c'$ $\max(n^{1-2/\alpha}, n^{-2})$ for $c^2 = -2c'$	n^{-1}
$\alpha = 1$	$n^{1-\beta/\alpha}$	n^{-1}	$n^{-1} \ln n$	$n^{-1} \ln n$
$1 < \alpha < 2$	$\max(n^{1-\beta/\alpha}, n^{1-2/\alpha})$	$n^{1-2/\alpha}$	$n^{1-2/\alpha}$	$n^{1-2/\alpha} \ln n$
$\alpha = 2$	$[\ln h(n)]^{-1}$	$[\ln h(n)]^{-1}$	$[\ln h(n)]^{-1}$	$[\ln h(n)]^{-1}$

Table 1: Convergence rates to a stable law as obtained in Section 3

$\frac{1}{h(n)} \sum_{i=1}^{n} X_i$ for $\alpha = 2$, has an even probability density $q_n(y)$ which converges to $q(y; \alpha, c, 0)$ with c given in (2.3a)-(2.3c). The difference $q_n(y) - q(y; \alpha, c, 0)$ is of the order given in Table 1.

In [2] the density is considered in which $A = 2c$ and $B = 2c'$ in (3.1). Again, for some specific values of α and β, this leads to different rates of convergence than we have found in Table 1. This is because there is some cancellation of terms in (3.4) for this choice of A and B. We indicate one such case in Table 1, for $\beta = 2\alpha$ and $0 < \alpha < 1$.

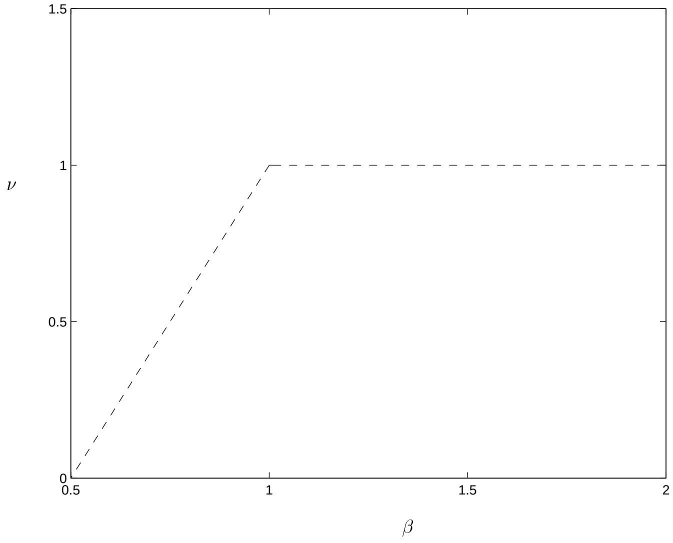


Figure 2: The exponent $\nu(\alpha, \beta)$ of the rate of convergence of q_n to q when the density $p(x)$ satisfies (3.1). The rate of convergence is given in Table 1. The figure is based on the first two entries in the first row of the table: $\nu = \beta/\alpha - 1$ for $\alpha < \beta < 2\alpha$ and $\nu = 1$ for $\beta > 2\alpha$ with $\alpha = 1/2$.

4 Convergence rates for non-symmetric densities

The method for determining the rates of convergence can be straightforwardly extended to non-symmetric densities. For this case we consider $p(x)$ such that

$$p(x) = \begin{cases} \frac{P}{|x|^{1+\alpha}}, & x > a \\ \frac{Q}{|x|^{1+\alpha}}, & x < -a \end{cases} . \tag{4.1}$$

When $p(x)$ is not symmetric, we consider the density of Y_n defined by

$$\begin{aligned} &\text{a.} \qquad & Y_n &= n^{-1/\alpha} \sum_{i=1}^{n} [X_i - M_n(\alpha)], \quad 0 < \alpha < 2 \\ &\text{b.} \qquad & Y_n &= \frac{1}{h(n)} \sum_{i=1}^{n} [X_i - M_n(2)], \quad \alpha = 2 \\ &\text{c.} \qquad & Y_n &= n^{-1/2} \sum_{i=1}^{n} [X_i - M_n(\alpha)], \quad \alpha > 2. \end{aligned} \tag{4.2}$$

For $\alpha > 1$ we choose $M_n(\alpha)$ to be the mean of X_i. However, for $\alpha \leq 1$, X_i has no mean. Then we choose the value for $M_n(\alpha)$ given in (4.9) in order that the densities of the Y_n converge. This choice is also indicated by the demonstration of the domain of stability of the Levy distribution in Gnedenko and Kolmogorov [11].

