Simulations of Option Pricing

By: Sarah Thompson

INTRODUCTION

The goal of this thesis is to determine if the Binomial Tree model is a good estimator for the value of a European put option, and if it s a good estimator for the Optimal Exercise Boundary for the American put option.

An option is a contract that gives the holder the right to buy or sell an asset at a specified price, on a specified date. The specified price is called the exercise or strike price and the specified date is called the expiry date.[1] The call option gives the holder the right to buy an asset at the strike price, therefore the higher the asset price, the more valuable the call option. If the asset price falls below the strike price, the holder would most likely not exercise the option, but purchase the asset in the market. The put option gives the holder the right to sell an asset at the strike price, therefore the lower the asset price the more valuable the option. If the asset price rises above the strike price, the holder would most likely not exercise the option, but sell the asset in the market.[2]

The value of an option can be calculated for anytime up to the expiry date, and holds probability information on what the asset price might do. As time reaches expiry, the uncertainty decreases and the option value approaches the value on expiry, which is simply the difference between the exercise price and the asset price, or zero, which ever is larger. For the call option, the value on expiry is, specifically, the current market price of the asset minus the exercise price, if greater than zero, since that is what the holder is saving by exercising the option and not buying the asset out in the market. Similarly, for the put option, the value is the exercise price minus the current market price for the asset, if greater than zero, since that s how much more the holder is making on the sale of the asset by not selling it in the market. The European option must be exercised on the expiry date, and the American option may be exercised at any time up to or on the expiry date.[3]

For the American option, where exercise can occur any time up to expiry, there is an additional question that asks when is it more profitable to exercise the option early than to hold on to it. This is called the Optimal Exercise Boundary, and if the difference between the current asset price and the exercise price is greater than the value of the option calculated for that time then it is said to more profitable to exercise early. Once the Optimal Exercise Boundary is found, all asset prices greater than, for a call option, or less than, for a put option, the boundary should all be exercised early. If the current asset price is not on the early exercise side of the boundary, the holder of the option should wait to exercise the option until early exercise is optimal, exercise the option on expiry, or never exercise the option.

The value of an option can be calculated using either the Binomial Tree method or the Black Scholes method, which are explained in detail below. The Optimal Exercise Boundary can be estimated using the Binomial Tree and an equation derived by J.D. Evans, R. Kuske, and Joseph B. Keller, also described below. Once calculated, different variables of the option value will be varied to determine how accurately the different models calculate the same value. The focus of these calculations is the European Put Option value, and the American Put Option Optimal Exercise Boundary.

METHODOLOGY

Black-Scholes Method

The Black-Scholes method is an exact calculation of the value of a European option for a predetermined stream of asset prices. The result of this calculation was the basis for determining how accurate an estimator the Binomial Tree method is.

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[1] Paul Wilmott, Sam Howison, and Jeff Dewynne, The Mathematics of Financial Derivatives (New York: Cambridge University Press. 1995), 4-6.
[2] Wilmott, 6.
[3] Wilmott, 13.

The stream of asset prices is generated using a lognormal distribution, or lognormal random walk. The main characteristic of the lognormal distribution is that the natural log of the random variable has a normal distribution. Another property used in describing asset prices is the Markov Property, which states that only the current value is used in determining future values.[4] Since we assume that the past history of an asset is reflected in its current price, this shows it satisfies the Markov Property.[5] The final assumptions used are that the expected return of the asset is the risk free interest rate and we cannot predict future values of the asset.[6]

So taking all that into account, the following formula is developed:

$$S_{n+1} = S_n + rS_n \Delta t + \sigma S_n \Delta X$$

with Δt equal to the length of time between steps, ΔX a standard normal random variable with mean 0 and variance Δt, σ equal to the asset s volatility, and S_n is the asset price at time n.

With further analysis, taking the natural log of S_n, we find that:

$$\ln S_n \sim \phi \ \ln S_0 + (\mu - \frac{\sigma^2}{2})(t-n), \sigma\sqrt{t-n}$$

and

$$\varepsilon[S_n] = Se^{\mu(t-n)} \qquad {}^{7}$$

where ε is the expected value, and t is expiry. This confirms that the expected return of the asset is the risk free interest rate, μ, later renamed r.

The variables and initial conditions used were as follows:

$S_0 =$	Initial asset price	= \$50
$r(=\mu) =$	Risk-free interest rate	= 0.1
$q =$	Dividend rate	= 0
$qt =$	Time of dividend	= 0
$\sigma =$	Asset volatility	= 0.4
$t =$	Length of time	= .41095 years ≈ 150 days

NOTE: If $q > 0$, and discrete, then $S_{0'} = S_0 * (1-q) * e^{-qt*r}$
If $q > 0$, and continuous, then replace r with $(r-q)$

It was assumed that the risk-free interest rate and asset volatility remained constant throughout the specified time period.

