Tilings of zonotopes: Discriminantal arrangements, oriented matroids, and enumeration

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Most of all, however, I owe a great deal to my adviser, Vic Reiner, without whose endless supply of good problems and good humor I might never have had the patience to complete this tome.

Dedication

This thesis is dedicated to my parents, Don and Elaine Bailey, without whom none of the things in my life would have been possible.

Abstract

A *d-zonotope* may be thought of as either the image of some projection of the n-cube into $\mathbb{R}^d$, with $n \geq d$, or as the Minkowski sum of n vectors in $\mathbb{R}^d$ containing some basis for $\mathbb{R}^d$. A *rhombohedral tiling* of a d-zonotope Z is a decomposition of Z into a non-disjoint union of cells, each of which is a translation of the Minkowski sum of some independent d-subset of the generating vectors for Z. The tiling is said to be *coherent* if it arises as the projection of the "top face" of some $(d+1)$-zonotope onto Z. The primary goal of this work is to explore and compare the enumeration and structure of

- the set of all tilings of Z.
- the subset of **coherent** tilings of Z.

To any zonotope Z, one may associate a particular hyperplane arrangement, called the *discriminantal arrangement* $\mathcal{D}(Z)$, which is useful in understanding the coherent tilings of Z. Understanding the structure of the set of **all** tilings of Z is best approached using the theory of oriented matroids.

In this work, we use these notions to

- Prove a new instance of Stembridge's "$q = -1$ phenomenon."
- Justify an idea of the French physicists Destainville, Mosseri and Bailly for counting tilings inductively.
- Resolve affirmatively a new case of the generalized Baues problem for tilings of d-zonotopes generated by multiple copies of $(d+1)$ vectors.
- Classify the 3-zonotopes which are generated by multiple copies of 5 vectors and which have all tilings coherent, and provide formulas enumerating their tilings.
- Characterize the free, factored, inductively factored and supersolvable hyperplane arrangements among a certain class of gain-graphic arrangements.

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1 History and main results

The following is a standard undergraduate-level counting problem:

How many partitions λ (viewed as Ferrers shapes) fit into an r by s box, where r and s are positive integers?

When one views the problem from the correct perspective, it is immediately seen to be a problem of enumerating lattice paths in an r by s grid.

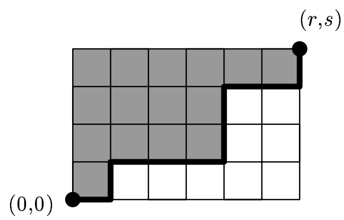


Figure 1: A Ferrers shape in an r by s grid

Since the boundary of each Ferrers shape defines a unique path from $(0,0)$ to (r,s), one may answer the question by enumerating all such paths which are "monotone increasing." Notice each path consists of $r+s$ unit segments, exactly r of which are horizontal and s of which are vertical. Thus each path is completely determined by the position of its horizontal (or equivalently, its vertical) segments, and so there are $\binom{r+s}{r}$ distinct partitions which fit inside an r by s box.

A considerably more difficult enumeration problem occurs when one generalizes the question to *plane partitions*. A convenient phrasing of this higher-dimensional question is:

How many distinct ways are there to stack unit cubes "flush into the corner" of a rectangular box with integer side lengths r, s, t?

The answer, along with a q-analogue, was originally given by MacMahon [Mac] in 1899:

$$N(r,s,t) = \frac{H(r+s+t)H(r)H(s)H(t)}{H(r+s)H(r+t)H(s+t)}$$

where $H(n) = (n-1)!(n-2)!\cdots 2!$ is the hyperfactorial function. Both of the above questions may be phrased as questions about *zonotopal tilings*.

A *zonotope* $Z(V)$ is (a translate of) the Minkowski sum of a vector set V, and a *zonotopal tiling* of Z is the decomposition of Z into a union of smaller zonotopes. A one-dimensional zonotope is simply a line segment, while a two-dimensional zonotope is a centrally symmetric $2n$-gon. To see how each of the partition questions is equivalent to a tiling question, one only needs to view the partitions from a point in general position.

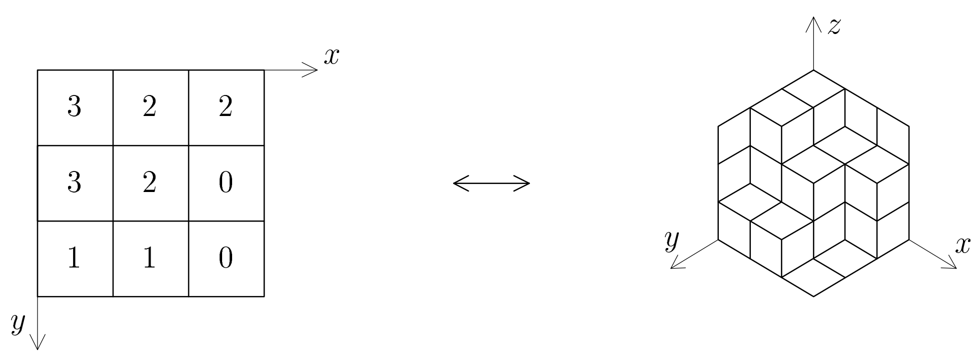


Figure 2: A stack of cubes induces a rhombic tiling of a hexagon

When the boundary of a Ferrers shape is viewed from a point in general position (in 2-space), one sees a line segment of length $r+s$ (modulo a slight deformation), broken up into r "horizontal" segments and s "vertical" segments. When one views a plane partition from a point in general position, things are a bit more interesting. Figure 2 illustrates the correspondence between a particular plane partition and the corresponding tiling of a hexagon, which is an example of a two-dimensional zonotope. In the left-hand view, the stack of unit cubes has been projected along the $(0, 0, 1)$ vector, with the integer entries indicating the heights of each smaller stack. In the right-hand view, the stack has been projected along the $(1, 1, 1)$ vector. Note that both MacMahon's problem and the Ferrers diagram problem can be phrased as counting tilings of d-zonotopes generated by multiple copies of $(d+1)$ vectors.

In the century since MacMahon's work, people have considered many other questions concerning plane partitions, especially the question of enumerating those tilings of a hexagon which are invariant under certain symmetries. It appears that even now, the work on these questions is not quite complete (for an exhaustive account of current results, see [Ste]). Naturally, many people have also attempted to give simple, closed formulas enumerating higher-dimensional partitions, but have failed even in the case of *solid* partitions, that is, stacks of 4-dimensional hypercubes "flush into the corner" of an r by s by t by u hyperbox. It seems that a more general approach is in order.

A straightforward generalization of MacMahon's work (given the historical difficulty of pursuing the question in higher dimensions) is to try to understand tilings of more general two-dimensional zonotopes. This is the approach Elnitsky [El] takes in his thesis, as he considers tilings of certain restricted classes of octagons. Elnitsky successfully q-counts the tilings for two related classes of octagons, and also completely enumerates those tilings invariant under each of several symmetries. In particular, he proves two instances of Stembridge's "$q = -1$ phenomenon" [Ste], which asks when the number of tilings of a $2n$-gon which are invariant under $180°$ rotation may be obtained by evaluating some q-count at $q = -1$. The principal technique Elnitsky uses is one in which certain *zones* of the zonotope are collapsed in a prescribed way, simplifying the problem. A similar technique for higher-dimensional zonotopes is suggested by the physicists Destainville, Mosseri and Bailly [DMB].

Much of the motivation for the study of higher-dimensional zonotopes comes not from MacMahon's work, but from a surprising connection between zonotopal tilings and *oriented matroids* (see [BLSWZ]). While there is a natural correspondence between zonotopes and *arrangements of hyperplanes* (see [OT]), and thus with certain oriented matroids, it was not until the appearance of the Bohne-Dress Theorem in 1989 (see [BD], [RZ]) that there began to be a greater interest in the study of zonotopal tilings. Simply stated, the Bohne-Dress Theorem describes a bijective correspondence between tilings of a zonotope $Z(V)$ and single-element liftings of the corresponding *oriented matroid* $\mathcal{M}(V)$.

Another relatively new area which both motivates and facilitates the study of higher-dimensional zonotopal tilings is that of *discriminantal arrangements* of hyperplanes. Discriminantal arrangements were first defined by Manin and Schechtman [MS] in 1986 as a generalization of the *braid arrangement* of type A_{n-1}. This definition was itself broadened by Bayer in 1993 [Ba]. For a particular zonotope $Z = Z(V)$, it turns out that there is a bijection between the collection of *coherent rhombohedral* tilings of Z and *chambers* in the corresponding discriminantal arrangement $\mathcal{D}(V)$ (see [BS]). Loosely stated, a zonotopal tiling of a rank d zonotope Z is *coherent* if it may be obtained by "looking at" some rank $(d+1)$ zonotope. For example, a coherent tiling of a regular hexagon of side length 1 may be obtained by viewing a cube from a point in general position. However, the tiling of the regular hexagon of side length 3 pictured in Figure 2 is incoherent.

This paper extends Elnitsky's results slightly and formalizes the method suggested by Destainville, Mosseri and Bailly, but our principal goal is to characterize and give enumeration formulas for three-dimensional zonotopes which have the property that all of their rhombohedral tilings are coherent.

The following chapter contains the necessary background and definitions for the work to follow. In the third chapter, we give a q-count for the tilings of one of the classes of octagons considered by Elnitsky, and show that this class of octagons provides another example of Stembridge's "$q = -1$ phenomenon," in that the number of tilings invariant under $180°$ rotation is obtained by evaluating the q-count at $q = -1$. The fourth chapter discusses a construction used by Elnitsky and by Destainville et al, and rigorously defines a method for enumerating tilings of a zonotope Z with "more zones" given the set of tilings for a fixed subzonotope Z' of Z with "fewer zones." This method relies on an oriented matroid construction of Sturmfels and Ziegler [SZ].

The fifth chapter introduces the counting methods which will be used to enumerate the set of rhombohedral tilings of a given zonotope Z, and gives a careful demonstration of these methods in the case of d-zonotopes generated by multiple copies of $(d + 1)$ vectors. The fifth chapter also contains a proof that this class of zonotopes satisfies the *generalized Baues conjecture*, in that a certain space of all zonotopal tilings of a zonotope in this class is homotopy equivalent to an $(n - d - 1)$-sphere.

The final two chapters contain a partial classification of those three-dimensional zonotopes Z which have the property that all rhombohedral tilings of Z are coherent, and enumeration formulas for several infinite classes. The approach for this classification is straightforward, following the work done by Edelman and Reiner [ER] in their classification of two-dimensional zonotopes with this property. Given a zonotope Z, we

a) Enumerate all tilings of Z.

b) Enumerate the **coherent** tilings of Z.

c) Compare.

To count **all** tilings of Z requires an oriented matroid argument, which includes a theorem of Las Vergnas [LV] and lattice-path enumerations along the lines of the one given on page 1.

To enumerate the **coherent** tilings of $Z = Z(V)$, we use the aforementioned result of Billera and Sturmfels, which states that these tilings are in bijective correspondence with the chambers of the discriminantal arrangement $\mathcal{D}(V)$. By a result of Zaslavsky [Za1], this enumeration may be accomplished by finding the roots of the *characteristic polynomial* $\chi(\mathcal{D}(V), t)$ for $\mathcal{D}(V)$. Since it happens that $\mathcal{D}(V)$ is a *free* arrangement (see [Te]) for all zonotopes $Z(V)$ under consideration, these roots are simply the *exponents* of $\mathcal{D}(V)$, by a result of Terao [OT]. The proof that the necessary classes of arrangements are free is given in the final chapter, using the notion of *gain graphs* introduced by Zaslavsky [Za3].

In summary, the main results of the paper are

- A new instance of Stembridge's "$q = -1$ phenomenon" (Corollary 3.3).
- Justification for an idea of Destainville et al [DMB] for recursively counting tilings of a d-zonotope for arbitrary d (Theorem 4.4).
- An affirmative answer for the case of the generalized Baues problem concerning tilings of d-zonotopes generated by multiple copies of $(d+1)$ vectors (Theorem 5.7).

- Characterization of the free, factored, inductively factored and supersolvable hyperplane arrangements among a certain family of *gain-graphic* arrangements (Theorems 7.3 and 7.8).

- Classification of those 3-zonotopes generated by multiple copies of 5 vectors for which all tilings are coherent, and formulas enumerating their tilings (see Figures 3, 4 and 5).

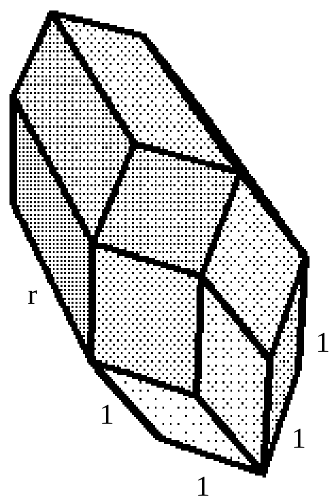


Figure 3: A zonotope with $2(r+4)!/4!$ distinct rhombohedral tilings (Theorem 6.1)

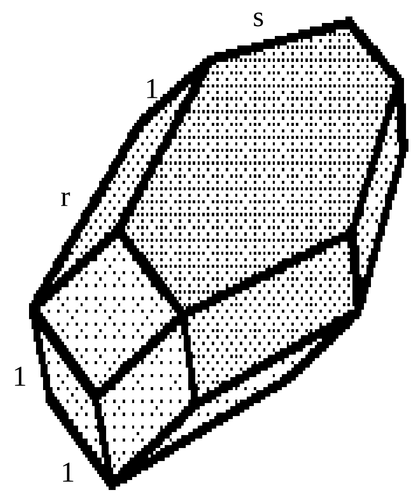


Figure 4: A zonotope with $2(r+s+1)!(r+s+2)!/(s+2)!(r+2)!$ distinct rhombohedral tilings (Theorem 6.3)

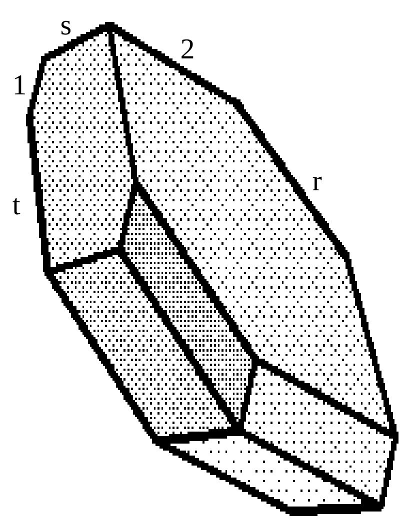


Figure 5: A zonotope with $2(r+s+t)!(r+s+t+1)!/(r+1)!(s+t+1)!$ distinct rhombohedral tilings (Theorem 6.4)

2 Background

A *d-zonotope* Z with n zones may be thought of either as the image of the n-cube under some affine projection into $\mathbb{R}^d$, or as (a translate of) the Minkowski sum of some n-set V of vectors in $\mathbb{R}^d$ which contains a basis. The set V is actually a multiset, as vectors may appear with multiplicity. Let $\overline{V}$ be the underlying set of V (that is, the maximal subset of **distinct** vectors in V). The set V is called the *generating set* of $Z = Z(V)$, and we say that V *generates* Z. It is clear that if Z is given, the elements of V correspond to extreme 1-cells of Z, and so V may be recovered. Therefore one may identify a zonotope $Z = Z(V)$ with its generating set V. Note also that both definitions immediately imply that every face of a zonotope is again a zonotope.

Our main objects of study are *rhombohedral tilings* of zonotopes. Given a zonotope $Z = Z(V)$, a *subzonotope* of Z is any zonotope $Z' = Z(V')$, where V' is a subset of V. A *tiling* T of a d-zonotope Z is the decomposition of Z into a union of d-subzonotopes, called the *tiles* of T, such that any two tiles t_1, t_2 intersect in a proper face of each. A tiling T is a *rhombohedral tiling* if each tile t is generated by a subset of V forming a basis of $\mathbb{R}^d$.

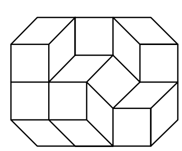


Figure 6: A rhombohedral tiling of a 2-zonotope

The principal tools we will use to study tilings are *arrangements of hyperplanes* (or simply *arrangements*) and *oriented matroids*. A *d-arrangement* $\mathcal{A}$ is a finite collection of codimension-one linear subspaces of $\mathbb{R}^d$ (see [OT]).

We do not rule out the possibility that the hyperplanes in an arrangement $\mathcal{A}$ might appear with multiplicity, and the reader should be aware that such collections are more commonly referred to as *multiarrangements*. The hyperplanes in $\mathcal{A}$ intersect in some linear subspace S of rank $0 \le s \le d-1$. Define the *rank* of a d-arrangement $\mathcal{A}$ to be $d-s$. If $s=0$, then $\mathcal{A}$ is an *essential* arrangement. One classic arrangement which will appear frequently in the sequel is the *braid arrangement* A_{n-1}, a rank $n-1$ arrangement in $\mathbb{R}^n$ defined by the hyperplanes normal to

$$\{\mathbf{e}_i - \mathbf{e}_j \,|\, 1 \le i < j \le n\}$$

where $\{\mathbf{e}_i\}$ is the collection of standard basis vectors in $\mathbb{R}^n$.