It is convenient to introduce $\xi_i = X_i - M_n$ in (4.2). Then Y_n has a characteristic function $S_\xi(k) = e^{-ikM_n} S(k)$ and for $\alpha \neq 2$,

$$q_n(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iky + n \ln S_\xi\left(\frac{k}{n^{1/\alpha}}\right)} dk \ . \tag{4.3}$$

For $\alpha = 2$, $n^{1/\alpha}$ is replaced by $h(n)$ here and below in (4.7). To determine $S(k)$ we split up the range of integration and write, for $\alpha \neq 2$,

$$S(k) = \int_{-\infty}^{\infty} e^{ikx} p(x) dx = \int_{-a}^{a} e^{ikx} p(x) dx + P \int_{a}^{\infty} \frac{e^{ikx}}{x^{1+\alpha}} dx + Q \int_{a}^{\infty} \frac{e^{-ikx}}{x^{1+\alpha}} dx \ . \tag{4.4}$$

To evaluate the last two integrals we introduce two functions $\tau_\alpha^\pm(x)$ and write

$$\int_0^\infty \frac{e^{\pm ikx}}{x^{1+\alpha}} dx = \int_0^\infty \frac{e^{\pm ikx} - \tau_\alpha^\pm(x)}{x^{1+\alpha}} dx - \int_0^a \frac{e^{\pm ikx} - \tau_\alpha^\pm(x)}{x^{1+\alpha}} dx + \int_a^\infty \frac{\tau_\alpha^\pm(x)}{x^{1+\alpha}} dx \ . \tag{4.5}$$

The $\tau_\alpha^\pm(x)$ are chosen as follows so that all the integrals in (4.5) exist:

$$\tau_\alpha^\pm(x) = \begin{cases} 1 & 0 < \alpha < 1 \\ 1 \pm ik \sin x & \alpha = 1 \\ 1 \pm ikx & 1 < \alpha < 2 \\ 1 \pm ikx - \frac{k^2}{2} x \sin x & \alpha = 2 \\ 1 \pm ikx - \frac{k^2}{2} x^2 & 2 < \alpha < 3 \\ 1 \pm ikx - \frac{k^2}{2} x^2 \mp i\frac{k^3}{6} x^2 \sin x & \alpha = 3 \\ 1 \pm ikx - \frac{k^2}{2} x^2 \mp i\frac{k^3}{6} x^3 & \alpha > 3 \ . \end{cases} \tag{4.6}$$

With this choice, we can evaluate the integrals in (4.4). Then we take the logarithm of the result to obtain

$$n \ln S_\xi\left(\frac{k}{n^{1/\alpha}}\right) = \begin{cases} -d_\alpha|k|^\alpha - \frac{d_\alpha^2}{2n}|k|^{2\alpha} + O(n^{-2/\alpha+1}) & 0 < \alpha < 1 \\ -(P+Q)|k|\pi/2 \mp i(P-Q)|k|\ln|k| + O(n^{-1}) & \alpha = 1 \\ -d_\alpha|k|^\alpha + B_2\frac{|k|^2}{n^{2/\alpha-1}} + O(n^{-1}) & 1 < \alpha < 2 \\ -\frac{|k|^2}{2}(P+Q) + \frac{C_2k^2 + \frac{P+Q}{2}k^2\log|k| \mp i(P-Q)k^2\pi/4}{\ln h(n)} + O(n^{-1}) & \alpha = 2 \\ -k^2m_2/2 + B_3\frac{|k|^3}{n^{1/2}} - \frac{|k|^\alpha}{n^{\alpha/2-1}}d_\alpha + O(n^{-1}) & 2 < \alpha < 3 \\ -k^2m_2/2 \pm i\frac{(P-Q)}{6}\frac{|k|^3\ln n}{n^{1/2}} + B_3\frac{|k|^3}{n^{1/2}} + O(n^{-1}) & \alpha = 3 \\ -k^2m_2/2 \mp im_3\frac{|k|^3}{6n^{1/2}} - \frac{|k|^\alpha}{n^{\alpha/2-1}}d_\alpha + O(n^{-1}) & \alpha > 3\ . \end{cases} \tag{4.7}$$

