

American options with dividends near expiry

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Abstract

Explicit expressions valid near expiry are derived for the values and the optimal exercise boundaries of American put and call options with dividends. The results depend sensitively on the ratio of the dividend yield rate D to the interest rate r . For $D > r$ the put boundary near expiry tends parabolically to the value rK/D where K is the strike price, while for $D \leq r$ the boundary tends to K in the parabolic-logarithmic form found for the case $D = 0$ by Barles *et al.* (1995) and by Kuske and Keller (1998). For the call, these two behaviors are interchanged—parabolic and tending to DK/r for $D < r$, as was shown by Wilmott *et al.* (1993), and parabolic-logarithmic and tending to K for $D \geq r$. The results are derived twice—once by solving an integral equation, and again by constructing matched asymptotic expansions.

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1 Introduction and formulation

For times near expiry, we shall derive explicit analytic expressions for the optimal exercise boundaries and for the values of American put and call options on an asset with a constant dividend yield rate D . The results show how the exercise boundary and the value of the option depend upon D , the interest rate r , and the volatility σ . In particular, they show that the behavior of the optimal exercise boundary changes significantly as D changes from less than to greater than r . They apply for times near expiry where the behavior is singular, and where numerical methods and other types of approximations are inaccurate [5]. Therefore the results provide starting values for numerical methods which can be used to calculate the option value and the optimal exercise boundary away from expiry.

The value $P(S, t)$ of an American put option on an asset with constant dividend yield rate D is a function of the asset price S and the time t . $P(S, t)$ and the optimal exercise boundary $S = S_f(t)$ satisfy the following conditions [6]:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D)S \frac{\partial P}{\partial S} - rP = 0, \quad \text{in } 0 < t < T_F, \quad S > S_f(t) \quad (1.1)$$

$$P = K - S_f, \quad \frac{\partial P}{\partial S} = -1, \quad \text{at } S = S_f(t) \quad (1.2)$$

$$P \sim 0, \quad \text{as } S \rightarrow +\infty \quad (1.3)$$

$$P(S, T_F) = \max(K - S, 0), \quad S_f(T_F^-) = \begin{cases} K & \text{if } D \leq r, \\ \frac{r}{D}K & \text{if } D \geq r. \end{cases} \quad (1.4)$$

In (1.4) K is the exercise or strike price, T_F is the time of expiry, the final value of P is the payoff, and the value of S_f is its limit as t tends to T_F from below.

By analysis of this problem for $T_F - t \ll 1$, i.e. for time near expiry, we shall obtain the following results for the optimal exercise boundary for the put:

$$D < r, \quad S_f(t) \sim K - K\sigma\sqrt{(T_F - t) \ln[\sigma^2 / (8\pi(T_F - t)(r - D)^2)]}, \quad (1.5)$$

$$D = r, \quad S_f(t) \sim K - K\sigma\sqrt{2(T_F - t) \ln[1 / (4\sqrt{\pi}D(T_F - t))]}, \quad (1.6)$$

$$D > r, \quad S_f(t) \sim \frac{r}{D}K \left(1 - \sigma\alpha_0\sqrt{2(T_F - t)}\right). \quad (1.7)$$

In (1.7), α_0 is a numerical constant determined by the transcendental equation (4.11). Equation (1.7) shows that for $D > r$ the boundary is parabolic, and as t tends to T_F , $S_f(t)$ tends to the value rK/D which is less than the strike price K . On the other hand, for $D < r$, (1.5) shows that

as t tends to T_F , $S_f(t)$ tends to K , but it is not parabolic. Instead, it resembles the parabolic–logarithmic boundary found in the absence of dividends by Barles et al. [1], Kuske and Keller [8], Stamicar *et al.* [10], and Chen *et al.* [3], and it reduces to that boundary for $D = 0$.

An expression for $S_f(t)$ for t near T_F which holds uniformly in D for $D \leq r$ is

$$S_f(t) \sim K - K\sigma\sqrt{2(T_F - t)}\alpha\left[\sigma^2(T_F - t)/2\right], \quad D \leq r, \quad (1.8)$$

Here the positive function $\alpha(\tau)$ is the solution of the equation (2.19). For $D = r$, (1.8) reduces to (1.6), while for $(r - D)^2/D^2 \gg \sigma^2(T_F - t)$, (1.8) reduces to (1.5). For $D > r$ the free boundary $S_f(t)$ for $\sigma^2(T_F - t)$ small can be found by solving (2.20), which can also be approximated by a transcendental equation. The solution tends to (1.7) for $\sigma^2(T_F - t) \gg (D - r)^2/D^2$.

Analogous results for the optimal exercise boundary of an American call option on an asset with dividends are

$$D > r, \quad S_f(t) \sim K + K\sigma\sqrt{(T_F - t)\ln[\sigma^2/(8\pi(T_F - t)(r - D)^2)]}, \quad (1.9)$$

$$D = r, \quad S_f(t) \sim K + K\sigma\sqrt{2(T_F - t)\ln[1/(4\sqrt{\pi}D(T_F - t))]}, \quad (1.10)$$

$$D < r, \quad S_f(t) \sim \frac{r}{D}K \left(1 + \sigma\alpha_0\sqrt{2(T_F - t)}\right). \quad (1.11)$$

The parabolic form (1.11) of the boundary for $D < r$ was obtained previously by Wilmott, Dewynne and Howison [11]. An expression for $S_f(t)$ for t near T_F which holds uniformly in D for $D \geq r$ is given by

$$D \leq r, \quad S_f(t) \sim K + K\sigma\sqrt{2(T_F - t)}\alpha\left[\sigma^2(T_F - t)/2\right]. \quad (1.12)$$

The positive function $\alpha(\tau)$ is the solution of the analog of equation (2.19).

The results in (1.5)–(1.12) and corresponding formulas for the values of the options are derived in the remainder of this paper by two different methods. The first method, based upon the representation of $P(S, t)$ as a convolution integral involving the fundamental solution of (1.1), leads to an integral equation for $S_f(t)$. This is the method which was used in [8]. The second method is the method of matched asymptotic expansions [2]. Each method has its advantages and disadvantages, as we will point out later. However, they both lead to the same results, so they confirm each other. These two complementary methods are commonly used in solving partial differential equations (pde’s) with moving boundaries [4].

Our result in [8] for the case $D = 0$ has an incorrect numerical factor, as was pointed out by John Chadam and also by Charles Knessl. The numerical factor 6 in (1.1), (1.3) and (1.4) of [8] should be

4, and the factor 3 in (4.2) should be 2. We have traced this to an incorrect asymptotic evaluation of the integral I_1 in that paper, which should be $I_1 = -\rho t/2 + o(t)$, and then $I = I_1 + I_2 = o(t)$. By a different method we did obtain the correct result in (5.6) of [8], but we considered that method to be just qualitatively correct.

