Optimal exercise boundary for an American put option

Rachel A. Kuske*
Department of Mathematics
University of Minnesota-Minneapolis
Minneapolis, MN 55455
e-mail: rachel@math.umn.edu

Joseph B. Keller
Departments of Mathematics and Mechanical Engineering
Stanford University
Stanford, CA 94305
email: keller@math.stanford.edu
Phone:(415) 723-0851
Fax: (415)725-4066

Running Title: Optimal exercise boundary for American put option.

Keywords: put option, exercise boundary, American option, free boundary

Abstract

The optimal exercise boundary near the expiration time is determined for an American put option. It is obtained by using Green's theorem to convert the boundary value problem for the price of the option into an integral equation for the optimal exercise boundary. This integral equation is solved asymptotically for small values of the time to expiration. The leading term in the asymptotic solution is the result of Barles et al. [1]. An asymptotic solution for the option price is obtained also.

*Supported in part by an NSF Mathematical Sciences Postdoctoral Research Fellowship.

1. Introduction and results

We shall determine the optimal exercise boundary $S = \beta(T)$ and the price $P(S,T)$ for an American put option on a share of price S at time T near the expiration time T_F. Our result for the boundary is, in parametric form with parameter α:

$$\frac{\sigma^2}{2}(T_F - T) \sim \left[\frac{\sigma^2 e^{-\alpha^2}}{6\pi^{1/2} r \alpha^2}\right]^2, \tag{1.1}$$

$$\beta(T) \sim K\left\{1 - 2\left[\frac{\sigma^2}{2}(T_F - T)\right]^{1/2}\alpha\right\}, \qquad \frac{\sigma^2}{2}(T_F - T) \ll 1. \tag{1.2}$$

Here K is the strike price, σ is the volatility, and r is the interest rate. To use these formulas we choose large values of α to ensure that the inequality in (1.2) will hold, compute T from (1.1) and $\beta(T)$ from (1.2).

To obtain $\beta(T)$ explicitly, we can solve (1.1) for α in terms of $t = \sigma^2(T_F - T)/2$. By taking logarithms in (1.1) and solving the resulting equation iteratively, we get after three iterations

$$\alpha(t) \sim \left\{\log\frac{\sigma^2}{6r\sqrt{\pi t}} - \log\left[\log\frac{\sigma^2}{6r\sqrt{\pi t}} - \log\left(\log\frac{\sigma^2}{6r\sqrt{\pi t}} - \log\alpha^2\right)\right]\right\}^{1/2}. \tag{1.3}$$

The use of (1.3) in (1.2), with the final $\log\alpha^2$ omitted, gives an explicit expression for $\beta(T)$. By using only the first term in (1.3), we get

$$\beta(T) \sim K\left\{1 - \left[2\sigma^2(T_F - T)\log\frac{\sigma^2}{6\pi^{1/2} r\left[\sigma^2(T_F - T)/2\right]^{1/2}}\right]^{1/2}\right\}. \tag{1.4}$$

This is the dimensionless form of the result of Barles et al. [1], with an improved constant in the argument of the logarithm.

Figure 1 shows $\alpha(t)$ obtained from (1.1) and Figure 2 shows $\beta(T)$ obtained from (1.1) and (1.2). Figure 3 shows $P(S,T)$ based upon (2.3) with $p(x,t)$ given by (3.2). To obtain (1.1) and (1.2) we derive an integral equation for the optimal exercise boundary $\beta(T)$, using the Green's function of the partial differential equation adjoint to that satisfied by P. Then we solve this equation asymptotically for T near T_F. We use the asymptotic solution for $\beta(T)$ in the integral representation (3.2) of p to get P. Our results can also be obtained directly without the use of the integral equation, as we show in section 5.