Here d_α is defined by

$$d_\alpha = \frac{\Gamma(1-\alpha)}{\alpha}\left[(P+Q)\cos\left(\frac{\pi\alpha}{2}\right) \mp i(P-Q)\sin\left(\frac{\pi\alpha}{2}\right)\right]\ . \tag{4.8}$$

In (4.7) and (4.8) the upper and lower signs correspond to k positive and k negative, respectively, and $M_n(\alpha)$ is given by

$$M_n = \begin{cases} (P-Q)\frac{a^{1-\alpha}}{1-\alpha} + \int_{-a}^{a} xp(x)dx & 0 < \alpha < 1\ , \\ (P-Q)\ln n + \int_{-a}^{a} xp(x)dx + (P-Q)\int_a^\infty \frac{\sin x}{x^2}dx & \alpha = 1\ , \\ m_1 & \alpha > 1\ . \end{cases} \tag{4.9}$$

For each case in (4.7) we determine $q_n(y)$ using (4.3). For $0 < \alpha < 1$ and $1 < \alpha < 2$ we find that $q_n(y)$ converges to $q(y;\alpha,c,\gamma)$ with the same error as in the symmetric case , $O(n^{-1})$ and $O(n^{1-2/\alpha})$ respectively. For $\alpha = 2$ the result is the same as in the symmetric case also: $q_n(y)$ converges to a Gaussian with error $O(1/\ln n)$. For $\alpha = 1$, $q_n(y)$ is given by

$$q_n(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}e^{-|k|[\pi/2(P+Q)\pm i(P-Q)\ln|k|]}\left(1 + O(n^{-1})\right)dk\ . \tag{4.10}$$

The first term in (4.10) is the limit density, and the error is $O(n^{-1})$.

The coefficient of k^3 in $S(k)$ does not vanish in the non-symmetric case. This does not affect the convergence rate for $2 < \alpha < 3$, which is $n^{-\alpha/2+1}$. But for $\alpha = 3$, we find from (4.7) and (4.3) that the convergence rate is $\ln n/n^{1/2}$. However when $P = Q$ the coefficient of the term $\ln n/n^{1/2}$ vanishes, so that the error is then $O(n^{-1/2})$, as in the symmetric case. For $\alpha > 3$ we find that the convergence rate is the well-known $n^{-1/2}$.

5 Illustration via simulation

We now illustrate the convergence and rate of convergence described in Theorem 1. To do so we consider an even probability density $p(x)$ satisfying (2.1) with $1 < \alpha < 2$ and set $Y_n = n^{-1/\alpha} \sum_{i=1}^{n} X_i$ as in (2.2a). We will use simulation to evaluate $E_n|Y_N|$, the expectation of the absolute value of Y_N with respect to q_n. We will compare it to $E|Y|$, the expectation of $|Y|$ with respect to q. The simulation indicates that $E_n|Y_n|$ converges to $E|Y|$ at the same rate $O\left[n^{-\nu(\alpha)}\right]$ given in Theorem 1 for the convergence of q_n to q, that is,

$$n^{-1/\alpha} E_n\left|\sum_{i=1}^{n} X_i\right| = E_n|Y_n| = E|Y| + O[n^{-\nu(\alpha)}]. \tag{5.1}$$