By taking normals, there is a natural correspondence between vector sets V in $\mathbb{R}^d$ and d-arrangements $\mathcal{A}(V)$, and so consequently there is a natural bijection between arrangements and zonotopes. In fact, for a fixed vector set V, the zonotope $Z(V)$ and the arrangement $\mathcal{A}(V)$ are geometrically polar duals to one another in the sense that each $(d-1)$-face of $Z(V)$ corresponds to a unique 1-ray arising as the intersection of hyperplanes in $\mathcal{A}(V)$. To make this correspondence precise, we turn to oriented matroids.

We will not define oriented matroids formally, but rather introduce only those elements of oriented matroid theory which are necessary for the work which is to follow. The standard reference for oriented matroids is the book by Björner et al [BLSWZ]. A discussion of oriented matroids which is specific to polytopes and zonotopes appears in chapters 6 and 7 of Ziegler's book [Zi]. Oriented matroids have many guises, but the one which is most useful here is the view of oriented matroids as a generalization of arrangements. To demonstrate how oriented matroids generalize arrangements, we first describe how to obtain the set of *covectors* $\mathcal{L} = \mathcal{L}(V)$ of the oriented matroid $\mathcal{M} = \mathcal{M}(V)$ associated with a particular d-arrangement $\mathcal{A} = \mathcal{A}(V)$, where the n elements of V are given an arbitrary ordering. The arrangement $\mathcal{A}$ decomposes $\mathbb{R}^d$ into a disjoint union of cones, where each k-cone is determined by some subarrangement of $\mathcal{A}$ with rank $d-k$. To each of these cones is associated a particular n-tuple in $\{0,+,-\}^n$. Specifically, the n-tuple X corresponding to the cone C is defined by $X_i = \text{sign}(c \cdot v_i)$, where c is any point in C and v_i is the i^{th} generating vector for $\mathcal{A}$. The collection of all such n-tuples is the set of *covectors* $\mathcal{L}(V)$ for the oriented matroid $\mathcal{M}(V)$ associated with the arrangement $\mathcal{A}(V)$. Figure 7 illustrates this correspondence for a 2-arrangement.

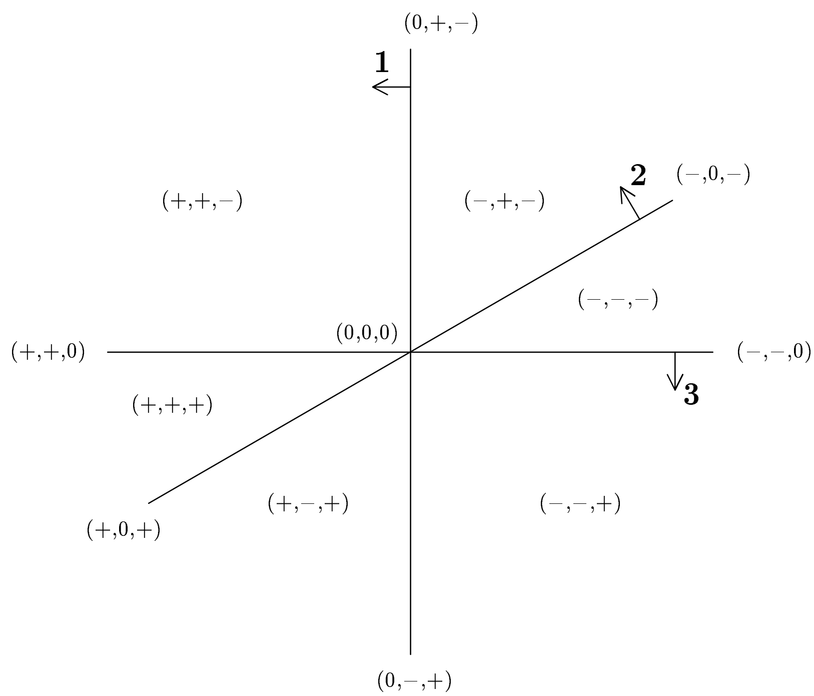


Figure 7: The covectors for the oriented matroid of a 2-arrangement

Oriented matroids allow one to distill an arrangement (vector configuration, zonotope) to its combinatorial essence. For example, it is easy to see that up to oriented matroid equivalence, there is exactly one 2-arrangement on m distinct vectors for any positive integer m. This allows the use of a convenient shorthand notation for discussing rank 2 vector configurations (arrangements, zonotopes), in that one may discuss "the" $(r_1, r_2, r_3, \ldots, r_m)$ 2-zonotope, where $|\overline{V}| = m$ and r_i indicates the multiplicity with which the ith vector in $\overline{V}$ appears. Implicit in this notation is the understanding that if one begins with the vector $v_1 \in \overline{V}$ and proceeds clockwise, one encounters $v_2, v_3, \ldots, v_m$ in order. The notation becomes somewhat more complicated for 3-zonotopes.

With the notion of covectors for $\mathcal{M}(V)$ in place, we can now explicitly define the correspondence between cones in the decomposition of $\mathbb{R}^d$ induced by $\mathcal{A}(V)$ and the faces of $Z(V)$. Specifically, suppose C is a cone induced by $\mathcal{A}$ with covector X. Define

$$X^- = \{i \,|\, X_i = -\} \qquad X^0 = \{i \,|\, X_i = 0\} \qquad X^+ = \{i \,|\, X_i = +\}.$$

Then the face of Z corresponding to C will be the Minkowski sum of those vectors $v_i \in V$ with $i \in X^0$, translated by $\displaystyle\sum_{i \in X^+} v_i - \sum_{i \in X^-} v_i$. This construction also demonstrates how to determine the covectors of $\mathcal{M}(V)$ directly from $Z(V)$.

It is important to point out that although every vector configuration V determines an oriented matroid $\mathcal{M}(V)$, not all oriented matroids arise in this way. Rather, any collection $\mathcal{L}$ of sign vectors $\mathcal{L} \subseteq \{0, +, -\}^n$ is the set of covectors of some oriented matroid $\mathcal{M}$ if a short list of axioms is satisfied. To list these axioms requires the definition of some terminology for sign vectors. Given two sign vectors X and Y, their *composition* $X \circ Y$ is defined by

$$(X \circ Y)_i = \begin{cases} X_i & \text{when} \quad X_i \neq 0, \\ Y_i & \text{otherwise.} \end{cases}$$

The *separation set* $S(X, Y)$ is the set of indices i such that $X_i = -Y_i \neq 0$.

Any collection of sign vectors $\mathcal{L} \subseteq \{0, +, -\}^n$ is the set of covectors of some oriented matroid provided the following four conditions are satisfied:

0) $\mathbf{0} \in \mathcal{L}$.

1) $X \in \mathcal{L}$ if and only if $-X \in \mathcal{L}$.

2) If X and Y are in $\mathcal{L}$, then $X \circ Y \in \mathcal{L}$.

3) If X and Y are in $\mathcal{L}$ and $j \in S(X, Y)$, then there exists $W \in \mathcal{L}$ such that $W_j = 0$ and $W_i = (X \circ Y)_i$ for all $i \notin S(X, Y)$.

It only requires the consideration of some few small examples to see that these covector axioms efficiently encode the essential combinatorial structure of a vector configuration. Nevertheless, a list of covectors is not yet minimal information for this task. One may define a partial order

$$\begin{tikzcd}[column sep=small] + \arrow[dr, dash] & & - \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

on each covector component, and extend it to a partial order on $\mathcal{L}$ by the product partial order $X \geq Y$ if and only if $X_i \geq Y_i$ for all i. The minimal nonzero covectors under this partial order are the *cocircuits* $\mathcal{C}^*$ of the oriented matroid. Thus in Figure 7, the set of cocircuits is

$$\mathcal{C}^* = \{\ (0, +, -),\ (-, 0, -),\ (-, -, 0),\ (0, -, +),\ (+, 0, +),\ (+, +, 0)\ \}.$$

Given the set of cocircuits $\mathcal{C}^*$ of an oriented matroid $\mathcal{M}$, the entire collection of covectors $\mathcal{L}$ can be recovered. Note that by covector axiom 1), only half of the given information is necessary. Thus we will usually consider only half of the cocircuits of a given oriented matroid. Also note that by the polar duality mentioned on page 12, cocircuits of $\mathcal{M}(V)$ correspond to maximal dimensional faces of $Z(V)$.

Any oriented matroid whose cocircuits may be obtained as the cocircuits of a vector configuration is said to be *realizable*. However, there are other types of topological arrangements, involving *pseudohyperplanes* or *pseudospheres*, which also give rise to oriented matroids. Often the oriented matroids arising from such arrangements are not realizable. A projectivized picture of the non-Pappus arrangement, the classic example of a non-realizable oriented matroid, is given in Figure 8.

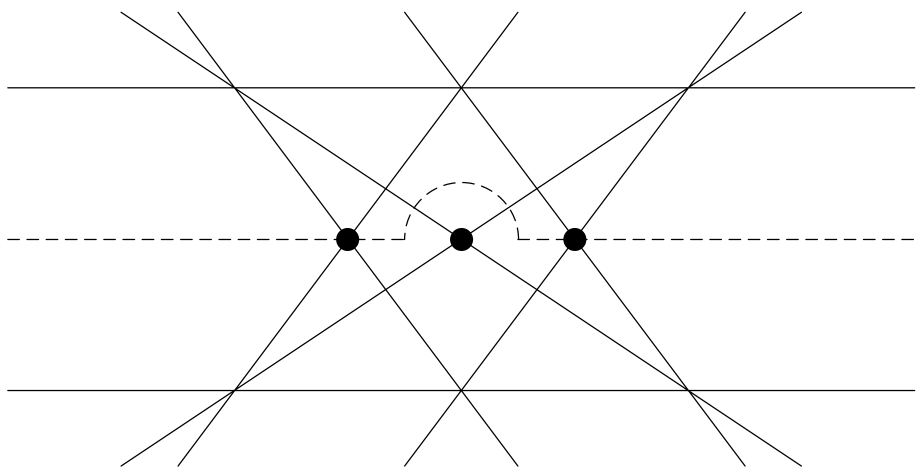


Figure 8: The non-Pappus arrangement

The line arrangement in Figure 8, excluding the dotted line, represents the intersection of a particular 3-arrangement with the upper half of a sphere. Thus one may assign a positive and negative side to all the lines and obtain the collection of covectors of an oriented matroid as in Figure 7. Each point where two or more lines intersect corresponds to a cocircuit. It is also true that if one includes a positive and negative side for the dotted line, the result is again the collection of cocircuits for an oriented matroid $\mathcal{M}$. However, Pappus' theorem states that the three black dots are collinear in every line arrangement which is combinatorially equivalent to the one in Figure 8. Thus there is no vector configuration which will have $\mathcal{M}$ as its oriented matroid. This example demonstrates that not all oriented matroids arise from hyperplane arrangements. However, Folkman and Lawrence [FL] have shown that all oriented matroids **do** arise from *pseudosphere arrangements* such as the one in Figure 8.

A subset S of the d-sphere S^d will be called a *pseudosphere* if S is homeomorphic to S^{d-1}. An *arrangement of pseudospheres* $\mathcal{A} = (S_e)_{e \in E}$ is a finite set of pseudospheres in S^d such that

1) Every nonempty intersection $S_A = \cap_{e \in A} S_e$ is (homeomorphic to) a sphere of some dimension, for $A \subseteq E$.

2) For every non-empty intersection S_A and every $e \in E$ such that $S_A \not\subseteq S_e$, the intersection $S_A \cap S_e$ is a pseudosphere in S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

The following partial statement of Folkman and Lawrence's result appears in [BLSWZ]:

Theorem 2.1 *[The Topological Representation Theorem] Let $\mathcal{L} \subseteq \{+, -, 0\}^E$. Then the following conditions are equivalent:*

i) $\mathcal{L}$ is the set of covectors of an oriented matroid of rank $d+1$.

ii) $\mathcal{L} = \mathcal{L}(\mathcal{A})$ for some signed arrangement $\mathcal{A} = (S_e)_{e \in E}$ of pseudospheres in S^{d+1+k}, such that $\dim(\cap_{e \in E} S_e) = k$.

We next consider zonotopal tilings. Suppose $Z = Z(V)$ is a d-zonotope. Up to choice of coordinates, it is possible to add a $(d+1)$st coordinate l_v to each $v \in V$ and add the basis vector $\mathbf{e}_{d+1}$ to V to obtain the generating set V' for a $(d+1)$-zonotope $Z' = Z(V')$. Let $\mathcal{F}$ denote the collection of *upper facets* of Z', those rank d faces corresponding to cocircuits with value + or 0 on $\mathbf{e}_{d+1}$, or informally, the rank d faces which are "visible" from a point with very large $d+1^{\text{st}}$ coordinate. Let π_{d+1} denote the projection of Z' along the basis vector $\mathbf{e}_{d+1}$. Then the collection $\{\pi_{d+1}(F) \mid F \in \mathcal{F}\}$ constitutes a tiling of Z. If a tiling T of a d-zonotope Z can be obtained in this manner for some choice of $\{l_v\}_{v\in V}$, then T is *coherent* (see Figure 9). Otherwise, T is *incoherent*. Similarly, if Z is such that T is coherent for all tilings T, then we say that Z itself is *coherent*, otherwise *incoherent*.

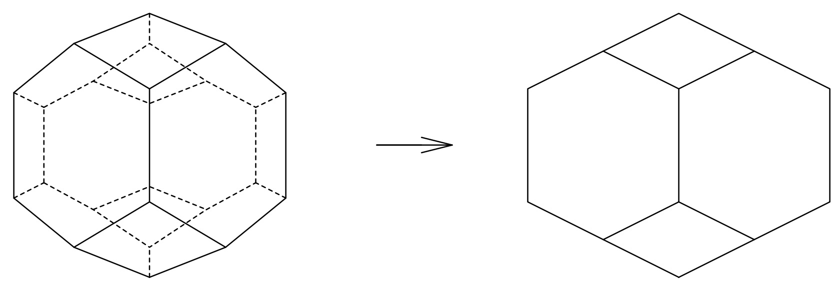


Figure 9: A coherent tiling of a hexagon is obtained by "looking at" a 3-zonotope

The principal tool for studying coherent rhombohedral tilings of a zonotope $Z(V)$ is the *discriminantal arrangement* $\mathcal{D}(V)$ (see [Ba]). Let $v_1, v_2, \ldots, v_n$ be an arbitrary ordering of the elements of V. Then $\mathcal{D}(V)$ is an n-arrangement defined by the minimally dependent sets of V as follows. The hyperplane $(a_1, a_2, \ldots, a_n)^\perp$ is in $\mathcal{D}(V)$ if and only if the set $V' = \{v_i \mid a_i \neq 0\}$ satisfies

$$\sum_{v_i \in V'} a_i v_i = 0$$

and V'' is independent for all proper subsets V'' of V'.

Billera and Sturmfels showed [BS]

Theorem 2.2 *Let V be a vector configuration. The set of coherent rhombohedral tilings of $Z(V)$ is in bijective correspondence with the set of* chambers, *or open cones of maximal dimension, in the arrangement $\mathcal{D}(V)$.*

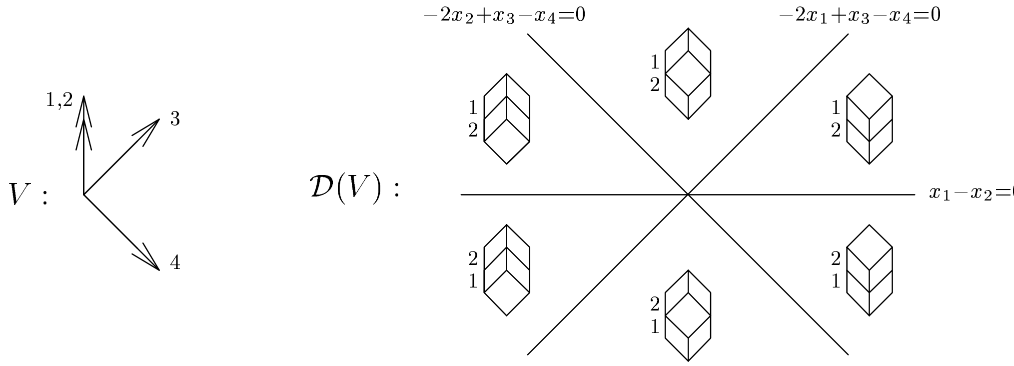


Figure 10: For a given vector configuration V, the chambers of $\mathcal{D}(V)$ correspond to the distinct rhombohedral tilings of $Z(V)$.

Given the ordering $v_1, v_2, \ldots, v_n$ of the elements of V, let l_i be the $(d+1)$st coordinate appended to v_i to obtain v_i', and let T be the coherent tiling of Z obtained under the projection π_{d+1}. The vector $(l_1, l_2, \ldots, l_n) \in \mathbb{R}^n$ is called the *lifting vector* for T, since it describes precisely how to "lift" each element of V into $\mathbb{R}^{d+1}$. Suppose $J \subset V$ is a minimal dependent set of cardinality $|J| \leq d+1$ (since $Z(V)$ is a d-zonotope, J cannot be any larger). Further, suppose that the lifting vector $(l_1, l_2, \ldots, l_n)$ is such that this dependence is preserved for $J' = \{v' \mid v \in J\}$. Then there exists a (possibly empty) collection of vectors $K' \subseteq V'$ such that $|J' \cup K'| = d+1$ and some extreme facet F of Z' is a translate of $Z(J' \cup K')$. Consequently, in the coherent tiling T obtained by π_{d+1}, the tile $\pi_{d+1}(F)$ is generated by $(d+1)$ elements of V, and so T is not a rhombohedral tiling. This shows that when passing from V to V', one must take care to avoid lifting vectors which preserve any minimal dependence among the elements of V. This is the essence of

Theorem 2.2: to obtain a coherent rhombohedral tiling T of $Z(V)$, the lifting vector $(l_1, l_2, \ldots, l_n)$ must not lie in any hyperplane defined by a minimal dependence among the elements of V. Rather, it must lie in some chamber defined by $\mathcal{D}(V)$.