Programmed using *C++*, this process was repeated 100 times, with the asset prices averaged to create a stable random stream of prices.

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[4] John C. Hull, Options, Futures, and Other Derivatives, Third Edition (New Jersey: Prentice-Hall, Inc. 1997), 209.
[5] Wilmott, 19.
[6] Wilmott, 18.
[7] Hull, 229.

The next step was to calculate the value of the put option using the stream of asset prices created. This was done using the Black-Scholes equation. The main assumptions used in the Black-Scholes equation are that there is no arbitrage, defined as the possibility to make instant risk-free profit[8], the expected return is the risk free interest rate, the stock price follows the lognormal random walk, and there are no transaction costs.[9]

The Black-Scholes equation can be derived by taking the expectation of the present value of the option value at expiry. We will derive the equation for the call option value, and then use the Put-call Parity to obtain the value for the put option. The value of the call option at expiry is,

$$C(S_T, t) = \max(S_T - E, 0) \qquad \text{for} \qquad t = T$$

and using the assumption that the expected return is the risk-free interest rate, we can rewrite this above equation for any time t as,

$$C = e^{-r(T-t)} \varepsilon \left[\max(S_T - E, 0) \right] \qquad (*) \qquad \text{such that} \qquad 0 \le t \le T$$

Recall that the asset price follows the lognormal distribution, and it turns out that the probability distribution function for S is

$$f(S) = \frac{1}{\sigma S \sqrt{2t\pi}} e^{-(\log(\frac{S}{S_0}) - (\mu - \frac{1}{2}\sigma^2)t)^2 / 2\sigma^2 t} \qquad \text{10}$$

So (*) becomes

$$C = \max([e^{-r(T-t)} \int_E^{\times} (S_T - E) f(S_T) dS_T], 0) \qquad (**)$$

Note that the limits on the integral go from E to $\times$, this is the case since if the asset price was below E, the value would be negative and the max function would use 0 instead. Evaluating (**) gives the following formula for the call option,

$$C = \max([SN(d_1) - Ee^{-r(T-t)} N(d_2)], 0)$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\times}^{x} \exp(-\frac{1}{2} y^2) dy \qquad \text{, the cdf for the normal distribution}$$

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$

8 Wilmott, 33.
9 Wilmott, 41-42.
10 Wilmott, 28

Now, to obtain the formula for the value of the put option, we use the Put-Call Parity, which describes the relationship between the put and call option. The Put-call Parity says that given a portfolio, Π, that contains an asset, S, call option, C, and put option, P, where the two options have the same expiry, the value of the portfolio, when exercised, is the following,

$$\Pi = S + P - C$$

Basically, your total portfolio value is the asset you have, S, plus the amount you could potentially earn for exercising the put, P, minus the amount potentially spent on exercising the call, C. At expiry the realized value of the portfolio is the following,

$$S + \max(E - S, 0) - \max(S - E, 0).$$

The present value of the portfolio is therefore,

$$S + P - C = Ee^{-r(T-t)} \quad \text{for} \quad 0 \le t \le T$$ [11]

Using the Put-call Parity along with the fact that

$$N(d) + N(-d) = 1$$

by the symmetry of the standard normal distribution, the value for the put option becomes

$$P(S,t) = E\exp(-r(T-t))N(-d_2) - SN(-d_1) \quad \text{for} \quad t < T$$ [12]

The variables and initial conditions used were as follows:

$S =$	Asset price	where $S_0 = 50$
$E =$	Exercise / Strike price	= 50
$r =$	Risk-free interest rate	= 0.1
$q =$	Dividend rate	= 0.0
$\sigma =$	Asset volatility	= 0.4
$T =$	Length of time of option	= .41095 years ≈ 150 days
$steps =$	Number of time steps	= 149
$\Delta t =$	Time between steps	= T/steps
$t =$	point in time of stock price	= $n\Delta t$ for S_n

Again, the risk-free interest rate and asset volatility were assumed to remain constant.

Once P, the put option value, is calculated, the actual option value is taken to be $\max(P,0)$, since the option would only be exercised if the value was greater than zero. These simulations were done using *Matlab* for the integration, and *C++* for the remainder.