If (5.1) holds, then by taking logarithms of the first and last expressions in it and letting $n \to \infty$, we get

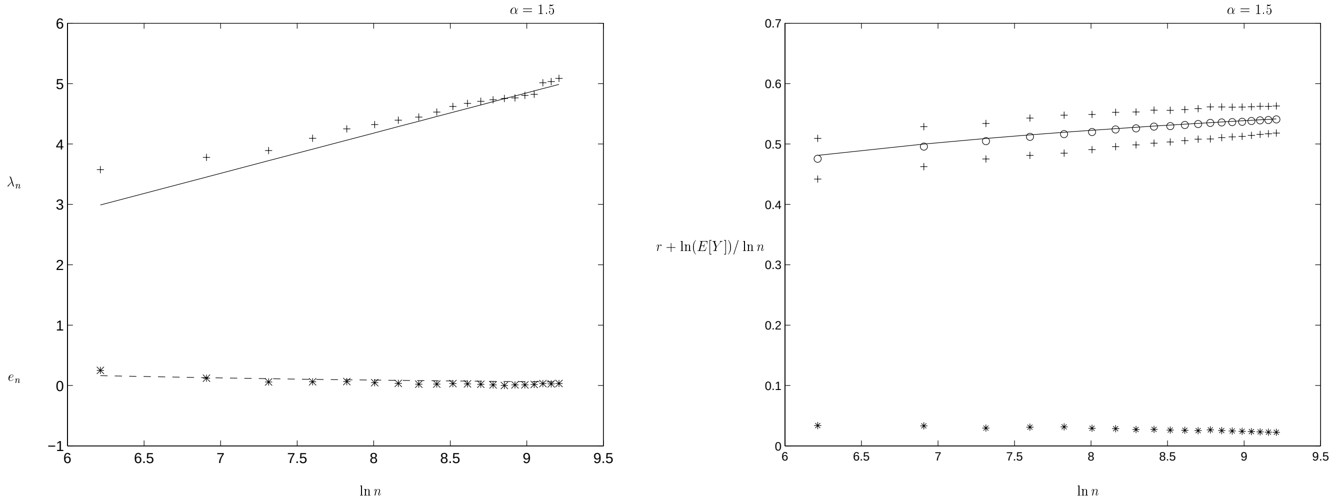
$$\ln E_n\left|\sum_{i=1}^{n} X_i\right| \sim \frac{1}{\alpha}\ln n + \ln E|Y| \qquad \text{as } n \to \infty. \tag{5.2}$$

From the second and third expressions in (5.1) we can also write, when (5.1) holds,

$$\Big|E_n\,|Y_N| - E\,|Y|\Big| = O\left[n^{-\nu(\alpha)}\right]. \tag{5.3}$$

We now set $\alpha = 3/2$ and choose a particular $p(x)$. Then for each $i = 1, \ldots, n$ we generate m random values X_{ij} from this distribution, and form the sample mean $m^{-1} \sum_{j=1}^{m} \left|\sum_{i=1}^{n} X_{ij}\right|$, which is an approximation to the expectation in (5.2). We also calculate $n^{-2/3}$ times the sample mean, which approximates $E_n|Y_n|$ in (5.3). In the upper part of Figure 3a, we plot $\ln\left(m^{-1} \sum_{j=1}^{m} \left|\sum_{i=1}^{n} X_{ij}\right|\right)$ versus $\ln n$ for values of n from 500 to 10,000 with $m = 100$. The solid line shows the right side of (5.2) as a function of n with $\alpha = 3/2$. $E|Y|$ was calculated using $q(y; 3/2, c, 0)$ with c given by (2.3a) with $\alpha = 3/2$. The sample points tend to the theoretical line as n increases. The lower part of Figure 3a shows the left side of (5.3) as a function of $\ln n$ with $E_n|Y_n|$ approximated by the sample mean. Since $\nu(3/2) = 1/3$, the dashed line shows $Cn^{-1/3}$ versus $\ln n$, which represents the leading term on the right side of (5.3). The constant C is calculated from the term in (2.16) proportional to $n^{-1/3}$. Again the sample points tend to the theoretical line as n increases. Plots such as those in Figure 3a can be used to determine α.