Previously McKean [2] and Van Moerbeke [3] used the integral equation method to analyze the price of an American call option. In that case the boundary is given by (1.2) with α a constant, so the boundary is parabolic for T near T_F. However for the put, α tends to infinity as T tends to T_F, as (1.1) shows, so the boundary

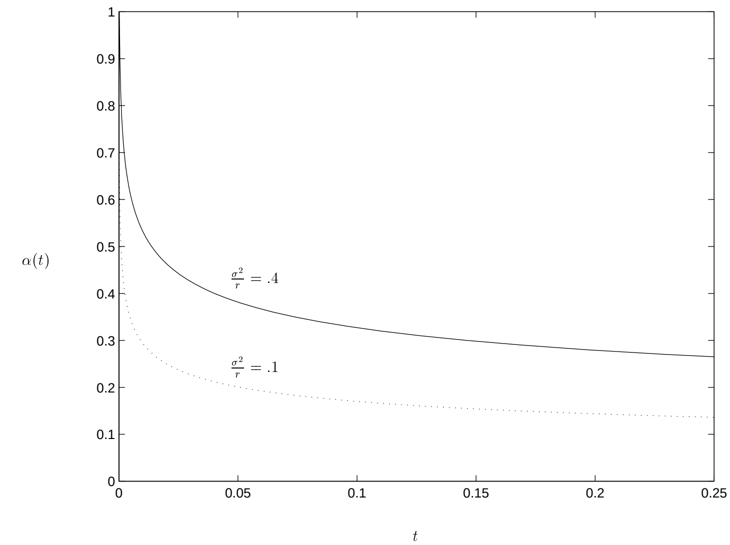

Figure 1: $\alpha(t)$ as a function of t obtained from (1.1) for $\sigma^2/r = .1$ (lower curve) and $\sigma^2/r = .4$ (upper curve).

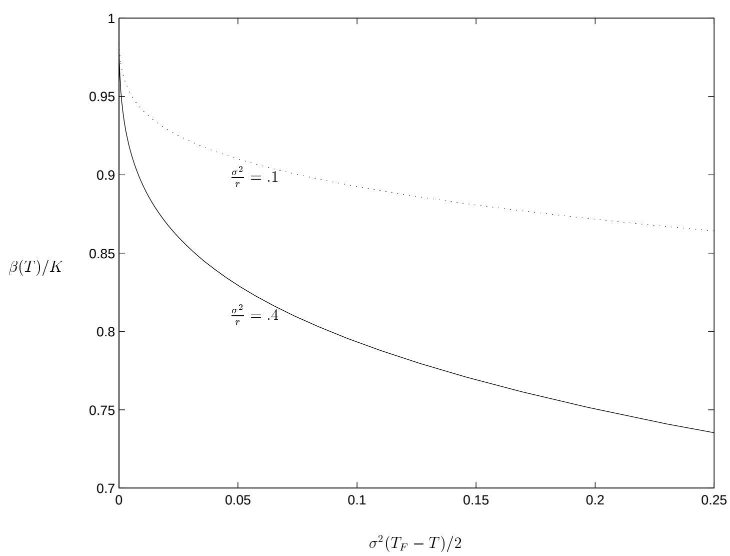

Figure 2: The optimal exercise boundary $\beta(T)/K$ as a func tion of $t = \sigma^2(T_F - T)/2$ given by (1.2) with α determined by (1.1) for $\sigma^2/r = .1$ (upper curve) and $\sigma^2/r = .4$ (lower curve).