Binomial Tree Method

The Binomial Tree Method uses the idea that at each small time step, the asset price can either increase or decrease, in other words there are two different possibilities at each point. So, Su is the new asset price if it increases, and Sd is the new asset price if it decreases. The probability

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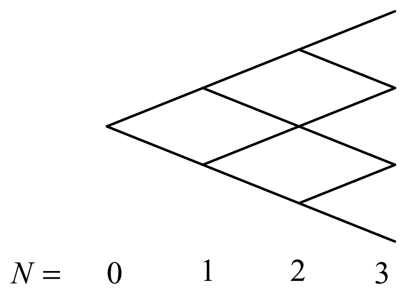
[11] Wilmott, 41.
[12] Wilmott, 79.

that the asset increases is p, therefore the probability it decreases is $(1-p)$. This process is repeated for very small Δt, until time T is reached. Following the Binomial Distribution, each probability can be calculated using its probability function:

$$P(S_n = a_{n,k}) = \binom{n}{k} p^k (1-p)^{n-k} \qquad for \quad k = 0,1,\ldots,n .$$ [13]

Where n is the time step in the Binomial Tree, k is one of $n+1$ possible asset prices for that time step, and $a_{n,k}$ is the asset price in the tree. The tree-like structure looks like the following,



where for time $N = n_0$, there are $n_0 + 1$ nodes. Each node has an asset price, option value, and probability, as shown above. The mean asset price at each time step would be the sum of the products, at each node for time n_0, of the asset price and corresponding probability. The asset prices are calculated using the following formula:

$$S(i+1) = Su^j d^{i-j} \qquad \text{where} \quad j = 0,1,\ldots,i \qquad \text{and} \qquad i = 0,1,\ldots,N .$$ [14]

Basically, at time $i+1$, the asset changed i times, and could be any combination of increasing and decreasing. One assumption used is that $ud = 1$, so if the asset increased and then decreased, it would be back at the original starting asset price. The formulas for p, u, and d are as follows:

$$p = \frac{e^{r\Delta t} - d}{u - d} \quad , \qquad u = e^{\sigma\sqrt{\Delta t}} \quad , \qquad d = e^{-\sigma\sqrt{\Delta t}} .$$

These are derived from the assumption that the expected return for an asset is the risk-free interest rate, therefore you can discount asset prices using that rate, and that the mean and variance of the change in asset price are as follows:

$$Se^{r\Delta t} = pSu + (1-p)Sd$$

$$S^2 e^{2r\Delta t}(e^{\sigma^2 \Delta t} - 1) = pS^2u^2 + (1-p)S^2d^2 - S^2[pu + (1-p)d]^2$$ [15]

The variables in the Binomial Tree, and initial conditions used are as follows:

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[13] Rick Durrett, Essentials of Stochastic Processes (New York: Springer-Verlag New York, Inc. 1999), 8.
[14] Hull, 343-345.
[15] Hull, 344-345.

$S =$	Initial asset price	= 50
$X =$	Exercise / Strike price	= 50
$r =$	Risk-free interest rate	= 0.1
$q =$	Dividend rate	= 0.0
$\sigma =$	Asset volatility	= 0.4
$T =$	Time length of option	= .41095 years = 150 days
$N =$	Number of intervals in tree	= 599
$\Delta t =$	Time between intervals in tree	= T / N
$u =$	Upward movement of asset	$= e^{\sigma\sqrt{\Delta t}}$
$d =$	Downward movement of asset	$= e^{-\sigma\sqrt{\Delta t}}$
$p =$	Probability of up movement	$= \dfrac{e^{r\Delta t} - d}{u - d}$

It is important to pick N large enough so Δt is small enough to provide for a good sample of asset prices in the Binomial Tree; specifically, looking up the predetermined asset prices in the Black-Scholes model is easier.

Once the asset prices are calculated for the whole tree, the values are then computed by starting at the last step in the tree and working backward, since the value on the expiry date is known. The formula for the option value at the final time is as follows:

$$f_{N,j} = \max(X - Su^j d^{N-j}, 0) \qquad j = 0,1,\ldots,N$$ [16]

and moving back one step at a time, the remainder of the option values are calculated as follows:

$$f_{i,j} = e^{-r\Delta t}\left\lfloor p f_{i+1,j+1} + (1-p) f_{i+1,j})\right\rfloor \qquad 0 \le i \le N-1, \quad 0 \le j \le i$$ [17]

The logic behind this calculation is for asset price (i, j), the probability it moves up is p, or the probability in one time step and has value $f_{i+1,j+1}$ is p. Then the probability it moves down and has value $f_{i+1,j}$ is $(1-p)$, and then discount it back one time step, Δt, for the present value of the option value.

To get the Optimal Exercise Boundary, for time step n, using the Binomial Tree, start from the bottom of the tree, $j = 0$, and search up until you find the first asset price such that:

$$X - Su^j d^{n-j} \ ? \ f_{i,j} \qquad 0 \le i \le N-1, \ j = 0,1,\ldots,N$$

Then take the previous stock price such that:

$$X - Su^{j-1} d^{n-(j-1)} < f_{i,j}$$

and linear interpolate to find an S such that:

$$X - S = f$$

16 Hull, 348.
17 Hull, 348.

This is the estimated Optimal Exercise Boundary. Please note that some initial conditions for this calculation were changed: the initial asset price and exercise price were reduced to 1, and the dividend rate was varied.