The result of averaging 300 replications of the preceding simulation is shown in Figure 3b. In this case, the solid line is the right side of (5.2) divided by $\ln n$ as a function of $\ln n$, and the open circles show the logarithm of the sample mean divided by $\ln n$. The + signs show the logarithm of the sample mean plus or minus one standard deviation as a function of $\ln n$. At the bottom of the figure, the standard deviation is shown as a function of $\ln n$.



(a) (b)

Figure 3: a) The solid curve is $\frac{1}{\alpha}\ln n + \ln E|Y|$ versus $\ln n$ with $\alpha = 3/2$ using $q(y; \frac{3}{2}, c, 0)$ to calculate $E|Y|$. The + signs are the sample values $\ln\left(m^{-1}\sum_{j=1}^{m}\left|\sum_{i=1}^{n} X_{ij}\right|\right)$ with $m = 100$ where X_{ij} are drawn from a distribution with even density $p(x)$ satisfying (2.1) with $\alpha = 3/2$. The dashed curve is $Cn^{-1/3}$ versus $\ln n$ and the asterisks are the values of $\Big|E_n\left|Y_N\right| - E\left|Y\right|\Big|$ where $E_n|Y_n|$ is approximated by the sample mean. b) The solid curve is $1/\alpha + (\ln E|Y|)/\ln n$ versus $\ln n$ with $\alpha = 3/2$. The open circles are the averages of 300 replications of the preceding simulation divided by $\ln n$. The asterisks at the bottom of the figure are the standard deviations of these 300 replications. The plus signs are the means plus or minus one standard deviation. The standard deviation is $O(10^{-2})$ and decreases as n increases.

Acknowledgment

We thank Prof. Javier Jimenez for bringing this problem to our attention and for some helpful discussion of it.

References

[1] Christoph, G., and Wolf, W. (1992) *Convergence Theorems with a Stable Limit Law*, Akademie Verlag.

[2] A. Juozulynas and V. Paulauskas, "Some remarks on the rate of convergence to stable laws", *Lith. Math. J.* **38** 1998, 335-347.

[3] Petrov, V. V. (1975) *Sums of Independent Random Variables*, Springer.

[4] Chandrasekhar, S., "Stochastic Problems in Physics and Astronomy", in *Noise and Stochastic Processes*, ed. N. Wax, (1954).

[5] Vlad, M.O., "Non-Gaussian asymptotic behavior of random concentration fields with long tails", *Int. J. Modern Phys. B* **8** (1994) 2489-2501.

[6] Jimenez, J. "Algebraic probability density functions in isotropic two-dimensional turbulence", *J. Fluid. Mech.* **313** (1996), 223–240.

[7] C.-K. Peng, J. Mietus, J.M. Hausdorff, S. Havlin, H.E. Stanley, A.L. Goldberger, "Long-range anticorrelations and non-Gaussian behavior of the heartbeat", *Phys. Rev. Lett.* **70** (1993) 1343-1346.

[8] Mantegna, R.N. "Levy walks and enhanced diffusion in the Milan stock exchange," *Physica A* **179** (1991), 232-242.

[9] Mantegna, R.N. "Levy processes in the New York stock exchange", *AIP Conference Proceedings* **285** (1993) 533-536.

[10] Feller, W. (1968) *An Introduction to Probability Theory and Its Applications* **Vol. II**, Wiley.

[11] Gnedenko, B. V. and Kolmogorov, A. N. (1954) *Sums of Independent Random Variables*, Addison-Wesley.