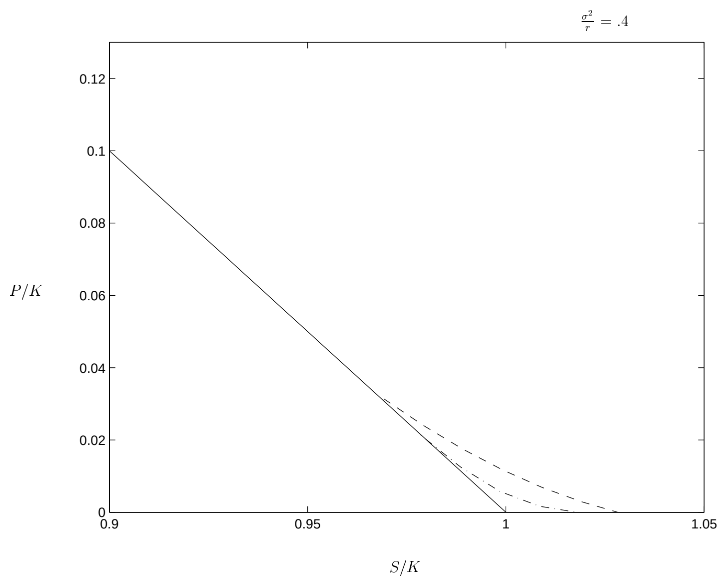

Figure 3: The price $P(S,T)/K$ as a function of S/K for $\sigma^2(T_F - T)/2 = .001$ (upper curve) and $\sigma^2(T_F - T)/2 = .0001$ (lower curve) for $\sigma^2/r = .4$. The curves are based upon (2.3) with p given by (3 .2). The solid straight line is the price at $T_F - T = 0$ and is given by $P/K = 1 - S/K$.

is not parabolic near $T = T_F$. In fact we show that the equation analogous to that derived in [3] for α, on the assumption that α is constant, has no solution.

There is a considerable literature on the optimal exercise boundary, both analytical and numerical. See for example Kim [4], Barone-Adesi and Elliott [5], Barles et al. [1] and Wilmott et al. [6]. A recent list of references, together with numerical approximations, is given by Aitsahlia and Lai [7].

The integral equation (3.3) for the boundary can be solved numerically, starting at T_F. Equations (1.1) and (1.2) provide the behavior of the boundary near the starting point. Once the boundary has been found, $P(S,T)$ can be obtained from (2.3) and (3.2).

2. Formulation

The boundary value problem for $P(S,T)$ and for the exercise boundary $S = \beta(T)$ can be formulated as follows (Wilmott, Dewynne, and Howison [6]):

$$\begin{aligned} -P_T &= \frac{\sigma^2 S^2}{2} P_{SS} + rSP_S - rP\,, \quad 0 < T < T_F, \qquad S > \beta(T) \\ \beta(T_F) &= K \\ P(S, T_F) &= 0, \quad S \geq K \\ P[\beta(T), T] &= K - \beta(T), \;\; P_S[\beta(T), T] = -1, \qquad 0 \leq T \leq T_F\,. \end{aligned} \tag{2.1}$$

For $S \leq \beta(T),\ P(S,T) = K - S$.

This problem (2.1) can be written in a simpler form by introducing the parameter $\rho = 2r/\sigma^2$, the new variables x and t, and the new dependent functions $p(x,t)$ and $b(t)$. They are defined by

$$S = Ke^x, \qquad T = T_F - \frac{2}{\sigma^2} t \tag{2.2}$$

$$P(S,T) = Ke^{-\rho t}[1 + p(x,t)] - S \tag{2.3}$$

$$b(t) = \log[\beta(T)/K]\,. \tag{2.4}$$

Then (2.1) becomes the following problem for the transformed price $p(x,t)$ and transformed exercise boundary $x = b(t)$:

$$p_t = p_{xx} + (\rho - 1)p_x, \qquad x > b(t), \quad 0 < t < \sigma^2 T_F/2 \tag{2.5}$$

$$b(0) = 0 \tag{2.6}$$

$$p(x,0) = e^x - 1, \qquad x \geq 0 \tag{2.7}$$

$$p[b(t),t] = e^{\rho t} - 1, \qquad 0 \leq t \leq \sigma^2 T_F/2\,. \tag{2.8}$$

$$p_x[b(t),t] = 0, \quad 0 \leq t \leq \sigma^2 T_F/2\,. \tag{2.9}$$

3. The integral equation for $b(t)$

The problem (2.5)– (2.9) can be converted into an integral equation for $b(t)$ by making use of the causal fundamental solution or Green's function for (2.5). This solution $G(x,t;\xi,s)$ is given by

$$G(x,t;\xi,s) = \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{[x-\xi+(\rho-1)(t-s)]^2}{4(t-s)}}, \qquad s < t. \tag{3.1}$$