The above calculations were all done using *C++* code.

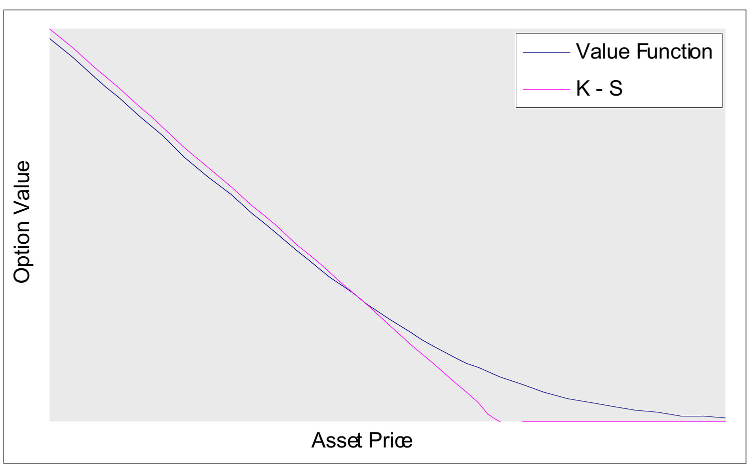
Optimal Exercise Boundary Equation
The Optimal Exercise Boundary equation for the American put option with dividends, derived by J. D. Evans, R. Kuske, and Joseph B. Keller, good only for $T_F - t \ll 1$, is as follows:

$$S_f(t) \sim K - K\sigma\left(\sqrt{(T_F - t)\ln[\sigma^2/(8(T_F - t)\pi(r-D)^2)]}\right) \qquad for \qquad D < r$$

$$S_f(t) = \frac{r}{D}K\left(1 - \sigma\alpha_0\sqrt{2(T_F - t)}\right), \qquad \alpha_0 \approx .495 \qquad for \qquad D > r$$ [18]

Where K is the exercise price, σ is the asset volatility, T_F is the expiry, D is the dividend rate, t is time being calculated for, and r is the risk-free interest rate. For any $S < S_f(t)$, it would be optimal to exercise the option early. Note that as $t \to T_F$ then $S_f(t) \to K$, or $S_f(t) \to \frac{r}{D}K$, and we know at expiry, the value of the option is $\max(K - S, 0)$, and therefore for any $S < K$ it would be optimal to exercise the option.

Graphically, what this equation determines it the point where the two functions graphed below intersect, at a specific time. That is, where the function of $K - S$ equals the calculated option value as a function of asset price.



The variables and initial conditions used were as follows:

$S =$	Asset price	where $S_0 = 1$
$K =$	Exercise / Strike price	= 1
$r =$	Risk-free interest rate	= 0.1
$D =$	Dividend rate	varied
$\sigma =$	Asset volatility	= 0.4

[18] J.D Evans, R. Kuske, and Joseph B. Keller. The Behavior Near Expiry for American Options With Dividends. March 31, 2000. (preprint).

$T =$ Length of time of option = .41095 years ≈ 150 days

Since the above formulas are good only for times very close to expiry, the times analyzed were from $(.386, .41095)$. This values were calculated using *C++* code.

Sensitivity Testing
The first objective in the sensitivity testing was to first see how the Binomial option values compared to the actual values, for the European put option, in the Black-Scholes model. One characteristic of the Binomial Tree model is that option prices are a function of both time and asset price. The initial asset price, for example, is not only found in the first step, but is found every two steps, since $Sud = S$, therefore when looking up option values in the Binomial Tree, both time and asset price need to be specified. In the Black-Scholes model the asset prices are predetermined, therefore the values are only a function of time.

Once the values in the initial case were compared, different variables were changed to compare to the initial case. The variables tested were the initial asset price, the exercise price, the risk-free interest rate, the volatility, and dividends.

This was done using lookup functions written in *C++* code.

In comparing the Binomial Tree and the Evans, Kuske, Keller equation, for the Optimal Exercise Boundary for American put options, only the dividend rate was varied. The boundary was simply calculated for each model, and then compared. Both methods were done in *C++*.

RESULTS — EUROPEAN PUT OPTION

Below are the results of the Binomial Tree calculation compared to the Black-Scholes calculation, broken down by variable, for the purpose of sensitivity testing. The synonymous case for each section is where S=E=50, r=10%, σ = 40%, and q = 0%. For each section, only the highlighted variable was varied, with all others remaining constant, with the exception of the Asset Price, where the Exercise Price was varied to always equal the Asset Price. Time steps are from time 0 to time 149, the expiry date, for a total of 150 time steps. The Binomial Tree was run using first 600 total time steps and then increasing to 900 total time steps to improve accuracy.