2 The dimensionless problem and the integral equation

It is convenient to define new dimensionless independent variables x and τ in place of S and t , and new dependent variables $p(x, \tau)$ and $x_f(\tau)$ in place of $P(S, t)$ and $S_f(t)$:

$$S = Ke^x, \quad t = T_F - \frac{2}{\sigma^2}\tau, \quad P = Ke^{-\rho\tau}p(x, \tau) + K - S, \quad S_f = Ke^{x_f}. \quad (2.1)$$

The parameters ρ and ν are defined by

$$\rho = \frac{2r}{\sigma^2}, \quad \nu = \frac{2D}{\sigma^2}, \quad (2.2)$$

which may be termed the dimensionless interest rate and dividend yield rate respectively. Then (1.1)–(1.4) can be rewritten in the dimensionless form

$$\frac{\partial p}{\partial \tau} = \frac{\partial^2 p}{\partial x^2} + (\rho - \nu - 1) \frac{\partial p}{\partial x} + e^{\rho\tau} (\nu e^x - \rho) \quad \text{in } 0 < \tau < \frac{\sigma^2}{2}T_F, x_f(\tau) < x \quad (2.3)$$

$$p = \frac{\partial p}{\partial x} = 0 \quad \text{at } x = x_f(\tau), \quad (2.4)$$

$$p \sim e^{\rho\tau} (e^x - 1) \quad \text{as } x \rightarrow +\infty, \quad (2.5)$$

$$p = \max(e^x - 1, 0) \quad \text{at } \tau = 0,$$

$$x_f(0) = x_0 = \begin{cases} 0 & \text{if } \nu \leq \rho, \\ -\log(\nu/\rho) & \text{if } \nu > \rho. \end{cases} \quad (2.6)$$

To solve this problem we introduce the causal fundamental solution or Green's function $G(x, \tau; \xi, s)$ for (2.3), given by

$$G(x, \tau; \xi, s) = \frac{1}{\sqrt{4\pi(\tau-s)}} e^{-\frac{[x-\xi+(\rho-\nu-1)(\tau-s)]^2}{4(\tau-s)}}, \quad s < \tau. \quad (2.7)$$

As in [8], we apply Green's theorem to $p(x, \tau)$ and G in a domain bounded by the optimal exercise boundary and the line $\tau = 0$ to obtain

$$\begin{aligned} p(x, \tau) &= \int_0^\infty (e^\xi - 1)G(x, \tau; \xi, 0)d\xi + \int_0^\tau e^{\rho s} \int_{x_f(s)}^\infty (\nu e^\xi - \rho)G(x, \tau; \xi, s)d\xi ds, \\ &= I^{(1)}(x, \tau) + I^{(2)}(x, \tau). \end{aligned} \quad (2.8)$$

From (2.4), it follows that on the optimal exercise boundary $p_\tau[x_f(\tau), \tau] = 0$, which Chen *et al.* [3] have found to be a most useful equation. Upon using (2.8) in it, we obtain the following integral equation for $x_f(\tau)$:

$$\frac{\partial I^{(1)}}{\partial \tau}[x_f(\tau), \tau] = - \lim_{x \rightarrow x_f(\tau)} \frac{\partial I^{(2)}}{\partial \tau}[x, \tau]. \quad (2.9)$$

In $I_\tau^{(2)}(x, \tau)$, we take the limit so that the singularity of G remains inside the domain of integration, rather than on its boundary.

The integrals $I^{(1)}$ and $I^{(2)}$ can be written as follows by using (2.7) for G :

$$\begin{aligned} I^{(1)}[x_f(\tau), \tau] &= \frac{1}{2} \left[e^{x_f(\tau) + (\rho - \nu)\tau} \operatorname{erfc} \left(- \left(\frac{x_f(\tau) + (\rho - \nu + 1)\tau}{2\sqrt{\tau}} \right) \right) \right. \\ &\quad \left. - \operatorname{erfc} \left(- \left(\frac{x_f(\tau) + (\rho - \nu - 1)\tau}{2\sqrt{\tau}} \right) \right) \right], \end{aligned} \quad (2.10)$$

$$\begin{aligned} I^{(2)}[x, \tau] &= \frac{1}{2} \left[\nu e^{x + (\rho - \nu)\tau} \int_0^\tau e^{\nu s} \operatorname{erfc} \left(\frac{x_f(s) - x - (\rho - \nu + 1)(\tau - s)}{2\sqrt{\tau - s}} \right) ds \right. \\ &\quad \left. - \rho \int_0^\tau e^{\rho s} \operatorname{erfc} \left(\frac{x_f(s) - x - (\rho - \nu - 1)(\tau - s)}{2\sqrt{\tau - s}} \right) ds \right]. \end{aligned} \quad (2.11)$$

Here $\operatorname{erfc} z$ denotes the complementary error function

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt \sim \frac{e^{-z^2}}{\sqrt{\pi}z} \quad \text{as } z \rightarrow +\infty. \quad (2.12)$$

To solve the integral equation (2.9), we write $x_f(\tau)$ in terms of x_0 defined in (2.6), and of a new unknown function $\alpha(\tau)$:

$$x_f(\tau) = x_0 - 2\tau^{1/2}\alpha(\tau). \quad (2.13)$$

The quotient $-x_f(\tau)/\tau^{1/2}$ in (2.10) is equal to $-x_0\tau^{-1/2} + 2\alpha(\tau)$. For $\nu > \rho$ the first term tends to $+\infty$ as $\tau \rightarrow 0$, and we will show that the second term is positive. Thus the quotient tends to $+\infty$ as $\tau \rightarrow 0$ for $\nu > \rho$. For $\nu \leq \rho$, $x_0 = 0$ so the quotient is just $2\alpha(\tau)$, and we will show that it tends to $+\infty$ as $\tau \rightarrow 0$. It follows that as $\tau \rightarrow 0$ both of the arguments of erfc in (2.10) tend to $+\infty$, so the asymptotic form in (2.12) is applicable. By using it in (2.10) we can write $I^{(1)}$ in the asymptotic form

$$I^{(1)}[x_f(\tau), \tau] \sim \frac{2\tau^{3/2}}{\sqrt{\pi}x_f^2(\tau)} e^{-x_f^2(\tau)/4\tau} \quad \text{as } \tau \rightarrow 0. \quad (2.14)$$

Then

$$\frac{\partial I^{(1)}}{\partial \tau}[x_f(\tau), \tau] \sim \frac{e^{-x_f^2(\tau)/4\tau}}{2\sqrt{\pi}\tau^{1/2}} \quad \text{as } \tau \rightarrow 0. \quad (2.15)$$

Next we rewrite the two arguments of erfc in (2.11) by using (2.13) for $x_f(s)$, and we also set $s = \tau z$. Then the arguments can be written as

$$\frac{x_f(s) - x - (\rho - \nu \pm 1)(\tau - s)}{2\sqrt{\tau - s}} = B(x, z, \tau) - \frac{(\rho - \nu \pm 1)}{2} \tau^{1/2} \sqrt{1 - z}, \quad (2.16)$$

where $B(x, z, \tau)$ is defined by

$$B(x, z, \tau) = \frac{[x_0 - x] / 2\tau^{1/2} - z^{1/2}\alpha(\tau z)}{\sqrt{1 - z}}. \quad (2.17)$$

Now we set $s = \tau z$ in (2.11) and use (2.16). Then both integrals in (2.11) have factors of τ . To evaluate $I^{(2)}$ through terms of order $\tau^{3/2}$, we expand the exponential functions and erfc to order $\tau^{1/2}$. In this way we obtain

$$\begin{aligned} I^{(2)}[x, \tau] &\sim (\nu e^x - \rho) \frac{\tau}{2} \int_0^1 \operatorname{erfc} B(x, z, \tau) dz + \frac{\tau^{3/2}}{2} \left[\nu e^{x_0} \frac{(\rho - \nu + 1)}{2} - \frac{\rho(\rho - \nu - 1)}{2} \right] \\ &\quad \cdot \int_0^1 \sqrt{1 - z} \frac{2}{\sqrt{\pi}} e^{-B^2(x, z, \tau)} dz \quad \text{as } \tau \rightarrow 0. \end{aligned} \quad (2.18)$$