It has a singularity at $x = \xi$, $t = s$. We apply Green's theorem to $G[x,t,\xi,s]$ and $p(\xi,s)$ in the region $\xi > b(s)$, $0 < s < t$. Then we use the values of p and p_x on the boundary $\xi = b(s)$, which are given by (2.8) and (2.9), and the initial value of p, given by (2.7). We also use the fact that p is bounded by its initial value, which follows from the maximum principle for (2.5). Then we obtain

$$\begin{aligned} p(x,t) \;\; &= \;\; -\int_0^t \left[-\frac{x-b(s)}{2(t-s)} + \frac{1}{2}(\rho-1) + b'(s) \right] (e^{\rho s}-1) G[x,t;b(s),s]ds \\ &\quad + \int_0^\infty (e^\xi - 1) G[x,t;\xi,0] d\xi \end{aligned} \tag{3.2}$$

This is a representation of p in terms of the boundary function $b(t)$.

We use (3.2) in the boundary condition (2.8) to get

$$\begin{aligned} e^{\rho t} - 1 \;\; &= \;\; -\int_0^t \left[-\frac{b(t)-b(s)}{2(t-s)} + \frac{1}{2}(\rho-1) + b'(s) \right] (e^{\rho s}-1) G[b(t),t;b(s),s]ds \\ &\quad + \int_0^\infty (e^\xi - 1) G[b(t),t;\xi,0] d\xi \ . \end{aligned} \tag{3.3}$$

Equation (3.3) is an integral equation for $b(t)$. We rewrite it by calling the first integral $I(t)$ and using (3.1) for G, obtaining

$$e^{\rho t} - 1 + I(t) \;\; = \;\; \frac{1}{\sqrt{2\pi}} \left[e^{b(t)+(1-(\rho-1))t} \int_{-\frac{b(t)+(3-\rho)t}{\sqrt{2t}}}^\infty e^{-z^2/2} dz - \int_{-\frac{b(t)-(\rho-1)t}{\sqrt{2t}}}^\infty e^{-z^2/2} dz \right]. \tag{3.4}$$

4. Asymptotic solution of the integral equation

We shall solve (3.4) asymptotically for $t \ll 1$. In the appendix we show that $I(t) \sim \rho t/2$ for $t \ll 1$. Then the left side of (3.4) is asymptotically $\rho t + \rho t/2 = 3\rho t/2$. Next we assume that $-t^{-1/2} b(t) \gg 1$ for $t \ll 1$, which we shall verify later. Then we use the asymptotic value of the error function,

$$\int_y^\infty e^{-z^2/2} dz = e^{-y^2/2}[y^{-1} + O(y^{-3})] \quad \text{for } y \gg 1 \tag{4.1}$$

to evaluate the right side of (3.4). Then (3.4) becomes

$$\frac{3\rho t}{2} \sim \frac{2t}{b^2(t)}\sqrt{\frac{t}{\pi}}e^{-\frac{b^2(t)}{4t}} \quad \text{for } t \ll 1 \ . \tag{4.2}$$

It is convenient to rewrite this equation for $b(t)$ in terms of $\alpha(t)$, defined by

$$b(t) = -2t^{1/2}\alpha(t) \ . \tag{4.3}$$

We use (4.3) for $b(t)$ in (4.2), set $\rho = 2r/\sigma^2$ and replace t by its definition in (2.2), which is $t = \sigma^2 (T_F - T)/2$. The result is (1.1). The first term of the expression (1.4) for $\alpha(t)$ shows that $-t^{-1/2}b(t) = 2\alpha(t) \gg 1$ for $t \ll 1$. This verifies the assumption we made in using (4.1).

Next we use (2.4) to express $\beta(T)$ in terms of $b(t)$, and then use (4.3) to express $b(t)$ in terms of α. This yields exactly (1.2). Thus we have obtained both (1.1) and (1.2). To get $P(S,T)$ we use (3.2) in (2.3).