In a similar vein, if T is a coherent tiling of a d-zonotope $Z = Z(V)$ obtained from $Z' = Z(V')$, then the oriented matroid $\mathcal{M}(V')$ is a *single-element lifting* of $\mathcal{M}(V)$. However, Z may also have an incoherent tiling T (the hexagonal tiling on page 2 is incoherent, for example). Bohne and Dress [BD] showed by passing to pseudosphere arrangements that T nevertheless corresponds to a single-element lifting $\mathcal{M}'$ of the oriented matroid $\mathcal{M} = \mathcal{M}(V)$:

Theorem 2.3 *[The Bohne-Dress Theorem] Let $Z = Z(V)$ be a zonotope. There is a bijection between the tilings of Z and single-element liftings of $\mathcal{M}(V)$.*

These last two theorems are some of the principal tools used to determine a partial classification of all coherent 3-zonotopes in Chapter 5. For the remainder of the paper, the term *tiling* will mean a rhombohedral tiling unless explicitly stated otherwise.

3 Stembridge's "$q = -1$ phenomenon" for an $(r, 1, s, 1)$ octagon

3.1 Introduction

One of the classic problems of enumeration is to count the number $N(r, s, t)$ of distinct ways of tiling a hexagon of integral side lengths r, s, t with rhombi of unit side length. It is well known that this problem is equivalent to counting the number of r by s *plane partitions* with parts bounded by t, which are weakly decreasing along rows and down columns.

As was previously mentioned, the answer, along with a q-analogue, was originally given by MacMahon [Mac] in 1899:

$$N(r, s, t) = \frac{H(r+s+t)H(r)H(s)H(t)}{H(r+s)H(r+t)H(s+t)}$$

where $H(n) = (n-1)!(n-2)!\ldots 2!$ is the hyperfactorial function.

In the years since MacMahon, people have been interested in a number of different questions concerning such tilings, among them

1) Counting the collection of such tilings which are invariant under certain group actions.

2) Obtaining a q-count of tilings invariant under these group actions, where the statistic measures the distance from a distinguished "base" tiling.

In 1992, Stembridge [Ste] made the observation that in all cases which involve the group action of *complementation* (first defined in [MRR]), an answer to the second question yields an answer to the first. That is to say, for any group action involving complementation, the number of tilings invariant under that action may be computed by substituting $q = -1$ in a particular q-count (for a summary of the current results in the attempt to complete the classification of these tilings, see [Ste]).

Of course, a hexagon of integral side lengths r, s, t (hereafter referred to as an (r, s, t) *hexagon*) is just one example of a two dimensional zonotope. Edelman and Reiner [ER] and Elnitsky [El] have given coherence results and enumeration formulas for other two dimensional zonotopes. The goal in this chapter is to provide a q-count $N_q(r, 1, s, 1)$ for the tilings of an $(r, 1, s, 1)$ octagon and to show that Stembridge's "$q = -1$ phenomenon" holds in this case.

3.2 Complementation and q-counts

The notion of *complementation* and the statistic for the q-count $N_q(r, 1, s, 1)$ are the essential definitions for this chapter. Since the case of an (r, s, t) hexagon partially motivates the study of more general two dimensional zonotopes, we use the example of the (r, s, t) hexagon to motivate our definition of these terms.

One of the most useful tools in the study of zonotopal tilings is to view a tiling of a d-zonotope as the result of projecting some higher-dimensional geometric object into $\mathbb{R}^d$. In the case of an (r, s, t) hexagon, the higher-dimensional object is a stack of cubes inside the box $[0, r] \times [0, s] \times [0, t]$ in $\mathbb{R}^3$, stacked in such a way that they are flush into the "corner" defined by the coordinate axes. There is an obvious bijection between such a stack of cubes and r by s plane partitions with parts bounded by t (see Figure 2).

Using this correspondence, we define the action of complementation on a tiling of a 2-zonotope. Let H be an (r, s, t) hexagon with tiling T. Let S be the corresponding stack of cubes in an $r \times s \times t$ box. The *complement* S^c of this stack of cubes is defined to be the collection of additional cubes necessary to completely fill the box. To obtain a legal stack of cubes (that is, one which is in the corner defined by the coordinate axes), reflect S^c through the point $(r/2, s/2, t/2)$. Finally, projecting this stack of cubes along the vector $(1, 1, 1)$ results in a tiling T^c of H. An example of complementation is given in Figure 11 for a $(2, 2, 2)$ hexagon.

T $\qquad\qquad$ T^c

Figure 11: A tiling of a $(2, 2, 2)$ hexagon and its complement

Note that, for a given S, the corresponding tiling T of H is determined by the 2-dimensional surface obtained by "looking at" the stack from a point in general position. This surface has been called the *membrane* of the tiling by Destainville, Mosseri and Bailly [DMB]. Since the membrane is precisely the intersection of S and S^c, it also determines T^c. Furthermore, it is clear that reflection through the point $(r/2, s/2, t/2)$ is equivalent to a $180°$ rotation of the membrane. Thus for the remainder of this chapter, the *complement* T^c of a tiling T of a 2-zonotope Z will be defined as the tiling obtained via a $180°$ rotation of Z.

In $N_q(r, s, t)$, the coefficient of q^n counts the number of tilings which correspond to a stack of cubes containing exactly n cubes. For example, in Figure 2, the tiling contributes a term of $q^{3+2+2+3+2+1+1} = q^{14}$. In Figure 11, the tilings T and T^c contribute terms q^2 and q^6, respectively.

Note that if a single cube is removed from a stack of cubes, then a *hexagon flip* occurs in the corresponding hexagon tiling. This is quickly verified by considering the two tilings of the $(1,1,1)$ hexagon.

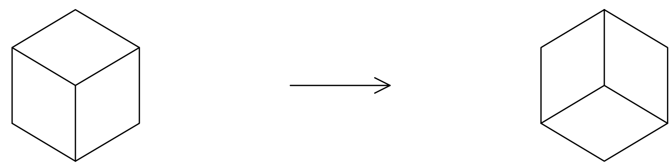


Figure 12: An example of a hexagon flip

Thus an equivalent method for obtaining $N_q(r,s,t)$ is to let the coefficient of q^n count the number of tilings which are obtained from the *base tiling* T_0 via a minimal sequence of n hexagon flips, where T_0 corresponds to the stack of zero cubes. We will use this interpretation of the statistic when deriving $N_q(r,1,s,1)$.

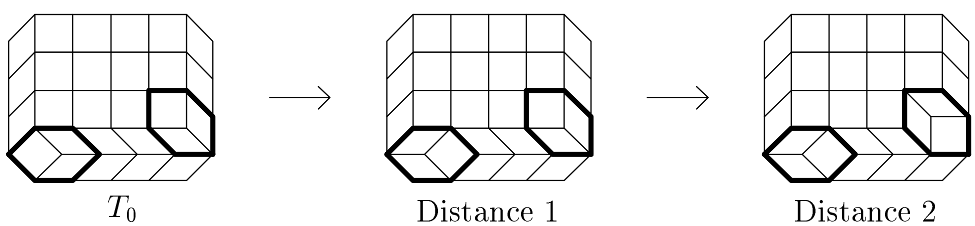


Figure 13: Hexagon flips in a tiling of a $(4,1,3,1)$ octagon

3.3 The case of an $(r,1,s,1)$ octagon

An $(r,1,s,1)$ octagon is one in which one views successively sides of length r, 1, s, and 1 in a walk around the perimeter of the octagon, where r and s are any positive integers. In particular, they are distinct from $(r,s,1,1)$ octagons (see Figure 14), for which Elnitsky [El] has already verified Stembridge's "$q=-1$ phenomenon."

Elnitsky has also computed $N(r,1,s,1)$:

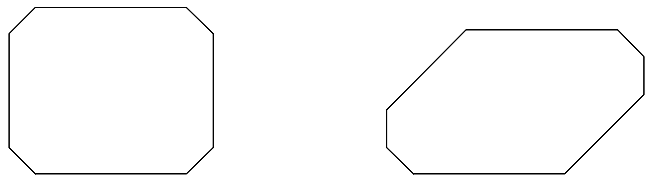


Figure 14: The difference between an $(r, 1, s, 1)$ octagon and an $(r, s, 1, 1)$ octagon

$$N(r, 1, s, 1) = \sum_{\substack{a+b=r \\ c+d=s}} \binom{a+c}{a}\binom{b+c}{b}\binom{a+d}{a}\binom{b+d}{b}.$$

Elnitsky [El] states that no closed form for this sum is known, but that P. Brock [Str] has determined the following recurrence for $N(r, 1, s, 1)$:

Proposition 3.1

$$N(r, 1, s, 1) - N(r, 1, s-1, 1) - N(r-1, 1, s, 1) = \binom{r+s}{r}^2.$$

We will derive a q-count for these tilings which immediately specializes to Elnitsky's result when $q = 1$.

Elnitsky points out that there is a natural bijection between tilings of an $(r, 1, s, 1)$ octagon and pairs of once-crossing paths in an r by s grid, where the point of intersection (called the *root*) is distinguished. Figure 15 demonstrates this bijection for a $(4, 1, 3, 1)$ octagon O. The key observation is that there is a unique way to contract O on each of the zones z_1, z_2 corresponding to the vectors with multiplicity one. Performing a single such contraction defines a unique path in a $(4, 3, 1)$ hexagon. Performing both contractions defines a unique pair of once-crossing paths on a 4 by 3 grid. The distinguished root vertex corresponds to the unique tile in O where z_1 and z_2 cross. In Figure 15, the root is located at $(2, 1)$.

Note that the crossing paths on the resulting r by s grid must be such that one path begins at $(0, 0)$ and ends at (r, s) and the other path begins at $(0, s)$ and ends at $(r, 0)$.

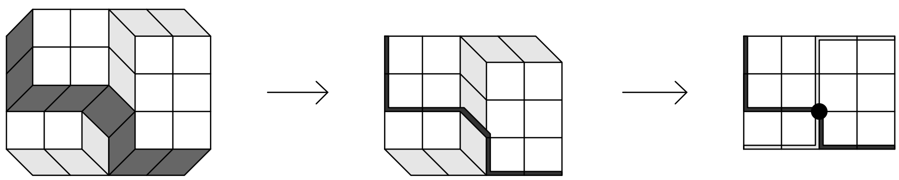


Figure 15: Tilings of $(r, 1, s, 1)$ octagons correspond to pairs of once-crossing paths on an $r \times s$ grid, together with a choice of "root"

It is a routine matter to verify, in terms of once-crossing paths on an r by s grid, that complementation is equivalent to a 180° rotation of the grid, and a hexagon flip corresponds either to "moving a path across a square" (if the root is not involved), or to moving the root along both paths. For example, in Figure 15, there is exactly one possible hexagon flip which involves the tile defined by the intersection of z_1 and z_2; performing this hexagon flip is equivalent to moving the root to the position $(2, 0)$ in the 4 by 3 grid.

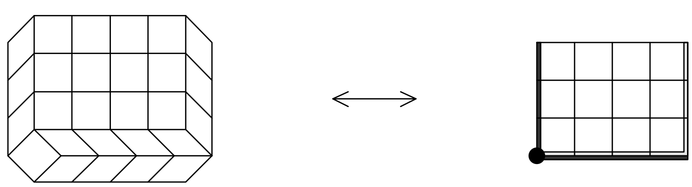


Figure 16: The base tiling T_0 and the corresponding pair of once-crossing paths

It is now a relatively straightforward matter to obtain $N_q(r,1,s,1)$. We begin by obtaining a q-count for all tilings which have their root at a fixed vertex (x_0, y_0), and then sum over all possible locations of root vertex. Note that moving the root from $(0,0)$ to (x_0,y_0) requires $x_0+(r+1)y_0$ hexagon flips, and thus contributes a factor of $q^{x_0+(r+1)y_0}$. Once the root is fixed, the grid is effectively broken up into four quadrants, of sizes x_0 by y_0, $r-x_0$ by y_0, x_0 by $s-y_0$, and $r-x_0$ by $s-y_0$. All that remains is to choose a path in each of the four smaller grids. It is well known that the number of possible paths from $(0,0)$ to (x,y) in an x by y grid is q-counted by $\binom{x+y}{y}_q$, where

$$[n] = 1 + q + q^2 + \cdots + q^{n-1} \qquad \text{and} \qquad [n]! = [n][n-1]\ldots[2][1]$$

and

$$\binom{n}{k}_q = \frac{[n]!}{[k]![n-k]!}.$$

Therefore, the q-count for tilings which have their root at (x_0, y_0) is given by

$$q^{x_0+(r+1)y_0}\binom{x_0+y_0}{x_0}_q\binom{(r-x_0)+y_0}{(r-x_0)}_q\binom{x_0+(s-y_0)}{x_0}_q\binom{(r-x_0)+(s-y_0)}{(r-x_0)}_q.$$

Summing over all possible choices of root, and making a convenient substitution of variable, yields

Theorem 3.2

$$N_q(r,1,s,1) = \sum_{\substack{a+b=r\\c+d=s}} q^{a+(r+1)c}\binom{a+c}{a}_q\binom{b+c}{b}_q\binom{a+d}{a}_q\binom{b+d}{b}_q.$$

From the theorem follows the corollary

Corollary 3.3 *Stembridge's "$q=-1$" phenomenon holds in the case of an $(r,1,s,1)$ octagon. Namely, the number of tilings of an $(r,1,s,1)$ octagon invariant under $180°$ rotation is obtained by evaluating the limit of $N_q(r,1,s,1)$ as $q \to -1$.*

Proof: Using the bijection between tilings of an $(r, 1, s, 1)$ octagon and once-crossing paths on an r by s grid, it is easy to see that the following statements characterize tilings invariant under 180° rotation:

- r and s must both be even.
- The root must be located at position $(r/2, s/2)$.
- The two once-crossing paths are determined by the paths in any two adjacent quadrants of the grid.

These conditions together imply that when r and s are even, there are $\binom{a+c}{a}^2$ tilings of an $(r, 1, s, 1)$ octagon which are invariant under 180° rotation, where $a = r/2$ and $c = s/2$, and when either r or s is odd the numbers of such tilings is zero.

When $N_q(r, 1, s, 1)$ is evaluated at $q = -1$, the symmetry of the q-binomial coefficients guarantees that the sum vanishes unless r and s are even. If r is odd, then the terms corresponding to a will cancel the terms corresponding to $r - a$, so r must be even. If r is even and s is odd, then the terms corresponding to c will cancel the terms corresponding to $s - c$, so s must be even also. Additionally, the reader can verify that the following lemma holds:

Lemma 3.4 *Taking the limit as $q \to -1$, the q-binomial coefficient $\binom{n}{k}_q$ is equal to*

$$\binom{n/2}{k/2} \quad \textit{for } n \textit{ even, } k \textit{ even} \qquad \binom{(n-1)/2}{k/2} \quad \textit{for } n \textit{ odd, } k \textit{ even}$$

$$0 \quad \textit{for } n \textit{ even, } k \textit{ odd} \qquad \binom{(n-1)/2}{(k-1)/2} \quad \textit{for } n \textit{ odd, } k \textit{ odd}$$

Thus when taking the limit of $N_q(r,1,s,1)$ as $q \to -1$, the sum breaks into three pieces:

$$\begin{aligned} N(r,1,s,1)_q|_{q=-1} = & \sum_{\substack{w \text{ even} \\ \text{in}[0,r]}} \sum_{\substack{y \text{ even} \\ \text{in}[0,s]}} \binom{\frac{w+y}{2}}{\frac{w}{2}} \binom{\frac{x+y}{2}}{\frac{x}{2}} \binom{\frac{w+z}{2}}{\frac{w}{2}} \binom{\frac{x+z}{2}}{\frac{x}{2}} \\ & - \sum_{\substack{w \text{ even} \\ \text{in}[0,r]}} \sum_{\substack{y \text{ odd} \\ \text{in}[1,s-1]}} \binom{\frac{w+y-1}{2}}{\frac{w}{2}} \binom{\frac{x+y-1}{2}}{\frac{x}{2}} \binom{\frac{w+z-1}{2}}{\frac{w}{2}} \binom{\frac{x+z-1}{2}}{\frac{x}{2}} \\ & - \sum_{\substack{w \text{ odd} \\ \text{in}[1,r-1]}} \sum_{\substack{y \text{ even} \\ \text{in}[0,s]}} \binom{\frac{w+y-1}{2}}{\frac{w-1}{2}} \binom{\frac{x+y-1}{2}}{\frac{x-1}{2}} \binom{\frac{w+z-1}{2}}{\frac{w-1}{2}} \binom{\frac{x+z-1}{2}}{\frac{x-1}{2}} \end{aligned}$$

where $w + x = r$ and $y + z = s$. Substituting $a = r/2$ and $c = s/2$ gives:

$$\begin{aligned} N(r,1,s,1)_q|_{q=-1} = & \sum_{m_1+n_1=a} \sum_{k_1+l_1=c} \binom{m_1+k_1}{m_1} \binom{n_1+k_1}{n_1} \binom{m_1+l_1}{m_1} \binom{n_1+l_1}{n_1} \\ & - \sum_{m_2+n_2=a} \sum_{k_2+l_2=c-1} \binom{m_2+k_2}{m_2} \binom{n_2+k_2}{n_2} \binom{m_2+l_2}{m_2} \binom{n_2+l_2}{n_2} \\ & - \sum_{m_3+n_3=a-1} \sum_{k_3+l_3=c} \binom{m_3+k_3}{m_3} \binom{n_3+k_3}{n_3} \binom{m_3+l_3}{m_3} \binom{n_3+l_3}{n_3} \\ & = N(a,1,c,1) - N(a,1,c-1,1) - N(a-1,1,c,1) \\ & = \binom{a+c}{a}^2. \end{aligned}$$

The final equality holds by Proposition 3.1. □

Direct calculation shows that Stembridge's "$q = -1$ phenomenon" fails in the case of a $(2,2,2,1)$ octagon. This suggests that the phenomenon has a threshold, based on the complexity of the oriented matroid of $Z(V)$, beyond which it fails. This is similar to the properties of coherence of $Z(V)$ and freeness of $\mathcal{D}(V)$, as discussed in [ER].