Asset Price
In testing the sensitivity of the asset price, the following initial prices were tested: 25, 50, 100, and 1000. The initial option values (time 0) were compared for all 4 cases, as well as a time closer to expiry. Overall, it appeared as though changing the asset price didn t affect the accuracy of the Binomial Tree prediction, and the Binomial Tree accurately predicted the initial value, and slightly overstated the value near expiry.

At time 0, these are values given by both models:

Asset Price	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
25	2.02768	2.02788	100.01%
50	4.05537	4.05576	100.01%
100	8.11074	8.11151	100.01%
1000	81.1074	81.1151	100.01%

Looking at the values as a function of asset price, it appears there exists a linear relationship between the two, for both models:

Black-Scholes model	**Binomial Tree model**
2.02768 * 2 = 4.05536	2.02788 * 2 = 4.05576
2.02768 * 4 = 8.11072	2.02788 * 4 = 8.11152
2.02768 * 40 = 81.1072	2.02788 * 40 = 81.1152

Looking at time 140 (of 149), the models give the following values:

Asset Price	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
25	0.59721	0.62205	104.16%
50	1.19441	1.24411	104.16%
100	2.38883	2.48821	104.16%
1000	23.88830	24.88210	104.16%

Again, the linear relationship can be seen between the asset price and the option value. The difference at this time, however, is that the Binomial Tree value is overstated more severely than at time 0.

Now, rerunning the Binomial Tree increasing the number of steps from 600 to 900, the values at time 140 given by the model are as follows:

Asset Price	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
25	0.59721	0.61330	102.69%
50	1.19441	1.22654	102.69%
100	2.38883	2.45308	102.69%
1000	23.88830	24.53080	102.69%

The increase of steps in the Binomial Tree has improved the accuracy of the option value.

In conclusion, the Binomial model is not affected by asset price, however, it s accuracy in calculating the option value appears to increase as the number of steps in the tree increase.

Risk-free Interest Rate

In looking at the risk-free interest rate, the following rates were looked at: .01, .05, .10, and .20. Again it appears that initially, the Binomial Tree accurately calculates the option value, and closer to expiry, the accuracy slightly worsens with improvement when the steps are increased.

At time 0, and initial asset price 50, these are values given by both models:

Risk-free Interest Rate	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
1%	4.98862	4.99052	100.04%
5%	4.55674	4.55792	100.03%
10%	4.05537	4.05576	100.01%
20%	3.17512	3.17415	99.97%

As can be seen here, as the interest rate increases, the Binomial Tree value goes from slightly over-calculating the option value to slightly under-calculating the option value.

Closer to expiry, here are values from both models:

Time	Asset Price	Risk-free Interest Rate	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
137	48.46 *	1%	2.31691	2.37735	102.61%
144	48.96	5%	2.49093	2.53838	101.90%
140	50.00	10%	1.19441	1.24411	104.16%
119	51.60	20%	1.29915	1.32409	101.92%

* Corresponding price in Binomial Model was 48.45

As can be seen, the option values are again overstated close to expiry. Two things to point out are first, for r = 1%, the price in the Binomial Model was actually lower than 48.46, this would suggest that the value found in the Binomial Model should be higher than the Black-Scholes value, therefore the 102.6% is slightly overstated.

When the Binomial Tree is rerun using 900 steps instead of 600, the results are as follows:

Time	Asset Price	Risk-free Interest Rate	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
142	48.32	1%	2.12586	2.17729	102.42%
133	49.15	5%	2.05248	2.09220	101.94%
140	50.00	10%	1.19441	1.22654	102.69%
128	51.74	20%	0.99924	1.02372	102.45%

It appears, again, that some accuracy was improved. The two cases, r = 5% and r = 20%, where accuracy was slightly worsened may be due to the fact that the observation times were later in the 900 step case than in the 600 step case. It is clear that the closer to the end you get in the Binomial Tree, less observation points are being used to determine the value, which can slightly diminish the accuracy.

Volatility

For volatility, the simulations were run using the following values: 0.1, 0.2, 0.4, and 0.8. One note, if volatility is too low then the uncertainty of what the asset price will be also decreases and will just earn the risk free interest rate. Therefore, for the current example, with asset price equal to strike price, most all the values of the option were zero if volatility was too low because it was most certain the asset price would increase.

At time 0, the models calculated the following values:

Volatility	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
10%	0.49502	0.49438	99.871%
20%	1.62506	1.62462	99.973%
40%	4.05537	4.05576	100.010%
80%	8.93969	8.94172	100.023%

In this case, it appears that for σ small, the Binomial Model slightly understates the value of the option, and as σ gets larger, it switches to slightly overstating the value.