5. Another derivation

Now we shall present another derivation of (1.2) and a modified form of (1.1) which may help us to understand why they arise. We begin by seeking a function $\tilde{p}(x,t)$ which satisfies the differential equation (2.5), the initial condition (2.7) for $x \geq 0$, and the initial condition $\tilde{p}(x,t) = 0$ for $x \leq 0$:

$$\begin{aligned} \tilde{p}_t &= \tilde{p}_{xx} + (\rho - 1)\tilde{p}_x, \quad 0 < t < \sigma^2 T_F/2 \\ \tilde{p}(x,0) &= e^x - 1, \quad x \geq 0 \\ \tilde{p}(x,0) &= 0, \quad x \leq 0 \ . \end{aligned} \tag{5.1}$$

The solution of this problem is

$$\tilde{p}(x,t) = \int_0^\infty (e^\xi - 1)G(x,t;\xi,0)d\xi. \tag{5.2}$$

We note that $\tilde{p}$ is just the second term on the right side of (3.2).

Next we seek the curve $\tilde{b}(t)$ on which (5.2) has the boundary value (2.8):

$$\tilde{p}[\tilde{b}(t),t] \ = \ e^{\rho t} - 1 \ . \tag{5.3}$$

When we use (5.2) for $\tilde{p}$ in (5.3) the resulting equation for $\tilde{b}(t)$ is

$$e^{\rho t} - 1 = \int_0^\infty (e^\xi - 1)G[\tilde{b}(t),t;\xi,0]d\xi. \tag{5.4}$$

This is just (3.3) with the first integral, $I(t)$, omitted. Proceeding as in Section 4, we evaluate the integral in (5.4) asymptotically. Then we write

$$\tilde{b} = -2t^{1/2}\tilde{\alpha}(t) \tag{5.5}$$

and we find from (5.4) that $\tilde{\alpha}(t)$ satisfies

$$\frac{e^{-\tilde{\alpha}^2}}{\tilde{\alpha}^2} \sim \frac{4r}{\sigma^2}\sqrt{\pi t}\ . \tag{5.6}$$

This is just (1.1) with the correct constant 6 replaced by 4. Therefore $\beta(T)$ computed from (2.4) with $b(t)$ replaced by $\tilde{b}(t)$, is still given by (1.1)–(1.4) with 6 replaced by 4. As (1.4) shows, the leading term in $\beta(T)$ is unaffected by this change. Furthermore (5.5) and (5.6) show that $\tilde{b}(0) = 0$, so (2.6) is satisfied.

We must still verify (2.9) so we evaluate (5.2) using (3.1) for G and we obtain for $t \ll 1$

$$\tilde{p}(x,t) = \frac{1}{\sqrt{2\pi}}\left[e^{x+(1-(\rho-1))t}\int_{-\frac{x+(3-\rho)t}{\sqrt{2t}}}^{\infty} e^{-z^2/2}dz - \int_{-\frac{x-(\rho-1)t}{\sqrt{2t}}}^{\infty} e^{-z^2/2}dz\right] . \tag{5.7}$$

¿From (5.7) we compute $\tilde{p}_x[\tilde{b}(t),t]$ to get

$$\begin{aligned}\tilde{p}_x(\tilde{b}(t),t) \quad &= \quad \frac{1}{\sqrt{2\pi}}\left[e^{\tilde{b}(t)+(2-\rho)t}\int_{-\frac{\tilde{b}(t)+(3-\rho)\sqrt{t}}{\sqrt{2t}}}^{\infty} e^{-z^2/2}dz\right] \\ &\quad + \sqrt{\frac{1}{4\pi t}}\left(e^{-\frac{(\tilde{b}(t)+[3-\rho])t)^2}{4t}+\tilde{b}(t)+(2-\rho)t} - e^{-\frac{(\tilde{b}(t)-[\rho-1]t)^2}{4t}}\right) .\end{aligned} \tag{5.8}$$

We use (4.1) to evaluate the integral in (5.8) and then use (5.5) for $\tilde{b}(t)$ with $\tilde{\alpha}$ given by the first term in (1.3). In this way we get

$$\tilde{p}_x[\tilde{b}(t),t] \sim \frac{1}{\sqrt{\pi}}\frac{e^{-\tilde{\alpha}^2}}{2\tilde{\alpha}} + \frac{2\tilde{\alpha}}{\sqrt{\pi}}e^{-\tilde{\alpha}^2} + o(\sqrt{t}) = O(\sqrt{t}\tilde{\alpha}^3) = O[\sqrt{t}|\log t|^{3/2}] \tag{5.9}$$

Thus $\tilde{p}_x[\tilde{b}(t),t]$ is asymptotically equal to zero for $t \ll 1$, so (2.9) holds asymptotically.