To write the integral equation (2.9) for $\tau \rightarrow 0$ we use (2.15) for $I_\tau^{(1)}$ and we take the τ derivative of (2.18) to get $I_\tau^{(2)}$. When $\nu \leq \rho$ then $x_0 = 0$, $x_f = -2\tau^{1/2}\alpha(\tau)$ and (2.9) becomes after multiplication by $2\tau^{1/2}$

$$\begin{aligned} \frac{e^{-\alpha^2(\tau)}}{\sqrt{\pi}} &\sim \lim_{x \rightarrow x_f(\tau)} \left[\left(\tau^{1/2}(\rho - \nu) + 2\nu\tau\alpha(\tau) \right) \left(\int_0^1 \operatorname{erfc} B(x, z, \tau) dz - \tau \frac{2}{\sqrt{\pi}} \int_0^1 B_\tau[x, z, \tau] e^{-B^2[x(\tau), z, \tau]} dz \right) \right. \\ &\quad - \left[\rho + \nu - (\rho - \nu)^2 \right] \left(\frac{3\tau}{2\sqrt{\pi}} \int_0^1 \sqrt{1 - z} e^{-B^2[x, z, \tau]} dz \right. \\ &\quad \left. \left. - \frac{2\tau^2}{\sqrt{\pi}} \int_0^1 \sqrt{1 - z} B[x, z, \tau] B_\tau[x, z, \tau] e^{-B^2[x, z, \tau]} dz \right) \right]. \end{aligned} \quad (2.19)$$

On the other hand, when $\nu \geq \rho$ then $x_0 = -\log(\nu/\rho)$ and (2.9) becomes

$$\begin{aligned} \frac{e^{-\left(\frac{-x_0}{2\sqrt{\tau}} + \alpha(\tau)\right)^2}}{\sqrt{\pi}} &\sim \rho\tau \lim_{x \rightarrow x_f(\tau)} \left[2\alpha(\tau) \left(\int_0^1 \operatorname{erfc} B(x, z, \tau) dz - \frac{2\tau}{\sqrt{\pi}} \int_0^1 B_\tau(x, z, \tau) e^{-B^2[x(\tau), z, \tau]} dz \right) \right. \\ &\quad \left. - \frac{3}{\sqrt{\pi}} \int_0^1 \sqrt{1 - z} e^{-B^2[x_f(\tau), z, \tau]} dz + \frac{4\tau}{\sqrt{\pi}} \int_0^1 \sqrt{1 - z} B[x_f(\tau), z, \tau] B_\tau[x_f(\tau), z, \tau] e^{-B^2[x_f(\tau), z, \tau]} dz \right]. \end{aligned} \quad (2.20)$$

In the last two integrals, the limit can be taken under the integral. Then in these integrals (2.17) and (2.13) show that

$$B[x_f(\tau), z, \tau] = [\alpha(\tau) - z^{1/2}\alpha(\tau z)] (1 - z)^{-1/2}. \quad (2.21)$$

In the following section we show that the third and fourth integrals on the right hand side of (2.19) are of higher order, but we have included them to demonstrate that for $\rho = \nu$, (2.19) and (2.20) are identical.

3 Solution of the integral equations

For $D \leq r$, i.e. $\nu \leq \rho$, the first integral on the right side of (2.19) tends to 0 as $\tau \rightarrow 0$, the next term tends to 2, and the third term is $o(\tau)$. (See Appendix A.) Therefore (2.19) yields to leading order,

$$\begin{aligned} \frac{e^{-\alpha^2(\tau)}}{\sqrt{\pi}} &\sim 2(\rho - \nu)\tau^{1/2}, & \nu < \rho, \\ \frac{e^{-\alpha^2(\tau)}}{\sqrt{\pi}} &\sim 4\nu\tau\alpha(\tau), & \nu = \rho. \end{aligned} \quad (3.1)$$

For $\nu < \rho$ the solution of (3.1) for $\alpha(\tau)$ is

$$\alpha^2(\tau) \sim \ln \frac{1}{2(\rho - \nu)\sqrt{\pi\tau}}, \quad \nu < \rho. \quad (3.2)$$

For $\nu = \rho$ the solution of (3.1) for $\alpha(\tau)$ is

$$\alpha^2(\tau) \sim \ln \frac{1}{4\sqrt{\pi\nu\tau}}, \quad \nu = \rho. \quad (3.3)$$

Then $S_f(t)$ is given by

$$S_f(t) = Ke^{x_f(\tau)} \sim K + Kx_f(\tau) \sim K - 2K\tau^{1/2} [\ln 1/2(\rho - \nu)\sqrt{\pi\tau}]^{1/2}, \quad D < r, \quad (3.4)$$

$$S_f(t) = Ke^{x_f(\tau)} \sim K + Kx_f(\tau) \sim K - 2K\tau^{1/2} \left[\ln \frac{1}{4\sqrt{\pi\nu\tau}} \right]^{1/2}, \quad D = r. \quad (3.5)$$

The two results (3.4) and (3.5) are both special cases of the uniform expression $x_f(\tau) = -2\tau^{1/2}\alpha(\tau)$ with $\alpha(\tau)$ the solution of (2.19). In terms of the original variables this is (1.8), while (3.4) and (3.5) are (1.5) and (1.6). When $D = 0$ so that $\nu = 0$, (3.4) is exactly the corrected leading order result of Kuske and Keller [8] and the leading order result of Chen *et al.* [3].

For $D > r$, i.e. $\nu > \rho$, (2.20) applies. Since $x_0 = -\log(\nu/\rho) < 0$, the left side is negligible for $\sqrt{\tau} \ll -x_0$. Then (2.20) can be satisfied by choosing $\alpha(\tau) = \alpha_0$ to be that constant which makes

the coefficient of τ on the right side vanish. Thus α_0 must satisfy

$$\begin{aligned} 2\alpha_0 \int_0^1 \operatorname{erfc} \left(\alpha_0 \frac{1-\sqrt{z}}{\sqrt{1-z}} \right) dz + \frac{2\alpha_0^2}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{1-z}} e^{-\alpha_0^2 \frac{(1-\sqrt{z})^2}{1-z}} dz = \\ \frac{3}{\sqrt{\pi}} \int_0^1 \sqrt{1-z} e^{-\alpha_0^2 \frac{(1-\sqrt{z})^2}{1-z}} dz + \frac{2\alpha_0^2}{\sqrt{\pi}} \int_0^1 \sqrt{\frac{1-\sqrt{z}}{1+\sqrt{z}}} e^{-\alpha_0^2 \frac{(1-\sqrt{z})^2}{1-z}} dz, \quad \nu > \rho. \end{aligned} \quad (3.6)$$

The unique positive solution of (3.6) is

$$\alpha_0 \approx .4517\dots, \quad \nu > \rho \quad (3.7)$$

Now

$$S_f(t) = K e^{x_f(\tau)} \sim \frac{r}{D} K \left[1 - 2\alpha_0 \tau^{1/2} \right], \quad D > r. \quad (3.8)$$

In terms of the original variables this is (1.7). The value (3.7) of α_0 is the same as that obtained for the American call with dividends $D < r$ in [11]. This is obvious from the analysis in the following section, in which the method of matched asymptotic expansions yields a transcendental equation (4.11) which is identical to that obtained in [11].

We have now obtained three results for $\alpha(\tau)$, namely (3.2) for $\nu < \rho$, (3.3) for $\nu = \rho$, and (3.7) for $\nu > \rho$. The first two are unified by taking α to be the solution of (2.19) which describes the transition as $\nu^2 \tau / (\rho - \nu)^2$ varies from 0 to a large value. The transition between (3.5) for $\nu = \rho$ and (3.7) for $\nu > \rho$ is given by the solution of (2.20), which can be solved numerically.