Closer to expiry, the models calculated the following values:

Time	Asset Price	Volatility	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
138	51.46	10%	0.01163	0.01416	121.81%
126	50.79	20%	0.55160	0.56514	102.46%
140	50.00	40%	1.19441	1.24411	104.16%
145	47.95	80%	2.82373	3.04475	107.83%

Again, the Binomial Model overstates the option value. For the case when σ = 10%, the percent of the Binomial Tree value compared to the Black-Scholes value is very high compared to the rest. However, the actual size of the difference is very small, only about .0025, so this must be taken into account when analyzing the accuracy of the Binomial Model.

When the Binomial Tree is rerun using 900 steps, the following values are found:

Time	Asset Price	Volatility	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
133	51.41	10%	0.02886	0.03183	110.30%
128	50.86	20%	0.49674	0.50927	102.52%
140	50.00	40%	1.19441	1.22654	102.69%
134	48.33*	80%	3.96463	4.04989	102.15%

* The actual price in the Binomial Tree was 48.32.

Here, the accuracy of the Binomial Tree was strongly improved for the case when σ = 10%, and also improved in the two cases σ = 40% and σ = 80%. For σ = 20% the accuracy stayed about the same.

Discrete Dividend Rate

Of the variables covered thus far, dividends prove to have the highest sensitivity in accuracy for the Binomial Model. The main reason for this is that the Binomial Tree model automatically assumes continuous dividends. The values tested for q were: 0, 0.01, 0.02, and .05. For all values of q listed, it was assumed that qt = .16438 ≈ 60 days.

At time 0, with asset price 50 the models calculated the following:

Dividend Rate	Modified Asset Price	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
0%	50.00	4.05537	4.05576	100.01%
1%	48.69	4.58652	4.67579	101.95%
2%	48.20	4.80006	4.98051	103.76%
5%	46.73	5.48775	5.97222	108.83%

As can be seen, as the dividend rate increases the Binomial Tree model increasingly over-calculates the option value. What makes the sensitivity of the discrete dividend rate more severe is the actual difference amount is much larger. With the other variables the difference was relatively small. Here, for the 5% case, the Binomial Tree model over-calculates the option value by about .48. In real world applications, this could have a much greater impact.

At time 140 the models calculate the following:

Dividend Rate	Asset Price	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
0%	50.00	1.19441	1.24411	104.16%
1%	48.69	1.91931	2.00812	104.63%
2%	48.20	2.24701	2.33801	104.05%
5%	46.73	3.38739	3.48312	102.83%

Here, for q = 5%, the Binomial Tree Model became more accurate at time 140 than at time 0, however, is still over-calculating the option value.

Rerunning the Binomial Tree with 900 steps, the following values were found:

Dividend Rate	Asset Price	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
0%	50.00	1.19441	1.22654	102.69%
1%	48.69	1.91931	1.98292	103.31%
2%	48.20	2.24701	2.30981	102.79%
5%	46.73	3.38739	3.46342	102.24%

Here, for all values of q tested, accuracy improved with increased steps in the Binomial Tree.

Exercise Price

The last variable tested was the Exercise or Strike Price. The values used were: 50, 51, 52, and 53. In this case, the Binomial Tree was again an accurate predictor. Close to expiry, as the exercise price increased, and the option value increased, the accuracy of the Tree improved. This was in part due to the larger size of the option value making the difference less severe.

At time 0 the models calculated the following:

Exercise Price	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
50	4.05537	4.05576	100.01%
51	4.53780	4.53893	100.02%
52	5.04946	5.05148	100.04%
53	5.58962	5.59251	100.05%

As the exercise price increases, the over-calculation of option value also increases, however, by only a very small percent.

At time 140, the models calculated the following:

Exercise Price	Asset Price	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
50	50.00	1.19441	1.24411	104.16%
51	50.00	1.75232	1.82967	104.41%
52	50.00	2.42760	2.49582	102.81%
53	50.00	3.20299	3.25639	101.67%

Here, we observe that as exercise price increases, the *Percent* decreases, which may be in part due to the increasing size of the value. Again, all the option values are over-calculated.

Rerunning the Binomial Tree with 900 steps gives the following results:

Exercise Price	Asset Price	Black-Scholes Value (a)	Binomial Tree Value (b)	Percent (b) / (a)
50	50.00	1.19441	1.22654	102.69%
51	50.00	1.75232	1.80665	103.10%
52	50.00	2.42760	2.47577	101.98%
53	50.00	3.20299	3.23971	101.15%

Again, the accuracy of the values increased for all cases.

RESULTS — AMERICAN PUT OPTION

Below are the comparisons for the American Put Option Optimal Exercise Boundary from the Binomial Tree and the Evans, Kuske, and Keller Equation. Five different dividend values were tested, and Δt was varied in the Binomial Tree for each case to see if accuracy was improved. Since the Evans, etal. equation is only valid for times very close to expiry, the times analyzed were from t = .386 to just before expiry.