The "$q = -1$ phenomenon" and complementation also make sense for an arbitrary d-zonotope $Z(V)$. The statistic measures distance from some chosen base tiling in the *mutation graph* for $Z(V)$ (choice of base tiling is arbitrary), and complementation is defined as follows. For a tiling T, let σ_T be the localization for $\mathcal{M}(V^*)$ corresponding to T (see chapter 5). Then T^c is the tiling corresponding to $-\sigma_T$. The $q = -1$ phenomenon for tilings of d-zonotopes with $d > 2$ remains largely unexplored.

4 Tilings of zonotopes from tilings of subzonotopes

4.1 Introduction

In the first chapter, we demonstrated the correspondence between plane partitions and rhombic tilings of a hexagon:

3	2	2
3	2	0
1	1	0

$\longleftrightarrow$

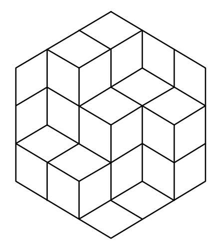


Figure 17: Figure 2 revisited

One may ask whether such a correspondence exists between stacks of $(d+1)$-dimensional cubes in an $r_1 \times r_2 \times \cdots \times r_{d+1}$ box and rhombic tilings of a d-zonotope. This question will be answered in chapter 5. One might also ask whether any similar correspondence exists for 2-zonotopes $Z(V)$ with $|\overline{V}| > 3$, or for arbitrary zonotopes in any dimension.

A first step toward answering this question is to notice that there is a different way to obtain the same correspondence between rhombic tilings of the hexagon and plane partitions. If the integer entries in an r by s plane partition are $0, 1, \ldots, t$, then there is a unique way to introduce t *worms* into the depiction of the plane partition. If one then "fattens the worms" along a fixed direction vector $\mathbf{v}$, the result is a tiling of an (r, s, t) hexagon. Since $\mathbf{v}$ is specified, it is clear that this approach defines a bijective correspondence. Figure 18 illustrates this idea, where the fixed direction vector is $\mathbf{v} = (-1, 1)$.

The first worm follows the boundary between zero entries and nonzero

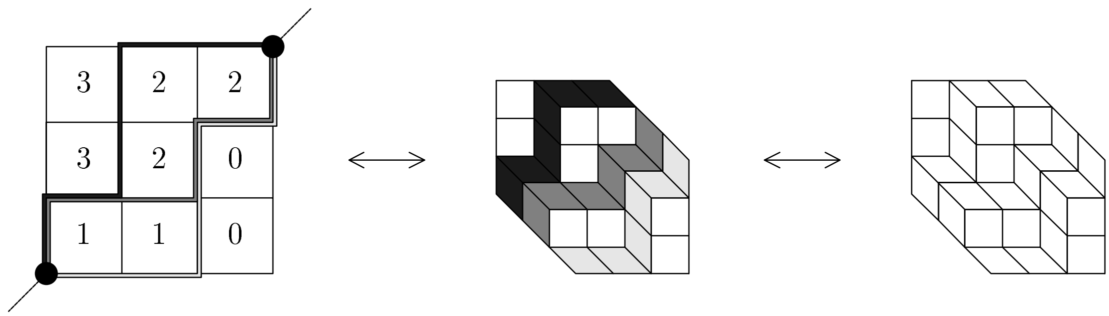


Figure 18: "Worms" define a bijection between plane partitions and tilings

entries (for this purpose, we consider the southeast corner of the unbounded region to have the entry 0, while the northwest corner has entry t). The second worm defines the boundary between those cells with entries 1 or less and the remaining cells, the third worm defines the boundary between those cells with entries 2 or less and the remaining cells, etc. It is easy to see that the resulting tiling is identical to the one obtained in Figure 17, modulo a 45° rotation and a slight deformation.

This view of the correspondence between partitions and tilings suggests a method for generalizing the correspondence, one which appears in work by Destainville, Mosseri and Bailly [DMB]. Specifically, the initial plane partition in Figure 18 is a *partition on a tiling*, in that it is the trivial tiling of a $(3,3)$ 2-zonotope, with integer entries on the tiles satisfying a partial order induced by $\mathbf{v}$. The integer entries are weakly decreasing along $\mathbf{v}$, and allow one to define a unique tiling of a $(3,3,3)$ 2-zonotope. Moreover, it is clear that every tiling of a $(3,3,3)$ hexagon corresponds to a tiling of a $(3,3)$ square, together with a unique choice of integer entries on each tile which obey the partial order induced by some direction vector $\mathbf{v}$.

Taking this view, let $(Z,T,\mathbf{v})$ be a triple, where Z is an $(r_1,\ldots,r_m)$ 2-zonotope, T is a tiling of Z, and the direction vector $\mathbf{v}$ is not a scalar multiple of any of the generating vectors of Z. A unique tiling of an $(r_1, \ldots, r_m, r_{m+1})$ 2-zonotope Z' is obtained by placing integer entries in $\{0, \ldots, r_{m+1}\}$ on the tiles of T in accordance with the partial order induced by $\mathbf{v}$. An example of this generalized correspondence is given in Figure 19.

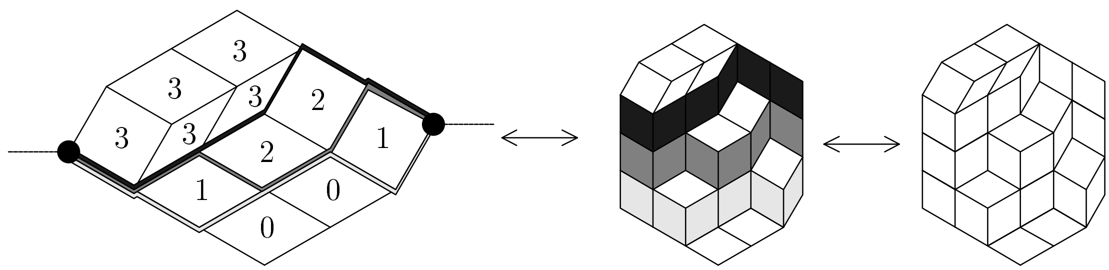


Figure 19: A partition on a tiling of a $(3, 2, 1)$ hexagon defines a unique tiling of a $(3, 3, 2, 1)$ octagon

To make this correspondence precise, and to carry this technique to higher-dimensional zonotopes, requires a suitable definition of the partial order $\succ$ on tiles induced by $\mathbf{v}$. This in turn requires an oriented matroid argument first presented by Sturmfels and Ziegler [SZ].

4.2 Oriented matroids and tilings

Most of the work in this section, with the exception of Theorem 4.4, is essentially a restatement of work done by Sturmfels and Ziegler ([SZ], section 3). Let $Z = Z(V)$ be a d-zonotope with tiling T. By the Bohne-Dress Theorem (Theorem 2.3), T corresponds to a unique single-element lifting of the oriented matroid $\mathcal{M}(V)$ of V by some element g. Let $\mathcal{M}$ denote the oriented matroid obtained as the result of this lifting. The pair $(\mathcal{M}, g)$ is an *affine oriented matroid.*

By the Topological Representation Theorem of Folkman and Lawrence (Theorem 2.1), the oriented matroid $\mathcal{M}$ may be represented as an arrangement $\mathcal{A}_{\mathcal{M}}$ of signed $(d-1)$-pseudospheres on the d-sphere in $\mathbb{R}^{d+1}$. Let S_g be the $(d-1)$-pseudosphere corresponding to g. Then S_g determines a positive hemisphere S_g^+ of the d-sphere, and the affine oriented matroid $(\mathcal{M}, g)$ may be viewed as the collection of all cocircuits Y of $\mathcal{M}$ satisfying $Y_g = +$. Cocircuits satisfying $Y_g = 0$ are said to be *at infinity*.

Recall that *coherent* tilings of a d-zonotope arise from "viewing" a $(d+1)$-zonotope. In essence, the Bohne-Dress Theorem says that, given $\mathcal{M}(V)$ and g, one obtains the corresponding tiling T of Z by "viewing" the affine pseudosphere arrangement $(\mathcal{M}, g)$. An example is given in Figure 20 for a hexagon with side length three.

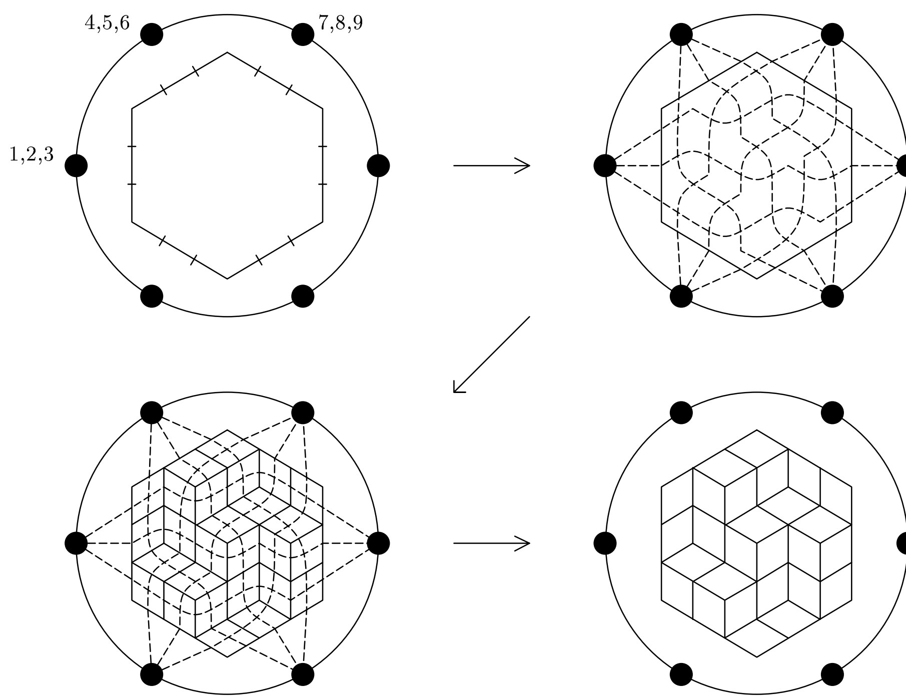


Figure 20: An affine pseudosphere arrangement for a lifting of $\mathcal{M}(Z)$ induces a tiling of Z.

Notice that under this correspondence, the tiles of T correspond to 0-cells in the pseudosphere arrangement, which in turn correspond to cocircuits of $\mathcal{M}$. In particular, a tiling T of a d-zonotope Z is rhombohedral if and only if each 0-cell in the pseudosphere arrangement is determined by the intersection of exactly d pseudospheres.

Having established this correspondence, we next consider *worms.* For a given oriented matroid $\mathcal{M}$, a *single-element extension* of $\mathcal{M}$ is simply the addition of a single pseudosphere to the arrangement $\mathcal{A}_{\mathcal{M}}$. In particular, the addition of a single worm in a zonotopal tiling induces a single-element extension not only of the oriented matroid $\mathcal{M}(V)$ associated with $Z(V)$, but also of $\mathcal{M}$, the oriented matroid obtained as a result of the single-element lifting of $\mathcal{M}(V)$ by g (since the worm influences the tiling of Z as well). Thus what we wish to consider are single-element extensions of the affine oriented matroid $(\mathcal{M}, g)$, which in turn arise from single-element extensions of $\mathcal{M}$.

When an oriented matroid $\mathcal{M}$ is extended by an element f, each cocircuit $Y \in \mathcal{C}^*$ naturally receives a signature $\sigma_f(Y) \in \{+,-,0\}$, depending on whether Y lies in S_f^+, S_f^-, or on S_f. Any cocircuit signature $\sigma : \mathcal{C}^* \to \{+,-,0\}$ which corresponds to a single-element extension is called a *localization.* Las Vergnas [LV] showed that a cocircuit signature σ is a localization for the oriented matroid $\mathcal{M}$ if and only if the restriction $\sigma|_R$ is a localization for every *rank 2 contraction* R of $\mathcal{M}$. A rank 2 contraction of an oriented matroid may be thought of as a collection of cocircuits which are contained in a 1-pseudosphere in the arrangement $\mathcal{A}_{\mathcal{M}}$.

Theorem 4.1 *Let $\mathcal{M}$ be an oriented matroid, and*

$$\sigma : \mathcal{C}^* \to \{+,-,0\}$$

a cocircuit signature, satisfying $\sigma(-Y) = -\sigma(Y)$ for all $Y \in \mathcal{C}^*$. Then the following statements are equivalent:

1) σ is a localization: there exists a single-element extension $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ such that

$$\{(Y, \sigma(Y)) | Y \in \mathcal{C}^*\} \subseteq \widetilde{\mathcal{C}}^*.$$

2) σ defines a single-element extension on every contraction of $\mathcal{M}$ of rank 2. That is, the signature on every rank 2 contraction is one of the types I, II and III shown in Figure 21.

3) The signature σ produces none of the three excluded subconfigurations (minors) of rank 2 on three elements, as given by Figure 22. $\square$

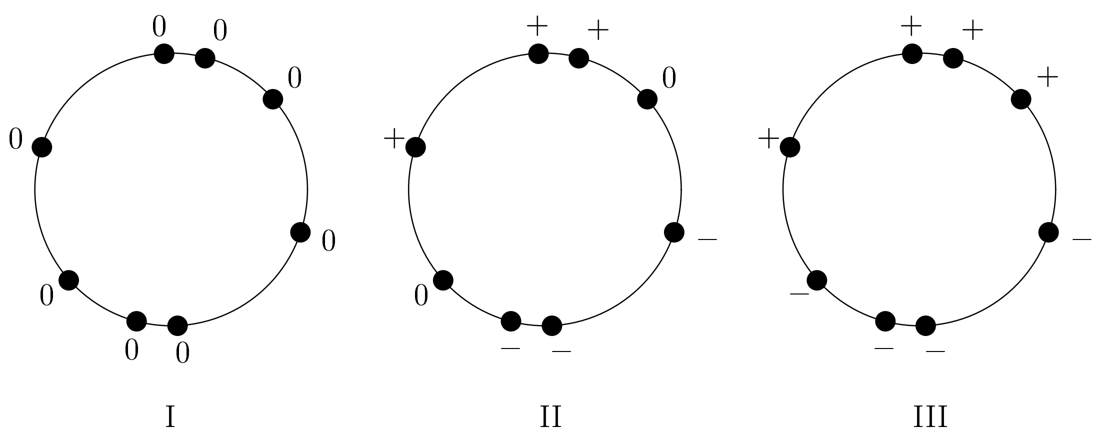


Figure 21: The three types of allowable cocircuit signature for a rank 2 oriented matroid

Let $\widetilde{\mathcal{M}}$ be the single-element extension of $\mathcal{M}$ by an element f, and let σ_f be the corresponding localization in $\mathcal{M}$. The triple $(\widetilde{\mathcal{M}}, g, f)$ is an *oriented matroid program.* For any oriented matroid $\mathcal{M}$ with g an element of $\mathcal{M}$, the *contraction* $\mathcal{M}/g$ may be thought of as the collection of cocircuits Y of $\mathcal{M}$ with $Y_g = 0$. In the pseudosphere picture, $\mathcal{M}/g$ corresponds to the pseudosphere arrangement in S_g defined by the collection of all intersections $S_f \cap S_g$ with $f \in \mathcal{M}$, $f \neq g$. The *extension set* $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ of $(\widetilde{\mathcal{M}}, g, f)$

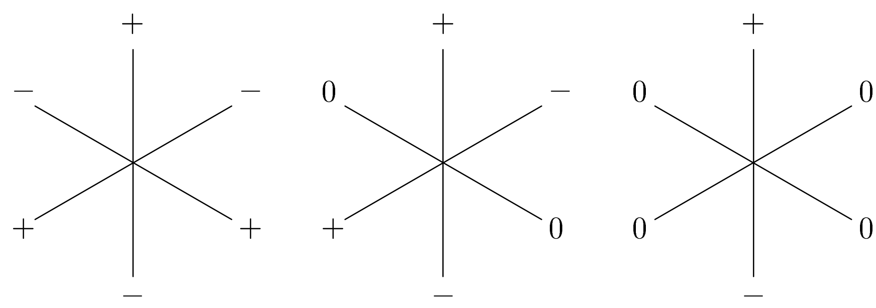


Figure 22: The three forbidden rank 2 cocircuit signatures

consists of all extensions $\mathcal{M} \cup f'$ of $\mathcal{M}$ by an element f' such that $\widetilde{\mathcal{M}}/g = (\mathcal{M} \cup f')/g$, or equivalently, $S_f \cap S_g = S_{f'} \cap S_g$ in $\mathcal{A}_{\mathcal{M}}$. The key point is that $\sigma_f(X) = \sigma_{f'}(X)$ for all cocircuits S in $\widetilde{\mathcal{M}}/g$. Informally, $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ is the collection of all possible worms which may be added to T (including worms whose addition results in a non-rhombohedral tiling!), where the image of f (equivalently of f') in $\widetilde{\mathcal{M}}/g = (\mathcal{M} \cup f')/g$ plays the same role in this construction that $\mathbf{v}$ does, above.