A Appendix

We shall now show that the remainder $R_n(y)$ in (2.14) satisfies (2.15). From (2.14), $R_n(y)$ is defined by

$$\begin{aligned} R_n(y) &= \frac{1}{2\pi}\left[\int_{-\infty}^{-n^{1/\alpha-\delta}} + \int_{n^{1/\alpha-\delta}}^{\infty}\right] e^{-iky} S^n(k/n^{1/\alpha})\, dk \\ &= \frac{1}{\pi} Re \int_{n^{1/\alpha-\delta}}^{\infty} e^{-iky} S^n(k/n^{1/\alpha})\, dk \ , \end{aligned} \tag{A.1}$$

for $0 < \delta < \min(1/\alpha, 2/\alpha - 1)$.

First we give the behavior of $S^n(k/n^{1/\alpha})$ for $k < n^{1/\alpha}$ and $k > n^{1/\alpha}$.

1. For $k < n^{1/\alpha}$,

$$S(k/n^{1/\alpha}) = 1 - c\frac{|k|^\alpha}{n} + O(n^{-2/\alpha}). \tag{A.2}$$

Reconsidering (2.9)-(2.10) (for $0 < \alpha < 2$), we have

$$S(k) = 1 - c|k|^\alpha - \int_0^a \frac{\cos kx - 1}{x^{1+\alpha}} dx \tag{A.3}$$

In Section 2 we expanded the integral (A.3) for k small, anticipating that we need the expansion of $S(k/n^{1/\alpha})$ for $k < n^{1/\alpha}$. Here we show that it is $O(n^{-2/\alpha})$. With the appropriate substitutions,

$$S(k/n^{1/\alpha}) \ = \ 1 - c|k/n^{1/\alpha}|^\alpha + \frac{1}{n}\int_0^{a/n^{1/\alpha}} \frac{\cos kz - 1}{z^{1+\alpha}} dz \tag{A.4}$$

Since

$$\lim_{n\to\infty} \frac{n^{-1}\int_0^{a/n^{1/\alpha}} \frac{\cos kz - 1}{z^{1+\alpha}} dz}{n^{-2/\alpha}} = \frac{a^{2-\alpha}k^2}{-2(2/\alpha+1)/\alpha}, \tag{A.5}$$

and $2/\alpha > 1$ for $0 < \alpha < 2$, we get (A.2) for $k < n^{1/\alpha}$.

2. For $k > n^{1/\alpha}$,

$$\frac{M_2 n^{1/\alpha}}{k} < \left|S(k/n^{1/\alpha})\right| < \frac{M_1 n^{1/\alpha}}{k} \tag{A.6}$$

for some constants M_1 and M_2.

Reconsider (2.9). Integrating by parts for large k

$$S(k) = \frac{1}{k}\left[2\sin kap(a) - \int_0^a \sin kxp'(x)dx + A\frac{\sin ka}{a^{1+\alpha}} + (1+\alpha)\int_a^\infty \frac{\sin kx}{x^{2+\alpha}}dx\right] \tag{A.7}$$

Assuming $p'(x)$ exists for $|x| < a$, we see that (A.6) holds for $k > n^{1/\alpha}$

Second, we show that that R_n decays exponentially with n as $n \to \infty$, using (A.2) and (A.6).

Defining I_n,

$$R_n(y) = \frac{1}{\pi}\mathrm{Re}\int_{n^{1/\alpha-\delta}}^\infty e^{-iky}S^n(k/n^{1/\alpha})dk \equiv \frac{1}{\pi}\mathrm{Re}(I_n) \tag{A.8}$$

we integrate I_n by parts once

$$\begin{aligned} I_n = \int_{n^{1/\alpha-\delta}}^\infty e^{-iky}S^n(k/n^{1/\alpha})dk \quad &= \quad \frac{e^{-iky}}{-iy}S^n(k/n^{1/\alpha})\Big|_{n^{1/\alpha-\delta}}^\infty \qquad \text{(A.9)} \\ &+ \frac{n}{n^{1/\alpha}}\int_{n^{1/\alpha-\delta}}^\infty \frac{e^{-iky}}{iy}S^n(k/n^{1/\alpha})\frac{S'(k/n^{1/\alpha})}{S(k/n^{1/\alpha})}dk. \end{aligned}$$