For Dividend Rate .02, the following results were found:

		Evans, Kuske,	Binomial Tree Results					
Time	Time left	and Keller	Δt = .000552		Δt = .000276		Δt = .000138	
t	T - t	Equation (a)	Value (b)	% (b)/(a)	Value (c)	% (c)/(a)	Value (d)	% (d)/(a)
0.3861	0.02482	0.878929	0.90728	103.23%	0.90794	103.30%	0.90817	103.33%
0.3917	0.01931	0.889650	0.91390	102.73%	0.91453	102.80%	0.91493	102.84%
0.3944	0.01655	0.895857	0.91731	102.39%	0.91846	102.52%	0.91842	102.52%
0.3972	0.01379	0.902838	0.92192	102.11%	0.92221	102.15%	0.92294	102.23%
0.3999	0.01103	0.910858	0.92767	101.85%	0.92728	101.80%	0.92734	101.81%
0.4027	0.00827	0.920373	0.93198	101.26%	0.93251	101.32%	0.93308	101.38%
0.4054	0.00552	0.932288	0.93784	100.60%	0.93925	100.75%	0.93975	100.80%
0.4082	0.00276	0.949026	0.94572	99.65%	0.94697	99.78%	0.94803	99.89%

It can be observed that as we approach expiry, the results of the two models become closer, with the Binomial Tree value consistently the higher of the two, except for the very last one. Additionally, as Δt was decreased in the Binomial Tree model, the percent did not significantly change, and actually increased in most cases.

For Dividend Rate .04, the following results were found:

		Evans, Kuske,	Binomial Tree Results					
Time	Time left	and Keller	Δt = .000552		Δt = .000276		Δt = .000138	
t	T - t	Equation (a)	Value (b)	% (b)/(a)	Value (c)	% (c)/(a)	Value (d)	% (d)/(a)
0.3861	0.02482	0.869833	0.90220	103.72%	0.90194	103.69%	0.90211	103.71%
0.3917	0.01931	0.881871	0.90823	102.99%	0.90895	103.07%	0.90925	103.10%
0.3944	0.01655	0.888783	0.91283	102.71%	0.91303	102.73%	0.91352	102.78%
0.3972	0.01379	0.896511	0.91793	102.39%	0.91802	102.40%	0.91786	102.38%
0.3999	0.01103	0.905333	0.92199	101.84%	0.92252	101.90%	0.92338	101.99%
0.4027	0.00827	0.915725	0.92743	101.28%	0.92936	101.49%	0.92896	101.45%
0.4054	0.00552	0.928637	0.93426	100.61%	0.93538	100.73%	0.93596	100.79%
0.4082	0.00276	0.946594	0.94283	99.60%	0.94456	99.78%	0.94487	99.82%

The comparisons in this case are similar, with the Binomial Tree value consistently higher, except for the last case, and as time gets very close to expiry the two models values become closer.

For Dividend Rate .08, the following results were found:

		Evans, Kuske,	Binomial Tree Results					
Time	Time left	and Keller	Δt = .000552		Δt = .000276		Δt = .000138	
t	T - t	Equation (a)	Value (b)	% (b)/(a)	Value (c)	% (c)/(a)	Value (d)	% (d)/(a)
0.3861	0.02482	0.839781	0.88445	105.32%	0.88409	105.28%	0.88423	105.29%
0.3917	0.01931	0.855979	0.89192	104.20%	0.89324	104.35%	0.89286	104.31%
0.3944	0.01655	0.865141	0.89712	103.70%	0.89749	103.74%	0.89759	103.75%
0.3972	0.01379	0.875268	0.90182	103.03%	0.90340	103.21%	0.90335	103.21%
0.3999	0.01103	0.886685	0.91022	102.65%	0.90914	102.53%	0.90948	102.57%
0.4027	0.00827	0.899944	0.91679	101.87%	0.91728	101.93%	0.91671	101.86%
0.4054	0.00552	0.916143	0.92551	101.02%	0.92480	100.94%	0.92527	101.00%
0.4082	0.00276	0.938989	0.93699	99.79%	0.93533	99.61%	0.93589	99.67%

Again, similar observations here, however, the two models appear to be a bit farther apart from each other than in the previous two cases.