Define the *graph* G_f of $\widetilde{\mathcal{M}}$ as follows. The vertices of G_f are the cocircuits Y of $\mathcal{M}$ with $Y_g = +$. These correspond to the 0-cells of the arrangement $\mathcal{A}_{\mathcal{M}}$ which lie in the affine space S_g^+. Two such vertices (Y^0, Y^1) are connected by an edge E in G_f if and only if they are connected by a 1-cell L in $\mathcal{A}_{\mathcal{M}}$ (that is, if Y^0 and Y^1 determine a rank 2 contraction of $\mathcal{M}$). Two more cocircuits Z and $-Z$ exist at the two points in $\mathcal{A}_{\mathcal{M}}$ at which S_g and L intersect. Either σ_f is zero on both Z and $-Z$, or $\sigma_f(Z) = -\sigma_f(-Z) \neq 0$. If the latter case holds, then direct E in the direction of increasing σ_f; otherwise, consider E to be a bidirected edge. Since adjacent vertices in G_f correspond to adjacent tiles in the tiling T, it is easy to see that G_f must be connected.

A *path* in G_f is a sequence of vertices $P = (Y^0, Y^1, \ldots, Y^k)$ such that the edge between Y^{i-1} and Y^i is either directed toward Y^i or a bidirected edge. A path P is *directed* if at least one edge in P is directed, and *undirected* otherwise.

Two vertices Y and Y' of G_f are said to be *equivalent* if there is a path from Y to Y' and a path from Y' to Y. A *strong component* of G_f is the induced subgraph of an equivalence class of vertices. A strong component is said to be *very strong* if it contains at least one directed edge. Let $\mathrm{SC} = \mathrm{SC}(\widetilde{\mathcal{M}}, g, f)$ denote the set of strong components, and let $\mathrm{VSC} = \mathrm{VSC}(\widetilde{\mathcal{M}}, g, f)$ be the subset of very strong components. There is a natural partial order on SC. For two strong components $\mathbf{c}$ and $\mathbf{c}'$, set $\mathbf{c} < \mathbf{c}'$ whenever there exists a directed path from a vertex Y in $\mathbf{c}$ to a vertex Y' in $\mathbf{c}'$. The set VSC is a subposet of SC, with the induced partial order. Figure 23 illustrates the posets of strong components corresponding to the tilings in Figures 17 and 18.

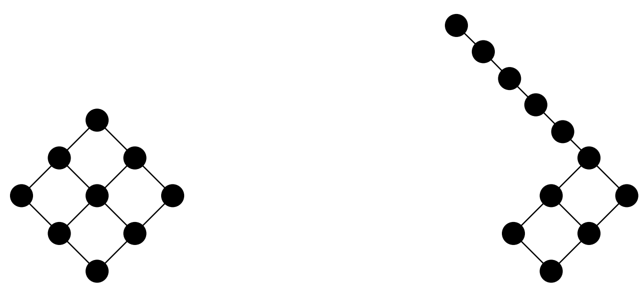


Figure 23: SC for the tilings in Figure 18 (left) and Figure 19

For both of these cases, the set VSC is empty and each individual tile is a strong component. This is always the case for 2-zonotopes Z for which $\mathbf{v}$ is distinct from the generating vectors of Z ([SZ], Corollary 4.5). Examples of 3-zonotopes with nonempty VSC set appear in Example 3.5 of [SZ] and Example 10.4.1 in [BLSWZ]. The following sequence of lemmas relates the extension space of an oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ to the order ideals in its poset SC of strong components.

Lemma 4.2 *([SZ Lemma 3.6]) Let* $\sigma \in \mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ *and* $\mathbf{c} \in \mathrm{SC}$.

1) The localization σ *has the same value on each cocircuit in* $\mathbf{c}$*, so that* $\sigma(\mathbf{c})$ *is well defined.*

2) The sets $I' := \{\mathbf{c} \in \mathrm{SC} | \sigma(\mathbf{c}) = -\}$ *and* $I := \{\mathbf{c} \in \mathrm{SC} | \sigma(\mathbf{c}) \neq +\}$ *are order ideals in the poset* SC.

3) For each very strong component $\mathbf{c} \in \mathrm{VSC}$*, we have* $\sigma(\mathbf{c}) \neq 0$.

Proof: By Theorem 4.1, one may conclude that σ is constant on bidirected edges and weakly increasing (in the order $- \prec 0 \prec +$) along directed edges of G_f. This proves parts (1) and (2). Furthermore, if σ assigns the signature 0 to any pair of cocircuits Y^0, Y^1 which are adjacent in G_f, then necessarily σ is identically zero on all cocircuits in the rank 2 contraction containing Y^0 and Y^1. In particular, the edge connecting Y^0 and Y^1 is bidirected. This observation, together with part (1), proves part (3). $\square$

Lemma 4.3 *([SZ Lemma 3.7]) Let* $I' \subseteq I$ *be order ideals of* SC *such that* $I \backslash I'$ *is an antichain in* SC *which does not intersect* VSC*. Then there is a unique localization* $\sigma \in \mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ *such that, for all* $\mathbf{c} \in \mathrm{SC}$,

$$\sigma(\mathbf{c}) = \begin{cases} - & \textit{if } \ \mathbf{c} \in I', \\ 0 & \textit{if } \ \mathbf{c} \in I \backslash I', \\ + & \textit{otherwise.} \end{cases}$$

Proof: The localization σ is determined by the requirements that

$$\sigma(Y) = \begin{cases} Y_f & \text{if } \ Y_g = 0, \\ - & \text{if } \ Y_g = + \ \text{ and } \ [Y] \in I', \\ 0 & \text{if } \ Y_g = + \ \text{ and } \ [Y] \in I \backslash I', \\ + & \text{if } \ Y_g = + \ \text{ and } \ [Y] \notin I, \end{cases}$$

where $[Y] \in \mathrm{SC}$ denotes the equivalence class of the cocircuit Y. By construction, σ is a localization on every rank 2 contraction of $\mathcal{M}$, and so by Theorem 4.1, σ is a localization on $\mathcal{M}$. The assumptions on $I \backslash I'$ are equivalent to the fact that there is no directed edge both of whose vertices are in $I \backslash I'$. $\square$

Lemmas 4.2 and 4.3 demonstrate that there is a bijection between elements of $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ (worms which induce a tiling of a single-element extension Z' of Z from a tiling of Z) and pairs of order ideals (I, I') in SC. Furthermore, since we are only concerned with rhombohedral tilings, we only wish to consider those elements of $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ corresponding to localizations σ which are never equal to zero, or equivalently to pairs (I, I') with $I = I'$. Thus there is a bijection between *uniform* elements of $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ (those worms which result in a rhombohedral tiling of Z') and order ideals in SC. All that remains is to discuss a convenient notation for the addition of multiple worms to a tiling T.

Given a poset P, Stanley [Sta3] defines a *P-partition* of n to be an order-reversing map $\pi : P \to \mathbb{N}$ satisfying $\sum_{i \in P} \pi(i) = n$. For the sake of convenience, we will define our P-partitions to be **order-preserving**. P-partitions generalize partitions in that if P is a p-element chain, then a P-partition of n is an ordinary partition of n into at most p parts.

Theorem 4.4 *[cf. Section 2.4 of [DMB]] Let $(Z, Z', T, \mathbf{v})$ be a quadruple, where*

1) $Z(V)$ is a d-zonotope,

2) $Z' = Z(V')$ is a d-zonotope such that $\overline{V}' = \overline{V} \cup \{\mathbf{v}\}$,

3) T is a rhombohedral tiling of Z, and

4) $\mathbf{v}$ is a fixed element of $\mathbb{R}^d$ which is not a scalar multiple of the generating vectors of Z.

Suppose $\mathbf{v}$ *occurs with multiplicity* r *in* V'*, and let* $\mathrm{SC} = \mathrm{SC}(\widetilde{\mathcal{M}}, g, f)$ *be as described in the above construction, where in particular,* $f/g = \mathbf{v}$*. There is a bijection between* SC*-partitions of* r *and tilings* T' *of* Z' *with the property that contraction of* Z' *along all zones parallel to* $\mathbf{v}$ *results in the original tiling* T *of* Z*.*

Proof: Suppose π is an SC-partition of r. Let I be the order ideal in SC defined by $I := \{\mathbf{c} \in \mathrm{SC} \,|\pi(\mathbf{c}) \leq r-1\}$. Lemmas 4.2 and 4.3 show that I defines a uniform extension of $\mathcal{M}$ by a unique element f_1 of $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ — or equivalently, a unique tiling T_1 of the zonotope $Z_1 = Z(V_1)$, where $V_1 = V \cup \{f_1/g\} = V \cup \{\mathbf{v}\}$. Let G_{f_1} be the graph corresponding to the oriented matroid program $(\widetilde{\mathcal{M}}_1, g, f)$, where $\mathcal{M}_1 = \mathcal{M} \cup f_1$.

Let $G_{f_1} - G_f$ denote the induced subgraph of G_{f_1} on vertices not in G_f. Since G_{f_1} is obtained from G_f by adding the pseudosphere S_{f_1} to the arrangement $\mathcal{A}_{\mathcal{M}}$, and since by construction $S_{f_1} \cap S_g = S_f \cap S_g$, any two adjacent "new" vertices will be joined by a bidirected edge. Moreover, since all vertices in $G_{f_1} - G_f$ correspond to tiles which are all in the same zone of Z_1, the induced subgraph on $G_{f_1} - G_f$ is connected. Thus all vertices of the induced subgraph on $G_{f_1} - G_f$ lie in the same strong component $\mathbf{c}_1$ of $\mathrm{SC}_1 = \mathrm{SC}(\widetilde{\mathcal{M}}_1, g, f)$. Since f_1 is a uniform extension, it is clear that if $(\widetilde{\mathcal{M}}_1, g, f)$ is extended by an additional copy of f_1, then $\mathbf{c}_1$ receives the signature zero while all other strong components receive a nonzero signature. Therefore we must conclude that $\mathbf{c}_1 := \{Y \in \mathcal{C}_1^* | Y_{f_1} = 0\}$, where $\mathcal{C}_1^*$ is the set of cocircuits of $\mathcal{M}_1$. Finally, since all directed edges from elements $\mathbf{c}' \in I$ to elements $\mathbf{c}'' \notin I$ must pass through S_{f_1}, it is clear that $\mathbf{c} \prec \mathbf{c}_1$ in SC_1 if and only if $\mathbf{c} \in I$ as an element of SC, and similarly $\mathbf{c} \succ \mathbf{c}_1$ in SC_1 if and only if $\mathbf{c} \notin I$ as an element of SC.

Essentially, given an element f_1, one obtains the poset SC_1 by adding an *articulation point* $\mathbf{c}_1$ which lies below all elements of I^c and above all elements of I.

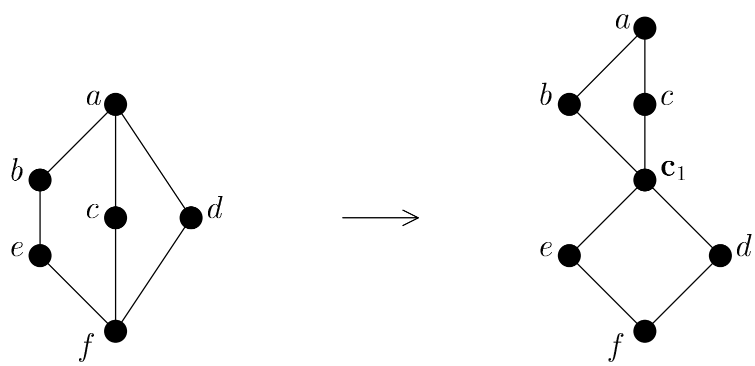


Figure 24: Passing from SC to SC_1

Given SC_1, it is clear that the SC-partition π of r induces an SC_1-partition π_1 of $r-1$ as follows.

$$\pi_1(\mathbf{c}) = \begin{cases} r-1 & \text{if } \mathbf{c} \notin I \\ \pi(\mathbf{c}) & \text{otherwise.} \end{cases}$$

This description includes $\pi_1(\mathbf{c}_1) = r-1$, since $\mathbf{c}_1 \notin I$. It is clear that one may iterate this process to obtain a unique sequence $(f_1, f_2, \ldots, f_r)$ of successive single-element extensions and consequently a uniquely determined tiling T' of Z' as described in the statement of the theorem.

Conversely, given a tiling T' of Z', it is easy to see how to obtain a unique SC-partition of r simply by contracting along each of the r zones parallel to $\mathbf{v}$ in turn, and for each tile of T' which does not have a copy of $\mathbf{v}$ as a Minkowski summand, keeping track of the number of times it lies "above" a contracted zone. □

Corollary 4.5 *Let Z, Z', $\mathbf{v}$ be as in the statement of Theorem 4.4, where $\mathbf{v}$ has multiplicity r in V'. Then the set $\mathcal{T}'$ of rhombohedral tilings of Z' decomposes*

$$\mathcal{T}' = \coprod_{\substack{tilings\ T \\ of\ Z}} \mathcal{T}'_T$$

where $\mathcal{T}'_T$ *is the subset of rhombohedral tilings of* Z' *which yield the tiling* T *of* Z *when all zones of* Z' *corresponding to* $\mathbf{v}$ *are deleted. Furthermore,* $\mathcal{T}'_T$ *is in bijective correspondence with the set of* $\mathrm{SC}(\widetilde{\mathcal{M}}, g, f)$*-partitions of* r*, where* $(\widetilde{\mathcal{M}}, g, f)$ *is the oriented matroid program corresponding to* $(Z, T, \mathbf{v})$*.*

Stanley [Sta3] shows how to decompose P-partitions according to linear extensions of P, leading to the formula

$$|\{P\text{-partitions of } r\}| = \sum_{\omega \in \mathcal{L}(P)} \binom{p + r - \mathrm{des}(\omega) - 1}{p}$$

where $\mathcal{L}(P)$ is the *Jordan-Hölder set* of P (see [Sta3], section 3.12), $\mathrm{des}(\omega)$ denotes the cardinality of the descent set of the permutation ω, and $p = |P|$. Combining this with Corollary 4.5 gives a formula for counting tilings similar to the one given in the Descent Theorem in section 5.5.2 of [DMB].

5 MacMahon zonotopes

5.1 Introduction

In this chapter and the next, we describe a method for determining whether a particular d-zonotope is coherent, and illustrate this method for several classes of zonotopes. The technique used is quite straightforward; given a d-zonotope Z:

a) Enumerate all tilings of Z using the Bohne-Dress Theorem (Theorem 2.3) and Las Vergnas' localization theorem (Theorem 4.1),

b) Enumerate the **coherent** tilings of Z using the result of Billera and Sturmfels (Theorem 2.2), along with techniques for counting chambers in hyperplane arrangements,

c) Compare.

Although this is the same technique employed by Edelman and Reiner [ER] in their classification of coherent 2-zonotopes, they had the advantage that the tiling counts for step (a) were already extant in the literature. This is the first time that the computational technique in step (a) has been explained, together with sample computations. It is also, so far as we know, the first time the Las Vergnas result has been used to solve such a problem.

In this chapter, we focus on the class of d-zonotopes $Z = Z(V)$ with the property that $\overline{V}$ consists of $d+1$ vectors in general position. We call them *MacMahon zonotopes* because they generalize the zonotopes arising from the plane partitions MacMahon studied. The case of MacMahon zonotopes serves as a gentle introduction to the counting methods used in step (a), and as an added bonus, the high degree of structure and symmetry in MacMahon zonotopes allows us to use the construction of Sturmfels and Ziegler [SZ] introduced in the last chapter to demonstrate that they satisfy the Baues conjecture (see [BKS]) — namely, that the Baues poset on the zonotopal tilings of a MacMahon zonotope has the homotopy type of a sphere.

5.2 Counting tilings and coherent tilings

Enumerating the **coherent** tilings of a d-zonotope $Z = Z(V)$ is relatively straightforward. By Theorem 2.2, the set of all coherent tilings of $Z(V)$ is in bijective correspondence with the chambers of $\mathcal{D}(V)$. In all cases considered below, $\mathcal{D}(V)$ is *free* with *exponents* $b_1, b_2, \ldots, b_m$. Terao showed [Te] that these exponents are the roots of the *characteristic polynomial* $\chi(\mathcal{D}(V), t)$ of $\mathcal{D}(V)$. Zaslavsky [Za1] showed that the number of chambers in an arrangement $\mathcal{A}$ is computed by $|\chi(\mathcal{A}, -1)|$. Thus we have

Theorem 5.1 *If $\mathcal{D}(V)$ is a free arrangement, then the number of coherent tilings of $Z(V)$ is counted by*

$$\prod(1 + b_i),$$

where $\{b_1, b_2, \ldots, b_m\}$ are the exponents of $\mathcal{D}(V)$.