We note that when α is given by (3.2)-(3.3) then $\alpha(\tau z) / \alpha(\tau) \sim 1$ for z away from zero, and when α is given by (3.7) then $\alpha(\tau z) / \alpha(\tau) = 1$. Therefore we make the approximation $\alpha(\tau z) / \alpha(\tau) \sim 1$ in (2.21) for $B[x_f(\tau), z, \tau]$ to obtain $B[x_f(\tau), z, \tau] \sim \alpha(\tau) B_0(z)$ where

$$B_0(z) = (1 - z^{1/2}) / \sqrt{1-z}. \quad (3.9)$$

Then both (2.19) and (2.20) become transcendental equations for $\alpha(\tau)$. The integrals in these equations can be evaluated numerically and the equations can be solved for $\alpha(\tau)$. In Figure 3.1 we show the value of α as a function of $r - D$, which demonstrates the transition between $D < r$ and $D > r$. In Figure 3.2 we show graphs of $\alpha(\tau)$ obtained in this way for four different values of r/D and we also show the graphs of the corresponding value of $x_f(\tau)$.

4 Matched asymptotic expansions for the put

We shall now use the method of matched asymptotic expansions to construct the small τ behavior of $p(x, \tau)$ and $x_f(\tau)$ for the American put. First we consider $D \leq r$, i.e. $\nu \leq \rho$, in which case $x_0 = 0$.

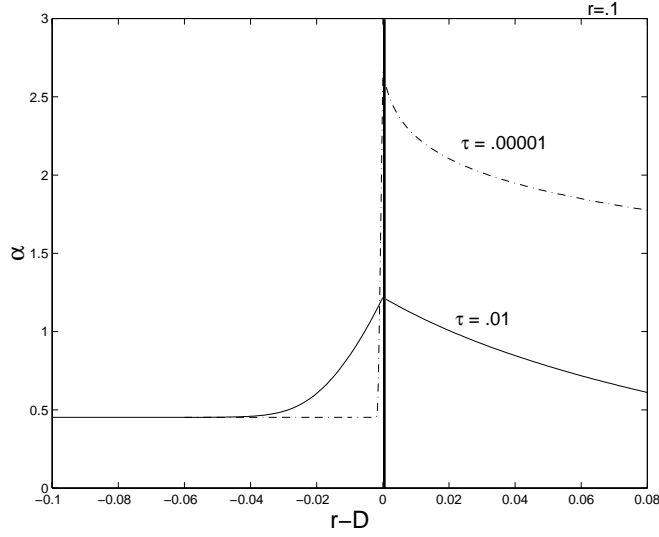


Figure 3.1: Graph of $\alpha(\tau)$ vs. $r - D$, for $r = .1$ and $\sigma = .25$ at two values of τ ($\tau = .01$ and $\tau = .00001$) obtained by solving the approximate forms of (2.19)-(2.20). For $r - D < 0$ and τ small, α is a constant independent of r and D , except for values of D such that $\log D/r \ll \sqrt{\tau}$.

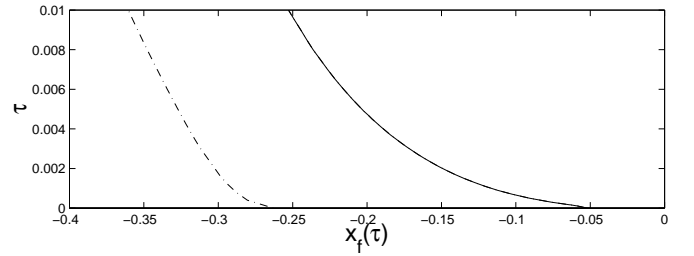
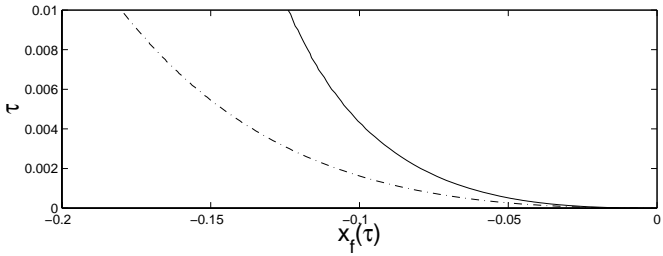
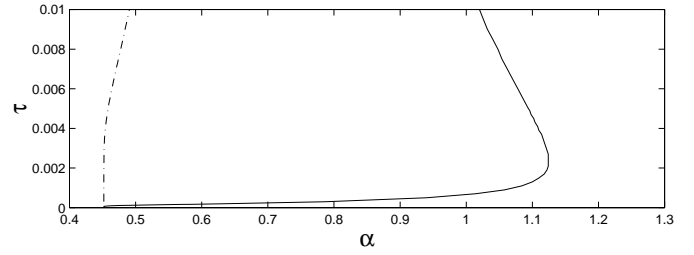
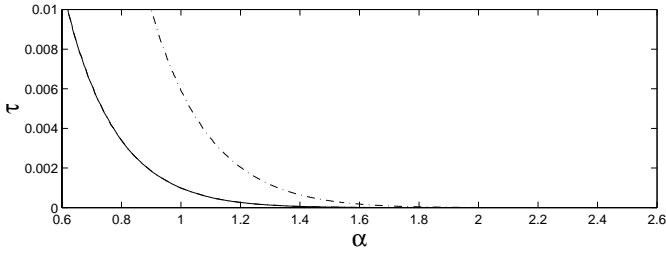


Figure 3.2a

Figure 3.2b

Figure 3.2: Figure 3.2a: The top graph is $\alpha(\tau)$ for $r = .1$, $D = .09$ (dotted line) and $D = .05$ (solid line). The bottom graph is $x_f(\tau)$ corresponding to $\alpha(\tau)$ in the top graph. Figure 3.2b: The top graph is $\alpha(\tau)$ for $r = .1$, $D = .15$ (dotted line) and $D = .105$ (solid line). The bottom graph is $x_f(\tau)$ corresponding to $\alpha(\tau)$ in the top graph.

As in the case of no dividends ($D = 0$) [8], the free boundary $x_f(\tau) \rightarrow 0$ as $\tau \rightarrow 0$, but it does not have parabolic behavior for $D < r$. This is easily verified by substituting a parabolic ansatz $x_f(\tau) \sim \alpha_0 \sqrt{\tau}$ into (2.9), and showing that there is no constant α_0 which satisfies the resulting equation. Instead $x_f(\tau)/\sqrt{\tau} \rightarrow -\infty$ as $\tau \rightarrow 0$ as in [8].

For $\tau \rightarrow 0$, the following three layer structure is derived in Appendix B.1. There is an outer expansion valid for $x > 0$:

$$p(x, \tau) = e^x - 1 + \rho(e^x - 1)\tau + O(\tau^2), \quad x > 0. \quad (4.1)$$

This expansion breaks down as $x \rightarrow 0^+$, and it begins to fail when $x = O(\sqrt{\tau})$. Then p is represented instead by the inner expansion

$$p(x, \tau) \sim \tau^{1/2} h_0 \left(\frac{x}{2\sqrt{\tau}} \right) + \tau h_1 \left(\frac{x}{2\sqrt{\tau}} \right) + \tau^{3/2} h_2 \left(\frac{x}{2\sqrt{\tau}} \right) + O(\tau^2), \quad x = O(\sqrt{\tau}). \quad (4.2)$$

Here

$$h_0(\zeta) = \frac{1}{\sqrt{\pi}} e^{-\zeta^2} + \zeta \operatorname{erfc}(-\zeta), \quad (4.3)$$

while $h_1(\zeta)$ and $h_2(\zeta)$ are defined by the boundary-value problems (B.5), (B.6) and (B.7), (B.8) respectively. They have the following asymptotic behaviors as $\zeta \rightarrow -\infty$:

$$\begin{aligned} h_0(\zeta) &\sim \frac{1}{2\sqrt{\pi}} e^{-\zeta^2} \left[\frac{1}{\zeta^2} + O(\zeta^{-4}) \right], & h_1(\zeta) &\sim (\nu - \rho) + O\left(\zeta^{-1} e^{-\zeta^2}\right), \\ h_2(\zeta) &\sim 2\nu\zeta + O\left(e^{-\zeta^2}\right). \end{aligned} \quad (4.4)$$

Near the moving boundary, where $x = x_f(\tau) + O(\tau)$, p has the behavior

$$p(x, \tau) \sim O(\tau^2), \quad x - x_f(\tau) = O(\tau). \quad (4.5)$$

Now we match the values of p_τ given by (4.2) and (4.5), recalling that $x_f(\tau)/\sqrt{\tau} \rightarrow -\infty$. To do so we use the asymptotic forms given in (4.4), and we obtain the transcendental equation

$$\begin{aligned} \frac{1}{2\sqrt{\pi\tau}} e^{-x_f^2/4\tau} + \nu - \rho, & \quad \nu < \rho, \\ \frac{1}{2\sqrt{\pi\tau}} e^{-x_f^2/4\tau} + \nu x_f, & \quad \nu = \rho. \end{aligned} \quad (4.6)$$

This is the same as (3.1).