Summarizing, we have constructed functions $\tilde{p}(x,t)$ and $\tilde{b}(t)$ which satisfy (2.5)-(2.8) exactly and which satisfy (2.9) asymptotically. Thus $\tilde{p}, \tilde{b}$ is an asymptotic solution of (2.5)-(2.9).

Replacing $p(x,t)$ by $\tilde{p}(x,t)$ in the expression (2.3) for $P(S,T)$ yields an expression similar to that for the value of the European put. In fact, if the expression for the European put, rather than $\tilde{p}(x,t)$ is used in (2.8) to determine $\tilde{b}(t)$, the resulting equation yields a solution which is also asymptotic to $b(t)$ for small t. This procedure was used in [1] to obtain a lower bound on the free boundary.

The result (1.4) shows that the boundary is not parabolic for t small. There has been some confusion about this point. It arises because the assumption that $b(t) \sim \alpha_0\sqrt{t}$ for $t \ll 1$, with α_0 constant, leads to an equation for α_0 for the American call in [2] and [3]. Suppose we assume that $b(t) \sim \alpha_0\sqrt{t}$ in the integral equation

(3.3) and evaluate both sides asymptotically for $t \ll 1$. The leading order terms yield

$$\begin{aligned} \rho t &\sim -\rho t \int_0^1 z \left[-\frac{\alpha_0(1-\sqrt{z})}{2(1-z)} + \frac{\alpha_0}{2\sqrt{z}} \right] \frac{\exp\left[-\alpha_0^2(1-\sqrt{z})^2/(4(1-z))\right]}{\sqrt{4\pi(1-z)}}\, dz \\ &+ \frac{1}{\sqrt{\pi}} \left[\int_{-\alpha_0-(3-\rho)\sqrt{t}}^{-\alpha_0+(\rho-1)\sqrt{t}} e^{-u^2} du + \alpha_0 \sqrt{t} \int_{-\alpha_0-(3-\rho)\sqrt{t}}^{\infty} e^{-u^2} du \right] . \end{aligned} \tag{5.10}$$

Here we have used the substitution $s = tz$ to eliminate t from the first integral on the right. The last term on the right in (5.10) is $O(\sqrt{t})$ while the left side is $O(t)$. Since the left and right sides do not have the same asymptotic behavior for $t \ll 1$, there is no solution α_0.

Acknowledgments

We thank Professor Jonathan Goodman for bringing this problem to our attention and for helpful discussions of it, and Dr. F.A. Aitsahlia and Professor Tse L. Lai for a preprint and discussion of their work.

References

[1] G. Barles, J. Burdeau, M. Romano, Nicholas Samsoen, Critical stock price near expiration, *Mathematical Finance* **5** pp. 77-95, (1995).

[2] H.P. McKean, Jr., Appendix A: A free boundary problem for the heat equation arising from a problem in mathematical economics. *Industrial Management Review*, **6**, pp.32-39 (1965).

[3] P. Van Moerbeke, On Optimal Stopping and Free Boundary Problems, *Arch. Ration. Mech. Anal.* **60**, pp. 101-148, (1976).

[4] I.J. Kim, The analytic valuation of the American options, *Rev. Fin. Stud.* **3** pp. 547-572 (1990).

[5] G. Barone-Adesi and R. Elliott, Approximations for the values of American options, *Stochastic Analysis and Applications* **9**, pp. 115-131, (1991).

[6] P. Wilmott,J. Dewynne, S. Howison, *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press, Oxford, 1993.

[7] F. Aitsahlia and T. Lai, Approximation for American options, preprint, 1996.