In most cases presented below, $\mathcal{D}(V)$ lies in one of two infinite classes of free arrangements. One of these families was shown to be free by Athanasiadis [Ath]. The other family will be proven free in the final chapter.

Enumerating **all** tilings of Z is somewhat more complicated. Recall that a d-zonotope $Z = Z(V)$ generated by n vectors ($n \geq d$) may be thought of as the image of the n-cube C under some affine projection π. In particular, we may take π to be a linear map of the n-cube into some rank d subspace W of $\mathbb{R}^n$, and so π may be represented as a $d \times n$ matrix. Then the columns of π may be taken for the generating set V of $Z(V)$ (up to projective equivalence). Let U be the orthogonal complement of W in $\mathbb{R}^n$, and let $\pi^\perp$ be the linear map of C into U satisfying $\pi \circ \pi^\perp = \pi^\perp \circ \pi = 0$. Then $\pi^\perp$ may be represented as an $(n-d) \times n$ matrix. Denote the columns of $\pi^\perp$ by V^*. The image of $\pi^\perp$ is an $(n-d)$-zonotope $Z^* = Z(V^*)$, the *dual* zonotope to Z.

Not only are Z and Z^* dual in the obvious (geometric) sense, but they are also dual in that the corresponding oriented matroids $\mathcal{M}(V)$ and $\mathcal{M}(V^*)$ are dual in the oriented matroid sense. Specifically, the circuits $\mathcal{C}$ of the oriented matroid $\mathcal{M}(V)$ are the cocircuits $\mathcal{C}^*$ of the oriented matroid $\mathcal{M}(V^*)$, and there is a bijective correspondence between single-element liftings T of $\mathcal{M}(V)$ and single-element extensions σ of $\mathcal{M}(V^*)$. By Theorem 2.3, we may conclude that there is a bijection between (not necessarily rhombohedral) tilings of the zonotope $Z = Z(V)$ and single-element extensions of the oriented matroid $\mathcal{M}(V^*)$. As noted in the last chapter, rhombohedral tilings correspond to *uniform* localizations, those localizations σ satisfying $\sigma(X) \neq 0$ for all $X \in \mathcal{C}^*$.

Therefore, what is required to enumerate the set of all tilings of Z is to

i) Determine all rank 2 contractions of $\mathcal{M}(V^*)$,

ii) For each rank 2 contraction R, determine a complete list of uniform localizations for the cocircuits of R,

iii) Use this collection of rank 2 localizations to determine a complete list of uniform localizations for the entire collection $\mathcal{C}^*$ of cocircuits of $\mathcal{M}(V^*)$.

By Theorem 4.1, the set of cocircuit signatures σ which induce a localization on every rank 2 contraction R is exactly the set of localizations for $\mathcal{C}^*$.

A *rank 2 contraction* of a realizable rank m oriented matroid $\mathcal{M} = \mathcal{M}(V)$ is a rank 2 subspace of the arrangement $\mathcal{A}(V)$ which arises from the intersection of $(m-2)$ independent elements of $\mathcal{A}(V)$ (here we assume $\mathcal{A}(V)$ is essential). For any pair X, Y of cocircuits in $\mathcal{C}^*$, define $O_{X,Y}$ to be the set of indices i such that $X_i = Y_i = 0$. Define $V^*_{X,Y}$ to be the set of vectors $\{v_i \in V^* \mid i \in O_{X,Y}\}$. Then X and Y define a rank 2 contraction R of the oriented matroid $\mathcal{M}(V^*)$ if and only if $\text{span}(V^*_{X,Y}) = n - d - 2$.

Thus in order to determine the rank 2 contractions of $\mathcal{M}(V^*)$, it is necessary to compute V^*. Since V and V^* arise from mutually orthogonal projections π and $\pi^\perp$ of the n-cube C, it is clear that the rows of π must be pairwise orthogonal with the rows of $\pi^\perp$. Given V, this orthogonality uniquely determines V^* up to (projective and) oriented matroid equivalence. Therefore, given a rank d vector configuration V with $|V| = n$, we may take V^* to be any rank $(n-d)$ configuration of n vectors such that, when viewed as matrices, the rows of V and V^* are pairwise orthogonal to one another.

5.3 Coherent MacMahon zonotopes

A d-zonotope $Z(V)$ is a *MacMahon* zonotope if $\overline{V}$ consists of $d+1$ distinct vectors in general position. It is clear that $\overline{V}$ is projectively equivalent to the *frame* in $\mathbb{R}^d$, namely the standard basis vectors together with the vector $(1, 1, \ldots, 1)$. Thus any MacMahon d-zonotope Z is uniquely determined by the multiplicities of its generating vectors, and it is reasonable to discuss "the" $\{r_1, r_2, \ldots, r_{d+1}\}$ MacMahon d-zonotope, where $\overline{V}$ is the frame.

Theorem 5.2 *The MacMahon $\{r_1, r_2, \ldots, r_{d+1}\}$ d-zonotope Z is coherent if and only if*

- $r_i \geq 2$ *for at most three indices i, and*
- $r_i \geq 3$ *for at most two indices.*

Furthermore, the MacMahon $\{r, s, 2, 1, \ldots, 1\}$ d-zonotope has exactly

$$\frac{2(r+s+1)!(r+s)!}{(r+1)!(s+1)!}$$

tilings, and the MacMahon $\{r, s, 1, \ldots, 1\}$ d-zonotope has exactly $(r+s)!$ tilings.

We begin by showing that the set of **all** tilings of the $\{r_1, r_2, \ldots, r_{d+1}\}$ MacMahon d-zonotope is in bijection with the number of ways of stacking $(d+1)$-cubes "flush into the corner" of an $r_1 \times r_2 \times \cdots \times r_{d+1}$ hyperbox. Specifically, we show

Proposition 5.3 *The collection of tilings of the $\{r_1, r_2, \ldots, r_{d+1}\}$ MacMahon d-zonotope is in bijection with the set*

$$\mathcal{J}\left(\prod_{i=1}^{d+1}[r_i]\right) \times \prod_{i=1}^{d+1} \mathcal{S}_{r_i},$$

where $[r_i]$ denotes the poset chain of length r_i, $\mathcal{J}(P)$ denotes the set of order ideals of the poset P, and $\mathcal{S}_n$ is the symmetric group on n elements.

The product of symmetric groups appears because two tilings t_1, t_2 which "look" the same are considered distinct if one is obtained from the other by reordering parallel zones. We will continue to enumerate tilings in this manner for the remainder of the paper.

Proof: Order the elements of $\overline{V}$ such that $v_1, v_2, \ldots, v_d$ are the standard basis vectors for $\mathbb{R}^d$, and $v_{d+1} = (1, 1, \ldots, 1)$. Then V may be represented by the $d \times n$ matrix:

$$V = \begin{pmatrix} \overbrace{1\,1\,1 \cdots 1}^{r_1} & \overbrace{0\,0\,0 \cdots 0}^{r_2} & \cdots & \cdots & \overbrace{0\,0\,0 \cdots 0}^{r_d} & \overbrace{1\,1\,1 \cdots 1}^{r_{d+1}} \\ 0\,0\,0 \cdots 0 & 1\,1\,1 \cdots 1 & \cdots & \cdots & 0\,0\,0 \cdots 0 & 1\,1\,1 \cdots 1 \\ \vdots\,\vdots\,\vdots \cdots \vdots & \vdots\,\vdots\,\vdots \cdots \vdots & \cdots & \cdots & \vdots\,\vdots\,\vdots \cdots \vdots & \vdots\,\vdots\,\vdots \cdots \vdots \\ 0\,0\,0 \cdots 0 & 0\,0\,0 \cdots 0 & \cdots & \cdots & 0\,0\,0 \cdots 0 & 1\,1\,1 \cdots 1 \\ 0\,0\,0 \cdots 0 & 0\,0\,0 \cdots 0 & \cdots & \cdots & 1\,1\,1 \cdots 1 & 1\,1\,1 \cdots 1 \end{pmatrix}$$

The discriminantal arrangement $\mathcal{D}(V)$ has as its set of defining vectors the columns of

$$\mathcal{D}(V) \;=\; (\; J_1 \mid J_2 \mid \cdots \mid J_{d+1} \mid \overline{A}\;)$$

consisting of

$$\sum_{i=1}^{d+1} \binom{r_i}{2} + \prod_{j=1}^{d+1} r_j$$

vectors in $\mathbb{R}^n$. Let h_i denote the i^{th} partial sum $\sum_{j=1}^{i} r_j$ (in particular, $h_0 = 0$ and $h_{d+1} = n$). Then J_i is the collection of all $\binom{r_i}{2}$ possible vectors whose only nonzero entries are a 1 and -1 located at distinct coordinates somewhere in the interval $[h_{i-1}+1, h_i]$ with 1 as the leading nonzero entry. The collection $\overline{A}$ is the set of all possible Cartesian products of basis vectors $\prod_{j=1}^{d} \mathbf{e}_{i_j} \times (-\mathbf{e}_{i_{d+1}})$, where $\{\mathbf{e}_{i_j}\}$, $1 \le i_j \le r_j$, are the standard basis vectors for $\mathbb{R}^{r_j}$, $1 \le j \le d+1$. The reader can verify that (up to a reordering of the vector multiset V), this union of vector sets defines $\mathcal{D}(V)$. For notational convenience, we make a change of coordinates such that $\overline{A}$ may be taken as the set of all products $\prod_{j=1}^{d+1} \mathbf{e}_{i_j}$.

Due to the oriented matroid duality discussed above, the collection of sign vectors $\mathcal{C}^*$ of the columns of $\mathcal{D}(V)$ are the cocircuits of $\mathcal{M}(V^*)$. Thus to prove the proposition, we must show that every element in $\mathcal{J}\left(\prod_{i=1}^{d+1}[r_i]\right) \times \prod_{i=1}^{d+1} \mathcal{S}_{r_i}$ corresponds to a unique uniform localization on $\mathcal{C}^*$, and that all uniform localizations are obtained in this manner.

Recall that a uniform localization σ is simply a cocircuit signature $\sigma : \mathcal{C}^* \to \{+,-\}$ with special properties. Specifically, for each rank 2 contraction R of $\mathcal{M}(V^*)$, σ must assign a signature to the cocircuits in R in a realizable manner. That is, σ must be a signature of the type in Figure 21(III) or equivalently, σ must **avoid** the uniform signature in Figure 22. We will consider only rank 2 contractions which contain three or more cocircuits, since that is the minimum number of cocircuits required for the obstructions in Figure 22. To determine the set $\mathcal{R}$ of such rank 2 contractions R, we must find all rank 2 spaces defined by pairs of cocircuits X, Y in $\mathcal{C}^*$ such that $V^*_{X,Y}$ has span $n - d - 2$, where $n = \sum r_i$.

First, we must find V^*. Recall that, up to oriented matroid equivalence, any collection of n vectors in $\mathbb{R}^{(n-d)}$ which are pairwise row orthogonal with V and which has full row rank will serve as V^*. Thus the computation of V^* is relatively straightforward. Simply reorder the columns of V to write

$$V = \left(\; I_d \mid M \; \right).$$

Then one may compute

$$V^* = \left(\; -M^T \mid I_{(n-d)} \; \right).$$

However, since row and column multiplication only *reorient* the corresponding oriented matroid, and do not alter any of the properties of concern to us, $-M^T$ may be replaced by M^T. Since the rows of $\mathcal{D}(V)$ are indexed by the columns of V, and since the rows of $\operatorname{sign}(\mathcal{D}(V)) = \mathcal{C}^*$ are indexed by the columns of V^*, it is important that the column ordering of V^* correspond to the column ordering of V. Therefore, we rearrange the columns of V^* to correspond to the column ordering of V given on page 46, to obtain the block matrix

$$V^* = \begin{pmatrix} B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_d & 0 \\ L_1 & L_2 & \cdots & L_d & I_{r_{d+1}} \end{pmatrix}$$

where

$$B_i = \left(\begin{array}{c|c} 1 & \\ \vdots & I_{r_i - 1} \\ 1 & \end{array} \right)$$

and L_i is the $r_{d+1} \times r_i$ matrix whose only nonzero entries are all ones in the first column.

In the case of $\mathcal{D}(V)$, the only relevant rank 2 contractions (those containing three or more cocircuits) are those which are defined by a triple of vectors in some J_i, or by a triple of vectors, two of which are in $\overline{A}$, and the third in some J_i. See Figure 25.

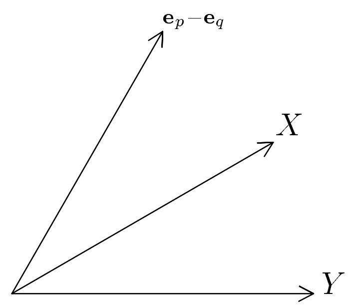


Figure 25: A rank 2 contraction containing three cocircuits

Here $X, Y \in \overline{A}$, $\mathbf{e}_p - \mathbf{e}_q \in J_i$ for some $1 \leq i \leq d+1$, and $(\mathbf{e}_p - \mathbf{e}_q) + Y = X$. For notational convenience, all future rank 2 contractions will be presented as

$$X_1 \vee X_2 \vee \cdots \vee X_m$$

to indicate that $X_1, X_2, \ldots, X_m$ all lie in a common rank 2 contraction R, and that for each triple $(i-1, i, i+1)$, there exist positive scalars a, b such that $aX_{i-1} + bX_{i+1} = X_i$. We say that a cocircuit signature σ *respects* R if $\sigma|_R$ is a localization on R.

Let $\mathcal{R}_0$ denote the collection of rank 2 contractions arising from triples of cocircuits $X_1 \vee X_2 \vee X_3$, where $X_1, X_2, X_3 \in J_i$ for some i, and let $\mathcal{R}_1$ denote the collection of rank 2 contractions of the kind shown in Figure 25. It is a routine matter to verify that this catalogues all rank 2 contractions. If σ respects all rank 2 contractions in $\mathcal{R}_0$, then σ induces an ordering on the coordinates of $\mathbb{R}^{r_i}$ for $1 \le i \le d+1$. This ordering is defined by

$$\sigma(\mathbf{e}_p - \mathbf{e}_q) = + \qquad \text{if and only if} \qquad \mathbf{e}_p > \mathbf{e}_q.$$

Since σ respects all $R \in \mathcal{R}_0$, these pairwise order relations may be extended to a linear order on the coordinates $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{r_i}$ (cycles cannot occur). Conversely, it is clear that every such coordinate ordering corresponds to a cocircuit signature σ which respects all $R \in \mathcal{R}_0$. Thus we may fix an ordering on the coordinates of $\mathbb{R}^{r_i}$ for each i and multiply the localization count by $r_1! r_2! \cdots r_{d+1}!$. It remains to show that for each coordinate ordering ρ in $\prod \mathcal{S}_{r_i}$, there are $\mathcal{J}(\prod [r_i])$ distinct localizations which induce ρ.

Without loss of generality, suppose the order relation on the coordinates of $\mathbb{R}^{r_i}$ is fixed to be $\mathbf{e}_p > \mathbf{e}_q$ if and only if $p < q$. Then each cocircuit $X_i \in J_i$ has $\sigma(X_i) = +$ for all $1 \le i \le d+1$. Consequently, when considering rank 2 contractions in $\mathcal{R}_1$, it follows that

$$(*) \quad \sigma(\mathbf{e}_p - \mathbf{e}_q) = + \qquad \text{implies} \qquad \sigma(X) \ge \sigma(Y)$$

(in the ordering $+ > 0 > -$), where $\mathbf{e}_p - \mathbf{e}_q$, X and Y are as in Figure 25. To see how this yields a bijection with elements of $\mathcal{J}(\prod_{i=1}^{d+1} [r_i])$, we adopt a different notation for the elements of $\overline{A}$. Recall $\overline{A}$ is the collection of all Cartesian products of the form

$$\prod_{j=1}^{d+1} \mathbf{e}_{i_j}$$

where $\mathbf{e}_{i_j}$ is any standard basis vector in $\mathbb{R}^{r_j}$. There is an obvious bijection between elements of $\overline{A}$ and $(d+1)$-tuples $(u_1, u_2, \ldots, u_{d+1})$, where $1 \le u_i \le r_i$ denotes the position of a unique nonzero entry among the coordinates in the interval $[h_{i-1}+1, h_i]$. Then the condition $(*)$ is equivalent to the statement:

$$\sigma((u_1, u_2, \ldots, u_{i-1}, \overline{u}_i, u_{i+1}, \ldots, u_{d+1})) \ge \sigma((u_1, u_2, \ldots u_{i-1}, u_i, u_{i+1}, \ldots, u_{d+1}))$$

$$\text{if and only if} \qquad \overline{u}_i \leq u_i.$$

Taking the set of such relations where $u_i = \overline{u}_i + 1$ for all $1 \leq i \leq d+1$, we obtain the cover relations for the lattice $\prod_{i=1}^{d+1}[r_i]$. In particular, those cocircuits X with $\sigma(X) = +$ form an order ideal I in the lattice. Thus for a fixed ordering ρ of the coordinates, each localization σ is determined by an order ideal I of cocircuits in $\prod_{i=1}^{d+1}[r_i]$ satisfying $\sigma(X) = +$ for all $X \in I$. $\square$

Since the tilings of a MacMahon zonotope are well-behaved, it is a fairly straightforward matter to determine which MacMahon zonotopes are coherent. One nice property of **incoherent** zonotopes is that they must always contain some "minimal" incoherent zonotope.