For $D > r$ and $\tau \ll 1$, the asymptotic expansion of $p(x, \tau)$ is derived in Appendix B.2.

For $\sqrt{\tau} \ll x_0$ and $x > x_0$, $p(x, t)$ has the outer expansion

$$p(x, \tau) = \max(e^x - 1, 0) + (\nu e^x - \rho)\tau + O(\tau^2), \quad x > x_0. \quad (4.7)$$

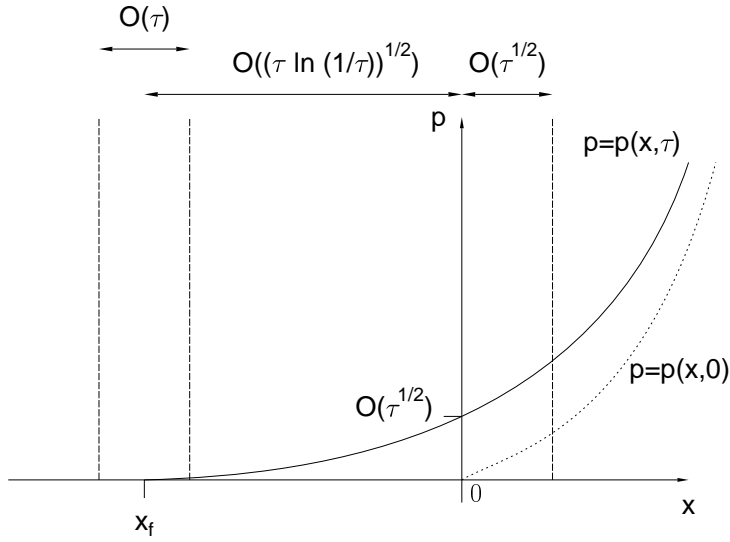


Figure 4.1: A schematic showing the asymptotic structure for small-time of the dimensionless value $p(x, \tau)$ of the American put in the case $D \leq r$ i.e. $\nu \leq \rho$. The construction of the asymptotic expansion uses an $O(\sqrt{\tau})$ inner layer about $x = 0$, in which $p(x, t)$ is $O(\sqrt{\tau})$. The location of the free boundary is $O(\sqrt{\tau \ln 1/\tau})$, so that it is outside the $O(\sqrt{\tau})$ inner layer. There is also an $O(\tau)$ layer at the free boundary, in which $p(x, \tau) = O(\tau^2)$.

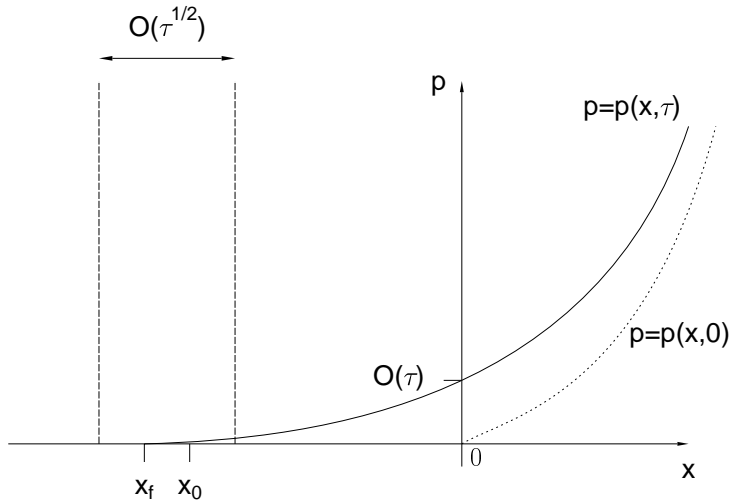


Figure 4.2: A schematic showing the asymptotic structure for small-time of the American put in the case $D > r$ i.e. $\nu > \rho$. In this case the construction of the asymptotic expansion uses only one $O(\sqrt{\tau})$ layer at the free boundary, in which $p(x, \tau)$ is $O(\tau^{3/2})$. At $x = 0$, $p(x, \tau) = O(\tau)$.

For $x - x_0 = O(\sqrt{\tau})$, $p(x, \tau)$ has the inner expansion

$$p(x, \tau) = \tau^{3/2} g\left(\frac{x - x_0}{2\sqrt{\tau}}\right) + O(\tau^2), \quad x - x_0 = O(\sqrt{\tau}). \quad (4.8)$$

Here $g(\zeta)$ is given by

$$g(\zeta) = 2\rho\zeta + 2\rho\alpha_0 \frac{i^3 \operatorname{erfc}(\zeta)}{i^3 \operatorname{erfc}(-\alpha_0)}, \quad i^3 \operatorname{erfc}(\zeta) = \frac{1}{6} \left(\frac{(1 + \zeta^2)}{\sqrt{\pi}} e^{-\zeta^2} - \left(\frac{3}{2}\zeta + \zeta^3 \right) \operatorname{erfc}(\zeta) \right). \quad (4.9)$$

The free boundary has the asymptotic form

$$x_f \sim x_0 - 2\alpha_0 \sqrt{\tau}. \quad (4.10)$$

When (4.8) is used in the boundary conditions on this free boundary, the following transcendental equation is obtained for α_0 :

$$-\alpha_0^3 e^{\alpha_0^2} \int_{\alpha_0}^{\infty} e^{-u^2} du = \frac{1}{4}(1 - 2\alpha_0^2). \quad (4.11)$$

This equation has the solution (3.7). It is also identical to the transcendental equation obtained for the free boundary of the American call with dividends, $D < r$ [11].

5 Matched asymptotic expansions for the put in the transition region

To analyze the solution in the transition region where $\nu - \rho$ is small and positive, we introduce the small parameter $\epsilon = -x_0 = \ln(\nu/\rho) > 0$ and rescale as follows:

$$x = \epsilon X, \quad x_f = -\epsilon L_f(T), \quad \tau = \epsilon^2 T. \quad (5.1)$$

Then we obtain the problem

$$\frac{\partial p}{\partial T} = \frac{\partial^2 p}{\partial X^2} + \epsilon(\rho - \nu - 1) \frac{\partial p}{\partial X} + \epsilon^2 e^{\epsilon^2 \rho T} (\nu e^{\epsilon X} - \rho), \quad \text{in } -L_f(T) < X, \quad T > 0 \quad (5.2)$$

$$p = \frac{\partial p}{\partial X} = 0 \quad \text{at } X = -L_f(T), \quad (5.3)$$

$$p \sim e^{\epsilon^2 \rho T} (e^{\epsilon X} - 1) \quad \text{as } X \rightarrow +\infty, \quad (5.4)$$

$$p = \max(e^{\epsilon X} - 1, 0), \quad L_f(0) = 1, \quad \text{at } T = 0. \quad (5.5)$$