Using (2.10),

$$\begin{aligned} S'(k) &= -2\int_0^a xp(x)\sin kxdx - A\int_a^\infty \frac{\sin kx}{x^\alpha}dx \qquad \text{(A.10)} \\ &= \frac{1}{k}\left[2ap(a)\cos ka + A\frac{\cos ka}{a^\alpha} - 2\int_0^a (xp(x))'\cos kxdx + A\alpha\int_a^\infty \frac{\cos kx}{x^{1+\alpha}}dx\right] \qquad \text{(A.11)} \end{aligned}$$

Note that the integrals in $S'(k)$ are of the same form as for $S(k)$. Then, for $k < n^{1/\alpha}$

$$\frac{S'(k/n^{1/\alpha})}{S(k/n^{1/\alpha})} = \frac{n^{1/\alpha}}{k}\left[\frac{N_1\cos ka - N_2\frac{|k|^\alpha}{n} + O(n^{-2/\alpha})}{1 - c\frac{|k|^\alpha}{n} + O(n^{-2/\alpha})}\right] \tag{A.12}$$

and for $k > n^{1/\alpha}$,

$$\frac{L_2 n^{1/\alpha}}{k} < \left|\frac{S'(k/n^{1/\alpha})}{S(k/n^{1/\alpha})}\right| < \frac{L_1 n^{1/\alpha}}{k} \tag{A.13}$$

for L_1, L_2, N_1, and N_2 constants.

Now we rewrite I_n,

$$I_n \quad = \quad I_n^a + e_I = \int_{n^{1/\alpha-\delta}}^{n^{1/\alpha+\delta}} e^{-iky}S^n(k/n^{1/\alpha})dk + e_I. \tag{A.14}$$

and similarly,

$$\begin{aligned}\int_{n^{1/\alpha-\delta}}^{\infty}\frac{e^{-iky}}{iy}S^n(k/n^{1/\alpha})\frac{S'(k/n^{1/\alpha})}{S(k/n^{1/\alpha})}dk &= J_n^a+e_J \qquad (\text{A.15})\\ &\equiv \int_{n^{1/\alpha-\delta}}^{n^{1/\alpha+\delta}}\frac{e^{-iky}}{iy}S^n(k/n^{1/\alpha})\frac{S'(k/n^{1/\alpha})}{S(k/n^{1/\alpha})}dk+e_J.\end{aligned}$$

Then, using (A.15) in (A.9), we have

$$I_n^a-n^{1-1/\alpha}J_n^a=\frac{e^{-iky}}{-iy}S^n(k/n^{1/\alpha})\bigg|_{n^{1/\alpha-\delta}}^{\infty}+e_I+e_J. \qquad (\text{A.16})$$

It follows from the behavior of $S(k)$ and the bounds (A.6) and (A.13) for $S'(k)/S(k)$ for $k\gg 1$ that e_I and e_J are exponentially small for large n; that is, $e_I\le C_In^{-\mu_In}$ and $e_J\le C_Jn^{-\mu_Jn}$ for C_I and C_J constants and $\mu_I>0$ and $\mu_J>0$. Using the behavior of $S'(k)/S(k)$ we can bound J_n^a as

$$\frac{K_1|I_n^a|}{n^{\delta}}\le|J_n^a|\le\frac{K_2|I_n^a|}{n^{-\delta}}, \qquad (\text{A.17})$$

for K_1 and K_2 constants. Using the behavior of $S^n(k)$ for $k\ll 1$ and $k\gg 1$,

$$\left|\frac{e^{-iky}}{-iy}S^n(k/n^{1/\alpha})\bigg|_{n^{1/\alpha-\delta}}^{\infty}\right|\le D_1\left[1-\frac{c}{n^{\delta\alpha}}+O(n^{-2\delta})\right]^n \qquad (\text{A.18})$$

for D_1 a constant. Combining (A.16), (A.17), and (A.18) implies that I_n^a decays exponentially with n as $n\to\infty$, as given in (2.15). Therefore we can neglect R_n as exponentially small in Section 2.