Lemma 5.4 *The zonotope $Z = Z(V)$ is coherent if and only if $Z' = Z(V')$ is coherent for every $V' \subseteq V$.*

Proof: The sufficiency is immediate. To show necessity, suppose Z' is a subzonotope of Z and let T' be an incoherent tiling of Z'. Using the results of chapter 4 (Theorem 4.4), T' may be "expanded" to some tiling T of Z. If Z is coherent, then there exists a zonotope $\widehat{Z} = Z(\widehat{V})$ which induces T. Moreover, the generating set $\widehat{V}$ of $\widehat{Z}$ is obtained via the lifting vector $(l_1, l_2, \ldots, l_{|V|})$. Let $E = \{(v, l_v) \in \widehat{V} \mid v \in V'\}$. Then $Z(E)$ is a zonotope which induces the tiling T', contradicting the hypothesis that T' is incoherent. $\square$

Notice that Lemma 5.4 is true even if Z and Z' are zonotopes of different dimensions. As a consequence, one can dismiss large infinite families of zonotopes as incoherent once some few relatively small obstructions are found.

Lemma 5.5 *Suppose Z is an $\{r_1, r_2, \ldots, r_{d+1}\}$ MacMahon d-zonotope. Z is incoherent if $r_i \geq 3$ for three distinct values of i.*

Proof: To show that a localization σ yields an incoherent tiling requires proving that no chamber of $\mathcal{D}(V)$ corresponds to σ. Chambers in $\mathcal{D}(V)$ yield signatures in the following manner. Recall that $\mathcal{C}^*$ may be obtained as the sign vectors of the columns of $\mathcal{D}(V)$. Let $(X_1, X_2, \ldots, X_m)$ be an ordering of the elements of $\mathcal{C}^*$, and $l = (l_1, l_2, \ldots, l_n)$ a lifting vector. Then the signature σ_l induced by l is given by $\sigma_l(X_i) = \operatorname{sign}(l \cdot c_i)$, where c_i is the ith column of $\mathcal{D}(V)$.

If Z is the $\{r_1, r_2, \ldots, r_{d+1}\}$ MacMahon d-zonotope with $r_i \geq 3$ for three distinct values of i, then by Proposition 5.3 $\Pi[r_i]$ contains a sublattice isomorphic to $[3] \times [3] \times [3]$. Therefore assume without loss of generality that $\Pi[r_i] = [3] \times [3] \times [3]$ and consider the class of localizations σ which induce the order on coordinates within each $\mathbb{R}^{r_i}$ of $\mathbf{e}_p > \mathbf{e}_q$ if and only if $p < q$. There is a tiling/localization σ corresponding to the order ideal

$$I = <(1,3,2), (2,1,3), (3,2,1), (2,2,2)>$$

in $\mathcal{J}([3] \times [3] \times [3])$ (this is the tiling shown in Figure 2). That is, $\sigma(X) = +$ for all $X \in I$. If σ is coherent, then there is a chamber in $\mathcal{D}(V)$ corresponding to σ. That is, there is a chamber C in $\mathcal{D}(V)$ such that for every $c \in C$, there are coordinates $\{c_1, c_2, \ldots, c_9\}$ corresponding to the cocircuits in $\Pi[r_i]$, for which following inequalities hold:

$$\left.\begin{array}{ll} c_1 + c_6 + c_8 > 0 & \text{corresponding to } (1,3,2) \\ c_2 + c_4 + c_9 > 0 & \text{corresponding to } (2,1,3) \\ c_3 + c_5 + c_7 > 0 & \text{corresponding to } (3,2,1) \end{array}\right. \Rightarrow \sum c_i > 0.$$

However, since σ may also be defined by the complementary filter

$$I^c = <(3,1,2), (2,3,1), (1,2,3)>,$$

a similar set of inequalities implies $\sum c_i < 0$. Thus no chamber of the discriminantal arrangement corresponds to the localization σ, and so σ is an incoherent tiling/localization. $\square$

A similar argument, this time using the order ideal and complementary filter

$$I =< (1,1,2,2), (2,2,1,1), (2,1,1,2) >,$$
$$I^c =< (1,2,1,2), (2,1,2,1), (1,2,2,1) >$$

gives

Lemma 5.6 *Suppose Z is a $\{r_1, r_2, \ldots, r_{d+1}\}$ MacMahon d-zonotope. Z is incoherent if $r_i \geq 2$ for four distinct values of i.*

All that remains is to show that all $\{r, s, 2, 1, 1, \ldots, 1\}$ MacMahon d-zonotopes are coherent, where possibly $r, s \geq 2$. Since the elements of $\overline{V}$ are in general position, it makes no difference which vectors appear with multiplicity. For such a zonotope Z, Proposition 5.3 states that there are

$$|\mathcal{J}([r] \times [s] \times [2]) \times \mathcal{S}_r \times \mathcal{S}_s \times \mathcal{S}_2|$$

total tilings of Z, since the additional singleton zones do not contribute any factors to the count. By MacMahon's original formula (see page 2), this number is

$$\frac{(r+s+1)!(r+s)!}{(r+1)!(s+1)!r!s!} \cdot 2r!s! \qquad = \qquad \frac{2(r+s+1)!(r+s)!}{(r+1)!(s+1)!}.$$

The discriminantal arrangement of Z is projectively equivalent to one of the arrangements studied by Athanasiadis [Ath], which interpolate between the cone over the braid arrangement of type A_{r+s-1} and the cone over the *Shi arrangement* of type A_{r+s-1}. In particular, $\mathcal{D}(V)$ corresponds to a complete bipartite transitive gain graph on vertex sets K_r and K_s (see chapter 7). Athanasiadis has shown (Corollary 7.10) that this class of arrangements is free with exponents

$$\{0, 1, r+1, r+2, \ldots, r+s-1, s+1, s+2, \ldots, r+s\}.$$

By Theorem 5.1, we conclude that all tilings of Z are coherent.

By Lemma 5.4, the MacMahon $\{r, s, 1, \ldots, 1\}$ d-zonotope Z' is coherent as well. All tilings of Z' are counted by

$$|\mathcal{J}([r] \times [s])| \cdot r!s!,$$

and it was shown on page 1 that $\mathcal{J}([r] \times [s])$ has cardinality $\binom{r+s}{s}$. This completes the proof of Theorem 5.2. $\square$

5.4 The Baues problem for MacMahon zonotopes

The Baues problem for a zonotope Z asks whether a certain poset structure on the set $\mathcal{T}$ of all zonotopal tilings of Z has the homotopy type of a sphere. The goal of this section is to answer this question affirmatively for MacMahon zonotopes Z. The only other family of zonotopes in arbitrary dimension d with arbitrarily many zones for which this is known is the family of *cyclic zonotopes* ([SZ]). We begin with a careful statement of the Baues problem for zonotopes.

The Bohne-Dress Theorem (Theorem 2.3) states that the collection $\mathcal{T}$ of (zonotopal, not necessarily rhombohedral) tilings of a d-zonotope $Z(V)$ is in bijection with the collection $\mathcal{E}(\mathcal{M}(V^*))$ of single-element extensions of the oriented matroid $\mathcal{M}(V^*)$ of the dual vector configuration V^*. Las Vergnas' theorem (Theorem 4.1), in turn, states that the collection $\mathcal{E}(\mathcal{M}(V^*))$ is in bijection with the collection of localizations $\sigma : \mathcal{C}^* \to \{+, -, 0\}$. There is a partial order on localizations given by

$$\sigma^1 \prec \sigma^2 \qquad \text{if and only if} \qquad \sigma^1(Y) \in \{0, \sigma^2(Y)\} \text{ for all } Y \in \mathcal{C}^*.$$

Thus $\mathcal{T}$ may be given a poset structure by the rule $T_1 \prec T_2$ if and only if $\sigma_{T_1} \prec \sigma_{T_2}$, where σ_{T_i} is the localization corresponding to T_i. We exclude the localization which gives the signature 0 to each cocircuit in $\mathcal{C}^*$. It may be checked that this corresponds to the partial order on tilings of $Z(V)$ under refinement.

For any given poset P, a *chain* of length m in P is a collection $p_1 \leq p_2 \leq \cdots \leq p_m$ of elements in P. There is a simplicial complex $\Delta(P)$ associated with P, called the *order complex* of P. The order complex $\Delta(P)$ is the abstract simplicial complex with vertex set equal to set of elements of P, and the set of simplices given by the chains in P.

The generalized Baues problem (see [BKS]) for zonotopes asks whether the simplicial complex $\Delta(\mathcal{T})$ has the homotopy type of a sphere. We will use the construction of Sturmfels and Ziegler [SZ] introduced in chapter 4 to prove

Theorem 5.7 *Suppose Z is an $\{r_1, r_2, \ldots, r_{d+1}\}$ MacMahon d-zonotope. Then the order complex $\Delta(\mathcal{T})$ associated to the poset $\mathcal{T}$ of tilings of Z has the homotopy type of an $(n-d-1)$ sphere, where $n = \sum r_i$.*

A few additional definitions are required before presenting the proof of the theorem. First, an oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ is *euclidean* if for every cocircuit Y of $\mathcal{M} = \widetilde{\mathcal{M}} \setminus f$ with $Y_g \neq 0$ there exists an extension $\sigma \in \mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ with $\sigma(Y) = 0$. As a result of Lemmas 4.2 and 4.3, the following proposition is immediate ([EM]):

Proposition 5.8 *An oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ is euclidean if and only if its poset of very strong components* VSC *is empty.*

An oriented matroid $\mathcal{M}$ is *strongly euclidean* if it has rank 1, or if it possesses an element g such that $\mathcal{M}/g$ is strongly euclidean and the program $(\widetilde{\mathcal{M}}, g, f)$ is euclidean for every extension $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$. Sturmfels and Ziegler show the following:

Theorem 5.9 *([SZ] Theorem 1.2) Let $\mathcal{M}$ be a strongly euclidean rank r oriented matroid. Then the extension poset $\mathcal{E}(\mathcal{M})$ is homotopy equivalent to the $(r-1)$-sphere.*

Thus by dualizing the generating set V for the MacMahon $\{r_1, r_2, \ldots, r_{d+1}\}$ d-zonotope, we may apply Theorem 5.9 to the poset $\mathcal{E}(\mathcal{M}(V^*))$ of the rank $(n-d)$ oriented matroid $\mathcal{M}(V^*)$ to determine that $\Delta(\mathcal{T})$ has the homotopy type of an $(n-d-1)$-sphere.

Let V be the vector configuration which generates an $\{r_1, r_2, \ldots, r_{d+1}\}$ MacMahon d-zonotope, and let $\mathcal{M} = \mathcal{M}(V^*)$ be the oriented matroid for V^*. Recall that the graph G_f of an oriented matroid program G_f has a very strong component $\mathbf{c}$ if for any two vertices X^1, X^2 in $\mathbf{c}$, there is a directed path from X^1 to X^2, and from X^2 to X^1 (possibly along bidirected edges), and there is at least one pair of vertices in $\mathbf{c}$ joined by a unidirected edge. In particular, the induced subgraph on the vertices in $\mathbf{c}$ has a *directed cycle* $(X^1, X^2, \ldots X^m)$ such that at least one of the edges in the cycle is unidirected.

In what follows, we will often identify cocircuits of $\mathcal{M}$ with the column vectors of $\mathcal{D}(V)$ corresponding to them. We recall the collection of defining vectors for each of V^* and $\mathcal{D}(V)$.

$$V^* = \begin{pmatrix} B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_d & 0 \\ L_1 & L_2 & \cdots & L_d & I_{r_{d+1}} \end{pmatrix}$$

where

$$B_i = \left(\begin{array}{c|c} 1 & \\ \vdots & I_{r_i - 1} \\ 1 & \end{array}\right)$$

and L_i is the $r_{d+1} \times r_i$ matrix whose only nonzero entries are all ones in the first column. Also

$$\mathcal{D}(V) \;=\; (\; J_1 \mid J_2 \mid \;\cdots\; \mid J_{d+1} \mid \overline{A} \;)$$

where J_i is the collection of all $\binom{r_i}{2}$ possible vectors whose only nonzero entries are a 1 and -1 located at distinct coordinates somewhere in the interval $[h_{i-1}+1, h_i]$ with 1 as the leading nonzero entry, and the collection $\overline{A}$ is the set of all possible Cartesian products of basis vectors $\prod_{j=1}^{d+1} \mathbf{e}_{i_j}$, where $\{\mathbf{e}_{i_j}\}$, $1 \leq i_j \leq r_j$, are the standard basis vectors for $\mathbb{R}^{r_j}$, $1 \leq j \leq d+1$.

Proof (of Theorem 5.7): Consideration of V^* demonstrates that one may use an inductive argument. If one chooses g to be any of the basis vectors $\mathbf{e}_i \in R^{n-d}$, then the oriented matroid of the contraction $\mathcal{M}/g$ is either

- Dual to the oriented matroid of a MacMahon $\{r_1, r_2, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_{d+1}\}$ d-zonotope for some r_i, or
- Dual to the oriented matroid for the trivial $\{r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_{d+1}\}$ zonotope in the case $r_i = 1$.

In the first case, $\mathcal{M}/g$ is strongly euclidean by induction. In the second case, it is clear that $\mathcal{M}/g$ is dual to an $r_1 \times r_2 \times \cdots \times r_{i-1} \times r_{i+1} \times \cdots \times r_{d+1}$ box. It will easily follow from the arguments below that $\mathcal{M}/g$ is strongly euclidean in this case as well. Thus it is sufficient to show that for some choice of coordinate vector $g = \mathbf{e}_i$, the directed graph G_f corresponding to the oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ has no directed cycle for every choice of extension f.

We will choose $g = \mathbf{e}_n$. Consider the structure of the undirected graph G underlying G_f. Since the choice of f only determines the direction of the edges in the graph, it is clear that for fixed g, the undirected graph G is independent of f. The vertices of G correspond to the collection of cocircuits X satisfying $X_g = +$. After replacing columns of $\mathcal{D}(V)$ with their negatives as necessary, the vertices of G may be partitioned into two classes:

$$A := \{\mathbf{e}_n - \mathbf{e}_p \,|\, p \in [h_d + 1, h_{d+1} - 1]\}$$

$$B := \left\{ \prod_{j=1}^{d} \mathbf{e}_{i_j} \times \mathbf{e}_n | 1 \leq i_j \leq r_j \right\}.$$

and all other cocircuits lie in S_g, the pseudosphere at infinity. Let f be any element such that $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$ is a single-element extension of $\mathcal{M}$. The approach for showing that G_f has no directed cycle for every f is as follows:

- Show that the induced subgraph A_f on the vertex set A contains no directed cycle,
- Show that the induced subgraph B_f on the vertex set B contains no directed cycle,
- Show that there is no directed cycle $(X^1, X^2, \ldots, X^m)$ containing vertices from both A_f and B_f.

An important observation arising from considering the collections $\mathcal{R}_0$, $\mathcal{R}_1$ of rank 2 contractions for $\mathcal{M}$ (see page 49) is that every vertex in A is adjacent to every other vertex of G. In particular, A is a complete graph on $r_{d+1} - 1$ vertices.

Lemma 5.10 *For every single-element extension f, the induced subgraph A_f on the vertex set A contains no directed cycle.*

Proof: Suppose A_f contains a directed cycle $\gamma = (X^1, X^2, \ldots X^m)$. Then A_f must contain a directed cycle of length three. If $m > 3$, then up to relabeling the vertices, the directed edge in γ is $\overline{e} : X^1 \to X^2$. By the above observation, there exists an edge e from X^2 to X^m. If e is directed $e : X^2 \to X^m$ or is bidirected, then (X^1, X^2, X^m) is a directed cycle of length three. If e is directed $e : X^m \to X^2$, then $(X^2, X^3, \ldots X^m)$ is shorter than γ and a directed cycle of length three is obtained by iterating this process.

Thus it is sufficient to show that A contains no directed cycle of length three. It is clear that up to isomorphism, there are three possibilities for a directed cycle $\gamma = (X_i, X_j, X_k)$ — γ either contains zero, one or two bidirected edges. Figure 26 shows the undirected induced subgraph $\overline{\gamma}$ of G, together with the pseudosphere at infinity and the rank 2 contractions which determine the edges of of $\overline{\gamma}$.