To solve, we assume that p has the expansion

$$p = \epsilon \hat{p}_0 + \epsilon^2 \hat{p}_1 + \epsilon^3 \hat{p}_2 + \dots. \quad (5.6)$$

Upon using (5.6) in (5.2)–(5.5) we are led to the following sequence of problems:

For \hat{p}_0

$$\frac{\partial \hat{p}_0}{\partial T} = \frac{\partial^2 \hat{p}_0}{\partial X^2} \quad \text{in} \quad -L_f < X < +\infty, \quad T > 0, \quad (5.7)$$

$$\hat{p}_0 = \frac{\partial \hat{p}_0}{\partial X} = 0 \quad \text{at} \quad X = -L_f, \quad (5.8)$$

$$\hat{p}_0 \sim X \quad \text{as} \quad X \rightarrow +\infty, \quad (5.9)$$

$$\hat{p}_0 = \max(X, 0), \quad L_f = -1 \quad \text{at} \quad T = 0. \quad (5.10)$$

For \hat{p}_1 ,

$$\frac{\partial \hat{p}_1}{\partial T} = \frac{\partial^2 \hat{p}_1}{\partial X^2} - \frac{\partial \hat{p}_0}{\partial X} \quad \text{in} \quad -L_f < X < +\infty, \quad T > 0, \quad (5.11)$$

$$\hat{p}_1 = \frac{\partial \hat{p}_1}{\partial X} = 0 \quad \text{at} \quad X = -L_f, \quad (5.12)$$

$$\hat{p}_1 \sim X^2/2, \quad \text{as} \quad X \rightarrow +\infty, \quad (5.13)$$

$$\hat{p}_1 = \begin{cases} X^2/2 & \text{if } X \geq 0, \\ 0 & \text{if } X < 0, \end{cases} \quad \text{at} \quad T = 0. \quad (5.14)$$

For \hat{p}_2

$$\frac{\partial \hat{p}_2}{\partial T} = \frac{\partial^2 \hat{p}_2}{\partial X^2} - \frac{\partial \hat{p}_1}{\partial X} - \rho \frac{\partial \hat{p}_0}{\partial X} + \rho(1 + X) \quad \text{in} \quad -L_f < X < +\infty, \quad T > 0, \quad (5.15)$$

$$\hat{p}_2 = \frac{\partial \hat{p}_2}{\partial X} = 0 \quad \text{at} \quad X = -L_f, \quad (5.16)$$

$$\hat{p}_2 \sim \frac{X^3}{6} + \rho XT \quad \text{as} \quad X \rightarrow +\infty, \quad (5.17)$$

$$\hat{p}_2 = \begin{cases} X^3/6 & \text{if } X \geq 0, \\ 0 & \text{if } X < 0, \end{cases} \quad \text{at} \quad T = 0. \quad (5.18)$$

These problems do not have similarity solutions like those in section 4. However, the solutions can be given in terms of a fundamental solution as in section 3. The significance of the scaling is that it reduces the problem to one in which the fundamental solution is relatively simple. Using it we can obtain the small τ behavior of the solution with results equivalent to those of the previous sections. For small T , $\sqrt{\tau} \ll x_0$ and we recover the parabolic behavior of the free boundary. For large T , $x_0 \ll \sqrt{\tau} \ll 1$, and we obtain a parabolic-logarithmic behavior which is similar to the result for $\nu < \rho$. In the previous sections we gave similarity solutions for \hat{p}_0 , \hat{p}_1 , and \hat{p}_2 in these two different ranges.

6 American call option

The problem for the value $C(S, t)$ of the American call option, and the optimal exercise boundary $S = S_f(t)$ is:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D) S \frac{\partial C}{\partial S} - rC = 0 \quad \text{in } 0 < t < T_F, \quad S < S_f(t), \quad (6.1)$$

$$C = S_f - K, \quad \frac{\partial C}{\partial S} = 1 \quad \text{on } S = S_f(t), \quad (6.2)$$

$$C = 0 \quad \text{at } S = 0, \quad (6.3)$$

$$C = \max(S - K, 0), \quad S_f = \begin{cases} \frac{r}{D}K & \text{if } D \leq r, \\ K & \text{if } D > r, \end{cases} \quad \text{at } t = T_F. \quad (6.4)$$

We introduce the change of variables (2.1), the dimensionless parameters (2.2) and

$$C = Ke^{-\rho\tau}c(x, \tau) + S - K. \quad (6.5)$$

Then we obtain the dimensionless problem

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (\rho - \nu - 1) \frac{\partial c}{\partial x} + e^{\rho\tau}(\rho - \nu e^x) \quad \text{in } 0 < \tau < \frac{\sigma^2}{2}T_F, \quad x < x_f(\tau), \quad (6.6)$$

$$c = 0, \quad \frac{\partial c}{\partial x} = 0 \quad \text{at } x = x_f(\tau), \quad (6.7)$$

$$c \sim 0 \quad \text{as } x \rightarrow -\infty, \quad (6.8)$$

$$c = \max(1 - e^x, 0) \quad \text{at } \tau = 0, \quad (6.9)$$

$$x_f(0) = x_0 = \begin{cases} \log(\rho/\nu) & \text{if } \nu \leq \rho, \\ 0 & \text{if } \nu > \rho. \end{cases} \quad (6.10)$$

This problem (6.6)–(6.10) can be analyzed by both the methods used for the put problem, so we shall just describe the results.

The small time behavior of the solution of the call problem for $D < r$ is similar to that of the put problem for $D > r$. It is given by Wilmott *et al.* (1993) and can be described as follows. As $\tau \rightarrow 0$ for $x < x_0$, c is given by the outer expansion

$$c = \max(1 - e^x, 0) + \tau(\rho - \nu e^x) + O(\tau^2), \quad x < x_0. \quad (6.11)$$

For $x - x_0 = O(\sqrt{\tau})$, c is given by an inner expansion with the leading order term

$$c \sim \tau^{3/2}g\left(\frac{x_0 - x}{2\sqrt{\tau}}\right), \quad x - x_0 = O(\sqrt{\tau}). \quad (6.12)$$

Here $g(\zeta)$ is given in (4.9), α_0 satisfies the transcendental equation (4.11), and the moving boundary is given by

$$x_f \sim x_0 + 2\alpha_0\sqrt{\tau}. \quad (6.13)$$

These results hold for $x_0 > 0$ with $x_0 = O(1)$.

In the case $D > r$ we obtain results analogous to those of section 2. The transcendental equation for the free boundary is

$$\begin{aligned} \frac{1}{2\sqrt{\pi\tau}}e^{-x_f^2/4\tau} + \rho - \nu &= 0, & \nu > \rho \\ \frac{1}{2\sqrt{\pi\tau}}e^{-x_f^2/4\tau} - \nu x_f(\tau) &= 0, & \nu = \rho. \end{aligned} \quad (6.14)$$

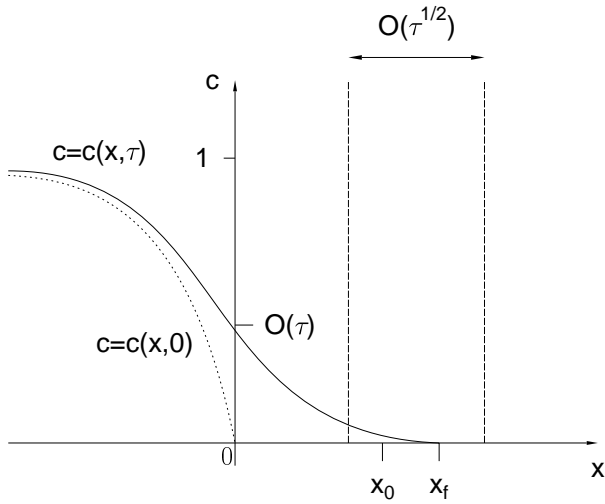


Figure 6.1: A schematic showing the asymptotic structure for small-time of the American call in the case $D \leq r$ i.e. $\nu \leq \rho$. Here the layer structure is similar to the case of the American put with $D \geq r$. Then the asymptotic expansion uses only one $O(\sqrt{\tau})$ layer at the free boundary, in which $c(x, \tau)$ is $O(\tau^{3/2})$. At $x = 0$, $c(x, \tau) = O(\tau)$.