[8] C.M. Bender and S.A. Orszag, Advanced mathematical methods for scientists and engineers, McGraw-Hill, NY 1978.

Appendix

We shall now show that in (3.3), $I(t) \sim \rho t/2$ for $t \ll 1$, as we stated in section 4. We divide the integral into two parts, $I_1(t)$ and $I_2(t)$:

$$\begin{aligned} I &= \int_0^\delta \left[-\frac{b(t)-b(s)}{2(t-s)} + \frac{1}{2}(\rho-1) + b'(s) \right] (e^{\rho s}-1) G[b(t),t;b(s),s] ds \\ &+ \int_\delta^t \left[-\frac{b(t)-b(s)}{2(t-s)} + \frac{1}{2}(\rho-1) + b'(s) \right] (e^{\rho s}-1) G[b(t),t;b(s),s] ds \equiv I_1 + I_2. \end{aligned} \tag{A.1}$$

First, we bound $|I_1|$:

$$\begin{aligned} |I_1| &= \left| \int_0^\delta \left[-\frac{b(t)-b(s)}{2(t-s)} + \frac{1}{2}(\rho-1) + b'(s) \right] (e^{\rho s}-1) G[b(t),t;b(s),s] ds \right| \\ &\leq \int_0^\delta \rho s \left| \frac{b(t)-b(s)}{2(t-s)} \right| \frac{t^{\gamma(t,\delta)}}{\sqrt{4\pi(t-s)}} ds + \int_0^t \rho s \left| \frac{1}{2}(\rho-1) \frac{t^{\gamma(t,\delta)}}{\sqrt{4\pi(t-s)}} \right| ds \\ &+ \int_0^t |b'(s)| \, \rho s \frac{t^{\gamma(t,\delta)}}{\sqrt{4\pi(t-s)}} ds \end{aligned} \tag{A.2}$$

where

$$\gamma(t,\delta) = \frac{1}{2} \left((\rho-1) \frac{t-\delta}{b(t)} \right)^2 > 0 \; . \tag{A.3}$$

Using the result that $b(t) \sim -\sqrt{2t \log t^{-1}}$ in (A.2) yields $|I_1| = o(t)$.

Next we consider I_2. For $0 < \delta < t$ we write the integral as

$$\int_\delta^t f(s;t) e^{-\lambda^2 \phi(s;t)} ds \tag{A.4}$$

where $\lambda^2 = \alpha^2(t)/4$ and $\phi(s;t) = \left[\frac{b(t)-b(s)+(\rho-1)(t-s)}{\alpha(t)(t-s)^{1/2}} \right]^2$. We evaluate this integral asymptotically for $\alpha \gg 1$, noting that the main contribution comes from the upper limit. (See e.g. Bender and Orszag [8].) Thus

$$\begin{aligned} I_2 &\sim \frac{1}{\sqrt{4\pi}} \left[\frac{1}{2} b'(t) + \frac{1}{2}(\rho-1) \right] \rho t \int_\delta^t \frac{1}{\sqrt{t-s}} e^{-\lambda^2 |\phi'(t;t)|(t-s)} ds \\ &\sim \frac{1}{\sqrt{4\pi}} \left[\frac{1}{2} b'(t) + \frac{1}{2}(\rho-1) \right] \rho t \frac{2}{\lambda \sqrt{|\phi'(t,t)|}} \int_0^{\lambda \sqrt{|\phi'(t;t)|(t-\delta)}} e^{-z^2} dz \; . \end{aligned} \tag{A.5}$$

Since $\lambda^2 |\phi'(t;t)| = (b'(t) + \rho - 1)^2/4$, the upper limit of the last integral in (A.5) tends to infinity as t tends to zero. Then the asymptotic behavior of I_2 is

$$I_2 \sim \frac{\sqrt{\pi}}{2} \frac{f(t;t)}{\lambda \sqrt{|\phi'(t;t)|}} \sim \frac{\rho t}{2} \text{ for } t \ll 1. \tag{A.6}$$

Thus $I(t) = I_1 + I_2 = o(t) + \rho t/2$ for $t \ll 1$, as we stated.