For Dividend Rate .12, the following values were found:

		Evans, Kuske,	Binomial Tree Results					
Time	Time left	and Keller	Δt = .000552		Δt = .000276		Δt = .000138	
t	T - t	Equation (a)	Value (b)	% (b)/(a)	Value (c)	% (c)/(a)	Value (d)	% (d)/(a)
0.3861	0.02482	0.796569	0.82441	103.50%	0.82456	103.51%	0.82441	103.50%
0.3917	0.01931	0.800910	0.82932	103.55%	0.82930	103.54%	0.82915	103.53%
0.3944	0.01655	0.803315	0.83138	103.49%	0.83139	103.49%	0.83112	103.46%
0.3972	0.01379	0.805931	0.83275	103.33%	0.83293	103.35%	0.83261	103.31%
0.3999	0.01103	0.808824	0.83336	103.03%	0.83348	103.05%	0.83317	103.01%
0.4027	0.00827	0.812108	0.83356	102.64%	0.83360	102.65%	0.83339	102.62%
0.4054	0.00552	0.816002	0.83361	102.16%	0.83380	102.18%	0.83359	102.16%
0.4082	0.00276	0.821079	0.83341	101.50%	0.83423	101.60%	0.83391	101.56%

In this case, all values for the Binomial Tree are higher than the values calculated in the other model. Again, as time gets closer to expiry, the two models values become closer.

For the last case, Dividend Rate .15, the following values were found:

		Evans, Kuske,	Binomial Tree Results					
Time	Time left	and Keller	Δt = .000552		Δt = .000276		Δt = .000138	
t	T - t	Equation (a)	Value (b)	% (b)/(a)	Value (c)	% (c)/(a)	Value (d)	% (d)/(a)
0.3861	0.02482	0.637255	0.66718	104.70%	0.66707	104.68%	0.66719	104.70%
0.3917	0.01931	0.640728	0.66706	104.11%	0.66712	104.12%	0.66766	104.20%
0.3944	0.01655	0.642652	0.66700	103.79%	0.66827	103.99%	0.66697	103.78%
0.3972	0.01379	0.644745	0.66686	103.43%	0.66827	103.65%	0.66704	103.46%
0.3999	0.01103	0.647059	0.66673	103.04%	0.66827	103.28%	0.66695	103.07%
0.4027	0.00827	0.649686	0.66704	102.67%	0.66827	102.86%	0.66704	102.67%
0.4054	0.00552	0.652802	0.66644	102.09%	0.66827	102.37%	0.66722	102.21%
0.4082	0.00276	0.656863	0.66653	101.47%	0.66827	101.74%	0.66704	101.55%

The results for this case are very similar to the case for dividend rate 12%, with the difference in the initial values a bit higher.

CONCLUSION

European Put Option Value
It can be concluded that the Binomial Model is a strong predictor of the European put option with the exception of one scenario. Overall, it appears to slightly over-calculate the option value, which can be improved by increasing the number of steps included in the Binomial Tree. The one exception would be with discrete dividend rates. If the dividend rate is high, the option value at time zero given by the Binomial Tree model will be more and more over-calculated. Closer to expiry, the values are a bit more accurate than at time zero. If the dividend rate is considered continuous, the accuracy of the Binomial Tree is synonymous with the interest rate case, since the updated interest rate is just the difference between the interest rate and the dividend rate. This concludes that one must consider whether the dividend rates for the asset are often enough to be considered continuous, and if not must keep in mind the weakness of the Binomial Tree under that scenario.

The Binomial Tree value can also be sensitive if volatility is very low. Again, when volatility is low, there is more certainty as to what the asset price will be over time. If it s most likely to increase above the exercise price, the values of the option will be very low, therefore the difference between the Black-Scholes value and the Binomial Tree value will be a higher percentage of the actual value.

Asset price appears to have no affect on the Binomial Tree Model, due to the fact that there is a linear relationship between initial asset price and option value. Overall, the Binomial Tree is an accurate predictor, with slight over-calculation, of the option value.

The risk-free interest rate has a small effect, if r is high, the Binomial Tree Model is more likely to initially understate the value of the option but as time goes on, ends up over-calculating the value. One other consideration is that if the risk-free interest rate is high and the exercise price of the option is relatively close to the initial asset price, most likely the values of the option will be very small.

The Exercise price also appears to have little affect on the Binomial Tree Model. As the Exercise price increases, the value of the option also increases, therefore is not as severely impacted by the difference between the two models values. Again, as the number of steps in the Binomial Tree increase, the accuracy of the option value also increases.

American Put Option Optimal Exercise Boundary
For the Optimal Exercise Boundary, it appears that the two model are relatively close, and become closer as time approaches expiry. This point makes sense since the equation by Evans, etal. is valid for times very close to expiry. Additionally, as the dividend rate varied, the difference between the two models only slightly varied. For dividend rates less than the risk-free interest rate, as the dividends increased, the difference in the two models increased slightly. Also, for dividend rates greater than the risk-free interest rate, the same pattern surfaced, for the higher dividend rate, the difference in the two models slightly increased.

When looking at these results, however, it must be kept in mind that the boundary values from the Binomial Tree were linearly interpolated from two asset prices on either side of the boundary. This method, should be accurate to a certain degree since very small time steps were used, however, the relationship between option values is not necessarily a linear one. One must also make a decision as to what values of $(T_F - t)$ are needed, and allow for less accuracy as that number grows.