7 Discussion

We have analyzed American put and call options on assets with dividends at times near expiry. The analysis shows that the local behavior of the optimal exercise boundary depends on whether the boundary reaches the strike price or not at expiry. A faster variation of the boundary near expiry occurs when it reaches the strike price. These results may be useful, since it is precisely near expiry that the optimal exercise boundary is difficult to track numerically from the partial differential equation formulation.

It is interesting to observe that the leading order terms in the transcendental equations (3.1) and (6.14) for the put and call optimal exercise boundaries can also be obtained by equating the value of the corresponding European option to the payoff function. This is so when the optimal exercise boundary reaches the strike price at expiry, i.e. when $D < r$ for the put and $D > r$ for the call. However, using the European options in this way is not correct for the put when $D > r$ or for the call when $D < r$.

Our results have been derived both by the method of integral equations and the method of matched asymptotic expansions. The integral equation method is similar to that used in [8] for an American option without dividends. An advantage of this method is that the integral equation is valid for all times, so that the behavior away from expiry can be obtained from it by iterative or numerical methods. The method of matched asymptotic expansions generates a sequence of reduced problems for the local behavior near expiry. The advantage of this method is that it is applicable to problems where the fundamental solution for the full problem is not available. Its limitation is that it is valid only for times near expiry. Other methods, such as Laplace transforms (Knessl [7]), Fourier transforms (Stamilar *et al.*, [10]), and cumulative distribution functions (Kwok [9]) have also been applied to these problems.

Acknowledgement

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A Asymptotic evaluation of the integrals

We shall evaluate the first three integrals in (2.19) asymptotically for τ tending to zero. They all involve $B(x, z, \tau)$ defined by (2.17):

$$B(x, z, \tau) = \frac{x_f(s) - x}{2\sqrt{\tau - s}} = \frac{-x/2 - \sqrt{\tau z}\alpha(\tau z)}{\sqrt{\tau}\sqrt{1 - z}}. \quad (\text{A.1})$$

The function $x_f(s) = -2\sqrt{s}\alpha(s)$ is monotone decreasing, equals zero at $s = 0$, and equals $-2\sqrt{\tau}\alpha(\tau)$ at $s = \tau$. Therefore there is a unique value $s_0(x)$ at which $x_f(s) = x$. If we define $\tau z_0(x) = s_0(x)$, then $B = 0$ at $z = z_0$ while $B > 0$ for $0 < z < z_0$ and $B < 0$ for $z_0 < z < 1$. Then (A.1) shows that $B \rightarrow \infty$ as $\tau \rightarrow 0$ when $0 < z < z_0$ and $B \rightarrow -\infty$ as $\tau \rightarrow 0$ when $z_0 < z < 1$. Thus for τ small we

can replace $\operatorname{erfc} B$ by its asymptotic forms for $B \rightarrow \pm\infty$ in the first integral in (2.19):

$$\begin{aligned} \int_0^1 \operatorname{erfc} B(x, z, \tau) dz &\sim \frac{1}{\sqrt{\pi}} \int_0^{z_0(x)} B^{-1} e^{-B^2} dz + \int_{z_0(x)}^1 \left(2 + B^{-1} \frac{e^{-B^2}}{\sqrt{\pi}} \right) dz \\ &= 2[1 - z_0(x)] + \frac{1}{\sqrt{\pi}} \int_0^{z_0(x)} B^{-1} e^{-B^2} dz \end{aligned} \quad (\text{A.2})$$

To evaluate the last integral in (A.2) and the second and third integrals in (2.19), we shall use Laplace's method. All three integrals contain the factor e^{-B^2} , which tends to zero as $\tau \rightarrow 0$ because $B^2 \rightarrow \infty$. Then the main contribution to the integrals comes from the neighborhood of $z_0(x)$ where $B = 0$. Then $B^2(z_0) = 0$ and $(B^2)_z = 0$ at z_0 , so near z_0 , $B^2(z) = \frac{1}{2} \partial_z^2 B^2(z_0)(z - z_0)^2 + \dots = [B_z(z_0)]^2 (z - z_0)^2 + \dots$. When we use this in the exponent, we obtain Gaussian integrals to which we can apply the formula,

$$\int_0^1 A(z) e^{-B^2(z)} dz \sim A(z_0) \int_0^1 e^{-B_z^2(z_0)(z-z_0)^2} dz \sim A(z_0) \frac{\sqrt{\pi}}{|B_z(z_0)|} \quad (\text{A.3})$$

This result does not apply to (A.2) since $A(z) = B^{-1}(z)$ which is infinite at z_0 . In that case we add and subtract the term $[B_z(z_0)(z - z_0)]^{-1}$ to $B^{-1}(z)$ to get

$$\begin{aligned} B^{-1} &= B^{-1} - [B_z(z_0)(z - z_0)]^{-1} + [B_z(z_0)(z - z_0)]^{-1} \\ &= \frac{B_z(z_0)(z - z_0) - B(z)}{B(z)B_z(z_0)(z - z_0)} + [B_z(z_0)(z - z_0)]^{-1} \sim -\frac{B_{zz}(z_0)}{2B_z^2(z_0)} + [B_z(z_0)(z - z_0)]^{-1} \end{aligned} \quad (\text{A.4})$$

Thus,

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \int_0^1 B^{-1} e^{-B^2} dz &\sim \frac{1}{\sqrt{\pi}} \int_0^1 \left(-\frac{B_{zz}(z_0)}{2B_z^2(z_0)} + [B_z(z_0)(z - z_0)]^{-1} \right) e^{-B_z^2(z_0)(z-z_0)^2} dz \\ &\sim -\frac{B_{zz}(z_0)}{2|B_z(z_0)|^3} \end{aligned} \quad (\text{A.5})$$

Here we have used the fact that $(z - z_0)^{-1}$ is odd about z_0 .

Now we take the limit as $x \rightarrow x_f(\tau)$. Then we just set $z_0 = 1$ in (A.2)-(A.5). The limit of the first integral in (2.19), given by (A.2), is asymptotic to the right side of (A.5) at $z_0 = 1$. From (A.1) we see that this expression is

$$\frac{2z_0}{[\alpha(\tau z_0) + 2\tau z_0 \alpha'(\tau z_0)]^2} = O(\alpha^{-2}). \quad (\text{A.6})$$

Similarly, we evaluate the second and third integrals in (2.19). For the second integral multiplied by $2\tau/\sqrt{\pi}$, (A.3) yields

$$2 \frac{\frac{-x}{4\tau^{1/2}} + z_0^{3/2} \alpha'(\tau z_0) \tau}{\frac{\alpha(\tau z_0)}{2\sqrt{z_0}} + \alpha'(\tau z_0) \sqrt{z_0} \tau}, \quad (\text{A.7})$$

which tends to 2 as $x \rightarrow x_f(\tau)$ and $z_0 \rightarrow 1$. For the third integral, (A.3) is $O(\tau)$, which we neglect in (3.1).

B Matched asymptotic expansions

B.1 Initial behavior of $p(x, \tau)$ for $D \leq r$.

To analyse the small-time behaviour of the system (2.3)–(2.6) we set

$$\tau = \theta T . \tag{B.1}$$

Here $T = O(1)$ and θ is an artificial small parameter. Then the problem for $p(x, t)$ becomes for $D < r$:

$$\begin{aligned} \frac{\partial p}{\partial T} &= \theta \left[\frac{\partial^2 p}{\partial x^2} + (\rho - \nu - 1) \frac{\partial p}{\partial x} + e^{\theta \rho T} (\nu e^x - \rho) \right] \\ p(x_f, T) &= \frac{\partial}{\partial x} p(x_f, T) = 0, \quad p(x, T) = e^{\theta \rho T} (e^x - 1), \quad \text{as } x \rightarrow \infty \\ p(x, 0) &= \begin{cases} 0, & x < 0, \\ e^x - 1, & x \geq 0. \end{cases} \end{aligned}$$

In the limit $\theta \rightarrow 0$ we obtain the following three layer structure:

An expansion in regular powers of θ gives the outer expansion valid for $x > 0$ and $x = O(1)$,

$$p = e^x - 1 + \theta \rho (e^x - 1) T + O(\theta^2), \quad x > 0. \tag{B.2}$$

To satisfy the boundary conditions we introduce a local expansion in the region $x = x_f(\tau) + \theta z$ with $z = O(1)$. There the problem for $p(x, \tau) = p(z, T)$ is given by

$$\begin{aligned} \theta \frac{\partial p}{\partial T} - \frac{dx_f}{d\tau} \frac{\partial p}{\partial z} &= \frac{\partial^2 p}{\partial z^2} + \theta (\rho - \nu - 1) \frac{\partial p}{\partial z} + \theta^2 e^{\theta \rho T} (\nu e^{x_f + \theta z} - \rho) \\ p(x_f, T) &= \frac{\partial}{\partial z} p(x_f, T) = 0, \end{aligned}$$

which yields

$$p = O(\theta^2). \tag{B.3}$$

Since the outer expansion breaks down when $x = O(\theta^{1/2})$, we require an inner region with the scaling $x = \theta^{1/2} X$ that bridges between the outer region and a region near x_f . Where $X = O(1)$ we write

$$p(x, \tau) = \theta^{1/2} P_0(X, T) + \theta P_1(X, T) + \theta^{3/2} P_2(X, T) + O(\theta^2). \tag{B.4}$$

This leads to the following sequence of problems:

(i) First for P_0 we have

$$\frac{\partial P_0}{\partial T} = \frac{\partial^2 P_0}{\partial X^2} \quad \text{in } -\infty < X < +\infty, \quad T > 0,$$

$$P_0(X, 0) = \max(X, 0), \quad \text{as } X \rightarrow -\infty, \quad P_0 \rightarrow 0, \quad \text{as } X \rightarrow +\infty, \quad P_0 \sim X.$$

This has the solution $P_0 = \sqrt{T} h_0(\zeta)$ where $\zeta = X/2\sqrt{T}$ and $h_0(\zeta)$ is given in (4.3).

(ii) For P_1 we obtain

$$\frac{\partial P_1}{\partial T} = \frac{\partial^2 P_1}{\partial X^2} + (\rho - \nu - 1) \frac{\partial P_0}{\partial X} + (\nu - \rho) \quad \text{in } -\infty < X < +\infty, \quad T > 0,$$

$$P_1(X, 0) = \begin{cases} \frac{1}{2}X^2, & X \geq 0, \\ 0, & X < 0, \end{cases} \quad \text{as } X \rightarrow -\infty, \quad \frac{\partial P_1}{\partial X} \rightarrow 0, \quad \text{as } X \rightarrow +\infty, \quad P_1 \sim \frac{1}{2}X^2.$$

This has the solution $P_1 = T h_1(\zeta)$ where $h_1(\zeta)$ satisfies,

$$\text{in } -\infty < \zeta < +\infty, \quad h_1'' + 2\zeta h_1' - 4h_1 = 2(1 - \nu - \rho)h_0' + 4(\nu - \rho), \quad (\text{B.5})$$

$$\text{as } \zeta \rightarrow -\infty, \quad h_1' \rightarrow 0, \quad \text{as } \zeta \rightarrow +\infty, \quad h_1 \sim 2\zeta. \quad (\text{B.6})$$

(iii) Finally for P_2 we have

$$\frac{\partial P_2}{\partial T} = \frac{\partial^2 P_2}{\partial X^2} + (\rho - \nu - 1) \frac{\partial P_1}{\partial X} + \nu X \quad \text{in } -\infty < X < +\infty, \quad T > 0,$$

$$P_2(X, 0) = \max\left(\frac{1}{6}X^3, 0\right), \quad \text{as } X \rightarrow -\infty, \quad \frac{\partial P_2}{\partial X} \rightarrow 0, \quad \text{as } X \rightarrow +\infty, \quad P_2 \sim \frac{1}{6}X^3 + \rho XT.$$

This has the solution $P_2 = T^{3/2} h_2(\zeta)$ where $h_2(\zeta)$ satisfies

$$\text{in } -\infty < \zeta < +\infty, \quad h_2'' + 2\eta h_2' - 6h_2 = 2(1 - \nu - \rho)h_1' - 8\nu\zeta, \quad (\text{B.7})$$

$$\text{as } \zeta \rightarrow -\infty, \quad h_2' \rightarrow 0, \quad \text{as } \zeta \rightarrow +\infty, \quad h_2 \sim \frac{4}{3}\zeta^3 + 2\rho\zeta, \quad (\text{B.8})$$

The forms of h_1 and h_2 as $X \rightarrow -\infty$, needed for matching purposes, are

$$h_1 \sim \nu - \rho + \frac{1}{2\sqrt{\pi}}(1 + \nu - \rho) \frac{e^{-\zeta^2}}{\zeta} \quad \text{as } \zeta \rightarrow -\infty,$$

$$h_2 \sim 2\nu\zeta + \frac{1}{4\sqrt{\pi}}(1 + \nu - \rho)^2 e^{-\zeta^2} \quad \text{as } \zeta \rightarrow -\infty.$$

B.2 Initial behaviour for the put with $D > r$

For $D > r$ we again use the scaling (B.1), and we obtain the outer expansion

$$p = \max(e^x - 1, 0) + (\nu e^x - \rho)T\theta + O(\theta^2), \quad x > x_0. \quad (\text{B.9})$$

An inner region near the initial position of the optimal exercise boundary is then required in order to satisfy the moving boundary conditions. We introduce the scalings

$$x = x_0 + \theta^{1/2} X, \quad p = \theta^{3/2} P_0 \quad x_f = x_0 + \theta^{1/2} L_0. \quad (\text{B.10})$$

This leads to the problem,

$$\text{in } L_0(T) < X < +\infty, \quad T > 0 \quad \frac{\partial P_0}{\partial T} = \frac{\partial^2 P_0}{\partial X^2} + \rho X, \quad (\text{B.11})$$

$$\text{at } X = L_0(T) \quad P_0 = \frac{\partial P_0}{\partial X} = 0, \quad (\text{B.12})$$

$$\text{as } X \rightarrow +\infty \quad P_0 \sim \rho X T, \quad (\text{B.13})$$

$$\text{at } T = 0 \quad P_0 = 0, \quad L_0 = 0. \quad (\text{B.14})$$

The boundary condition (B.13) follows from matching with the outer solution (B.9) for $x_0 < x < 0$.

Written in terms of the inner variables, the outer solution then has the behavior $p(x, t) \sim \rho X T$.

The problem (B.11)–(B.14) has the similarity solution

$$P_0 = T^{3/2} g(\zeta), \quad \zeta = \frac{X}{2\sqrt{T}}, \quad L_0 = 2\alpha_0 \sqrt{T},$$

with $g(\zeta)$ and α_0 given by (4.9) and (4.11).

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