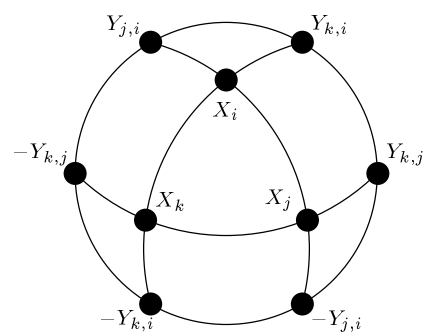


Figure 26: The induced subgraph $\overline{\gamma}$ of G, together with the pseudosphere S_g

The cocircuits with labels X_p denote cocircuits of the form $\mathbf{e}_n - \mathbf{e}_p$. The cocircuits at infinity with labels Y_{p_1,p_2} denote the cocircuits of the form $\mathbf{e}_{p_1} - \mathbf{e}_{p_2}$. For any choice of extension f, the signatures $\sigma_f(Y_{p_1,p_2})$ determine how the edges of γ are directed. For example, $\sigma_f(\mathbf{e}_i - \mathbf{e}_j) = +$ if and only if the edge between X_i and X_j is directed toward X_i.

Note that the cocircuits at infinity also define a rank 2 contraction R. We will demonstrate that for any directed cycle γ, the corresponding cocircuit signature on R is one of the forbidden signatures in Figure 22, and consequently cannot occur as the signature arising from any extension f. Without loss of generality, assume that the edge between vertices X_j and X_k is directed toward X_k. In Figure 27, the vector next to each cocircuit denotes that cocircuit's signature for each of the four possible directed cycles.

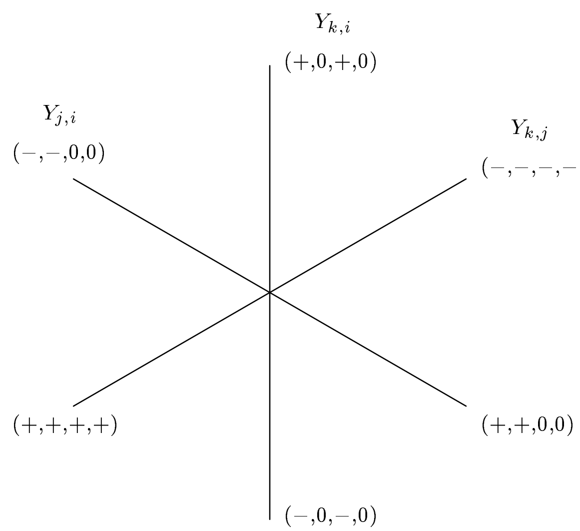


Figure 27: Every directed cycle γ yields a forbidden cocircuit signature on R

The reader may verify that for any fixed coordinate in the sign vectors above, the resulting cocircuit signature is one induced by a particular directed cycle in γ, and that it is one of the forbidden signatures given in Figure 22. Thus for every choice of extension f, the induced subgraph A_f of G_f has no directed cycle. $\square$

Lemma 5.11 *For every single-element extension f, the induced subgraph B_f on the vertex set B contains no directed cycle.*

Proof: In a manner similar to the collection of cocircuits in $\overline{A}$ in the proof of Proposition 5.3, the collection of vertices in B is in bijective correspondence with elements in the poset $\prod_{j=1}^{d}[r_j]$. Moreover, there is an obvious bijection between vertices in B and d-tuples $(\beta_1, \beta_2, \ldots, \beta_d)$, where $1 \leq \beta_j \leq r_j$ for $1 \leq j \leq d$. We will identify vertices in B with this collection of d-tuples.

Furthermore, upon consideration of the collection $\mathcal{R}_1$ of rank 2 contractions of $\mathcal{M}$, it is easy to see that vertices b, b' are adjacent in B if and only if

$$b = (\beta_1, \beta_2, \ldots, \beta_{i-1}, \beta_i, \beta_{i+1}, \ldots, \beta_d)$$
$$b' = (\beta_1, \beta_2, \ldots, \beta_{i-1}, \beta_i', \beta_{i+1}, \ldots, \beta_d)$$

that is, if and only if b and b' differ in exactly one coordinate. In particular, let $K^{\beta_1,\beta_2,\ldots,\hat{\beta}_i,\ldots,\beta_d}$ denote a collection of vertices in B with all values of β_j fixed for $j \neq i$. It is clear that the induced subgraph on each such $K^{\beta_1,\beta_2,\ldots,\hat{\beta}_i,\ldots,\beta_d}$ is a complete graph on r_i vertices. Using the same argument as in the proof of Lemma 5.10, it follows that no $K_f^{\beta_1,\beta_2,\ldots,\hat{\beta}_i,\ldots,\beta_d}$ contains a directed circuit for any choice of extension f.

Observation: Consider the elements

$$b_1 = (\beta_1, \beta_2, \ldots, \beta_{i-1}, k_1, \beta_{i+1}, \ldots, \beta_d)$$

and

$$b_2 = (\beta_1, \beta_2, \ldots, \beta_{i-1}, k_2, \beta_{i+1}, \ldots, \beta_d)$$

of B. As noted above, they are adjacent in B, and for any extension f, the direction of the edge e from b_1 to b_2 in B_f is determined by $\sigma_f(\mathbf{e}_{k_1} - \mathbf{e}_{k_2})$. Note that the direction of the edge e' between

$$b_1' = (\beta_1', \beta_2', \ldots, \beta_{i-1}', k_1, \beta_{i+1}', \ldots, \beta_d')$$

and

$$b_2' = (\beta_1', \beta_2', \ldots, \beta_{i-1}', k_2, \beta_{i+1}', \ldots, \beta_d')$$

is also determined by $\sigma_f(\mathbf{e}_{k_1} - \mathbf{e}_{k_2})$, and so e and e' are similarly directed.

In particular, then, suppose $\gamma = (b_1, b_2, \ldots, b_m)$ is a directed cycle in B. Without loss of generality, the edge e between b_1 and b_2 is directed $e : b_1 \to b_2$, with b_1, b_2 as above. That is, suppose b_1 and b_2 differ only in the i^{th} coordinate.

Suppose $m > q \geq 2$ is the least index for which the (possibly bidirected) edge e' from b_q to b_{q+1} has the property that the change occurs in the i^{th} coordinate. Specifically, suppose the i^{th} coordinate of b_q is k_2 and the i^{th} coordinate of b_{q+1} is k_3. Then using the above observation, we conclude that the edge between b_2 and

$$b_3' = (\beta_1, \beta_2, \ldots, \beta_{i-1}, k_3, \beta_{i+1}, \ldots, \beta_d)$$

is either directed from b_2 to b_3', or is bidirected.

Continuing in this fashion yields a directed cycle $\gamma' = (b_1, b_2, b_3', \ldots b_s')$ in $K^{\beta_1,\beta_2,\ldots,\hat{\beta}_i,\ldots,\beta_d}$, which is a contradiction. So it follows that all vertices $b_2, b_3, \ldots, b_m$ in γ have the same i^{th} coordinate. However, b_2 and b_m are each adjacent to b_1, and therefore they each differ from b_1 in exactly one coordinate, which must be the i^{th} coordinate. Since b_2 and b_m agree on the i^{th} coordinate, we conclude that $b_2 = b_m$ and so the pair $\{b_1, b_2\}$ constitute a directed cycle, which is absurd. Thus B cannot contain a directed cycle. $\square$

Lemma 5.12 *For every single-element extension f, G_f contains no directed cycle with vertices from both A and B.*

Proof: Suppose such a directed cycle γ exists. Then γ contains some vertex $a \in A$, and $\gamma = (a, X_1, X_2, \ldots, X_m)$. As noted above, a is adjacent to every X_j. We claim that it is once again sufficient to consider only directed cycles of length three.

To see this, note that the directed cycle γ contains at least one directed edge e. First, suppose this edge is $e : a \to X_1$. If the edge e' between a and X_2 is directed $e' : X_2 \to a$ or is bidirected, then $\gamma' = (a, X_1, X_2)$ is a directed cycle of length three. If e' is directed $e' : a \to X_2$, then $\gamma' = (a, X_2, \ldots, X_m)$ is a shorter cycle and the claim follows by induction. A similar argument follows if the directed edge in γ is $e : X_m \to a$.

If the directed edge in γ is $e : X_i \to X_{i+1}$, then consider the edge e' between a and X_i. If it is directed $e' : a \to X_i$, then $\gamma' = (a, X_i, X_{i+1}, \ldots, X_m)$ is a shorter cycle unless $X_i = X_1$, which returns to the earlier case. If e' is directed $e' : X_i \to a$, then $\gamma' = (a, X_1, \ldots, X_i)$ is a shorter cycle. Similar arguments apply if the edge e'' between a and X_{i+1} is directed. Finally, if e' and e'' are both bidirected, then (a, X_i, X_{i+1}) is a cycle of length three, which completes the proof of the claim.

Therefore the proof of the lemma reduces to checking all possible directed cycles γ on three vertices which contain at least one vertex from each of A and B. One possibility is presented in Figure 28, which has two vertices from B. The other case may be checked by the reader.

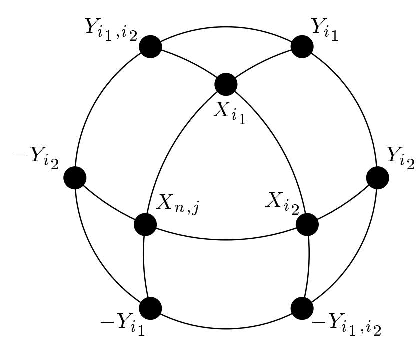


Figure 28: The induced subgraph $\overline{\gamma}$ of G, together with the pseudosphere S_g

In Figure 28, cocircuits X_{i_1} and X_{i_2} correspond to adjacent elements of

B which differ in the i^{th} position, with values i_1 and i_2, respectively. The vertex $X_{n,j}$ corresponds to the cocircuit $\mathbf{e}_n - \mathbf{e}_j$. The vertex Y_{i_1,i_2} corresponds to the cocircuit $\mathbf{e}_{i_1} - \mathbf{e}_{i_2}$, and the vertices Y_{i_1} and Y_{i_2} correspond to those elements of $\overline{A}$ which have all but their final coordinates equal to those of X_{i_1} respectively X_{i_2}, and the final nonzero coordinate for each lies in position $j \in [h_{d-1}+1, h_d - 1]$. As before, one only needs to verify that every possible directed cycle on vertices $X_{i_1}, X_{i_2}, X_{j,n}$ implies the existence of a forbidden cocircuit signature on the rank 2 contraction defined by $Y_{i_1}, Y_{i_2}, Y_{i_1,i_2}$. See Figure 29. This completes the proof of Lemma 5.12. $\square$

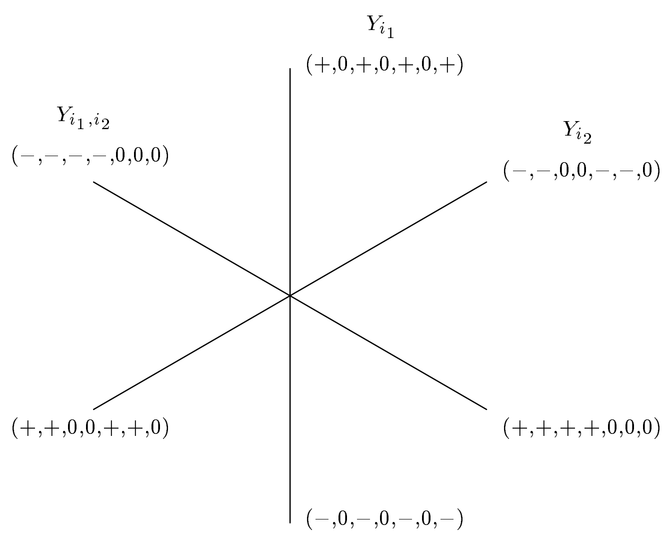


Figure 29: Every directed cycle γ yields a forbidden cocircuit signature

Lemmas 5.10, 5.11 and 5.12 prove that if $g = \mathbf{e}_n$, then G_f is euclidean for every choice of extension f. This fact, together with Theorem 5.9, completes the proof of Theorem 5.7. $\square$

6 A classification of coherent 3-zonotopes on five or fewer distinct generating vectors

6.1 Introduction

In 1996, Edelman and Reiner [ER] were able to give a completely combinatorial classification of coherent 2-zonotopes $Z = Z(V)$ in terms of $\overline{V}$ and the m-tuple $(r_1, r_2, \ldots, r_m)$ of multiplicities. The work in this chapter is a start toward such a classification for 3-zonotopes. Specifically, we provide a completely combinatorial classification of coherent 3-zonotopes $Z = Z(V)$ for those vector sets V with $|\overline{V}| \leq 5$.

Along the way, we discover a number of beautiful families of 3-zonotopes whose tilings are counted by simple product formulas like those of MacMahon and Elnitsky. In higher dimensions, there are more oriented matroid equivalence classes to consider, and so the convenient $(r_1, r_2, \ldots, r_m)$ notation is no longer meaningful.

The oriented matroid equivalence classes for $Z(V)$ with $|\overline{V}| \leq 5$ are as follows:

- $|\overline{V}| = 4$, which yields the MacMahon 3-zonotopes, as discussed in the last chapter.
- $|\overline{V}| = 5$ and the elements of $\overline{V}$ are in general position.
- $|\overline{V}| = 5$ and $\overline{V}$ contains exactly one 3-subset of coplanar vectors.
- $|\overline{V}| = 5$ and $\overline{V}$ contains exactly two 3-subsets of coplanar vectors, with exactly one vector common to each 3-subset.

The remaining possibilities, in which four or five vectors in $\overline{V}$ are coplanar, reduce to the rank 2 case.

The method we will use to obtain the tiling counts which follow is identical to the method used in the previous chapter. In the case where the elements of $\overline{V}$ are in general position, it is possible to say something about the coherence of a d-zonotope with $|\overline{V}| = d + 2$ for arbitrary d. The remaining arguments are only given for 3-zonotopes.

6.2 $d + 2$ vectors in general position in $\mathbb{R}^d$

Let $Z = Z(V)$ be a d-zonotope such that the $d + 2$ elements of $\overline{V}$ are in general position. Since the oriented matroid $\mathcal{M}(V^*)$ of the dual vector configuration V^* has rank 2, there is only one oriented matroid equivalence class of such zonotopes. Thus there is no loss of generality in assuming that the underlying set $\overline{V}$ for the generating multiset V of Z is the frame together with $(1, a_1, a_2, \ldots, a_{d-1})$, where $1 > a_1 > a_2 > \cdots > a_{d-1} \geq -1$, and $a_i \neq 0$ for all i.

In this section, we prove that all such zonotopes are coherent if at most one generating vector has multiplicity $r > 1$, and argue that this is a complete classification of coherent d-zonotopes in this class for $d = 3$.

Theorem 6.1 *Suppose $Z = Z(V)$ is a d-zonotope such that $\overline{V}$ consists of $d + 2$ vectors in general position. Then Z is coherent if at most one of the generating vectors appears with multiplicity $r > 1$. If Z satisfies this condition, then the tilings of Z are enumerated by*

$$\frac{2(d + r + 1)!}{(d + 1)!}$$

Furthermore, if $d = 3$, then this condition is both necessary and sufficient to characterize when Z is coherent.

Proof: Again, since the elements of $\overline{V}$ are in general position, it makes no difference which vector appears with multiplicity. We choose to have $(1, a_1, a_2, \ldots, a_{d-1})$ appear with multiplicity. Then V is:

$$V \;=\; \begin{array}{c} \hphantom{1\,0\,0\,\cdots\,0\;\;1\;\;}\overbrace{\hphantom{a_{d-1}\;a_{d-1}\;\cdots\;a_{d-1}}}^{r} \\ \left(\begin{array}{ccccc|ccccc} 1&0&0&\cdots&0&1&1&1&\cdots&1\\ 0&1&0&\cdots&0&1&a_1&a_1&\cdots&a_1\\ 0&0&1&\cdots&0&1&a_2&a_2&\cdots&a_2\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\cdots&\vdots\\ 0&0&0&\cdots&1&1&a_{d-1}&a_{d-1}&\cdots&a_{d-1} \end{array}\right) \end{array}$$

the dual vector configuration may be represented by the $(r+1)\times(r+d+1)$ matrix:

$$V^* = \left(\begin{array}{ccccc|c} 1&1&1&\cdots&1& \\ 1&a_1&a_2&\cdots&a_{d-1}& \\ \vdots&\vdots&\vdots&\cdots&\vdots& I_{r+1} \\ \vdots&\vdots&\vdots&\cdots&\vdots& \\ 1&a_1&a_2&\cdots&a_{d-1}& \end{array}\right)$$

and the discriminantal arrangement $\mathcal{D}(V)$ is the $(r+d+1)\times(1+r(d+1)+\binom{r}{2})$ matrix

$$\mathcal{D}(V) = \left(\ \overline{X} \mid B_0 \mid B_1 \mid B_{a_1} \mid B_{a_2} \mid \cdots \mid B_{a_{d-1}} \mid \ A'_{r-1}\ \right)$$

where $\overline{X}$ denotes the minimal dependence among vectors in the frame, B_x is the $(r+d+1)\times r$ matrix whose columns are all possible products $(1-x, a_1-x, a_2-x, \ldots, a_{d-1}-x, x)\times -\mathbf{e}_i$ for $1 \leq i \leq r$, and A'_{r-1} denotes the Cartesian product of $\mathbf{0}\in\mathbb{R}^{d+1}$ with the braid arrangement A_{r-1}.

Let X_x denote a cocircuit in B_x for $x \in \{0, 1, a_1, \ldots, a_{d-1}\}$, and in particular, let $X_{x,j}$ denote the unique cocircuit in B_x with nonzero entry in position $d+j+1$, so that $1 \leq j \leq r$. Let $\mathbf{e}_p - \mathbf{e}_q$ denote the appropriate cocircuit in A'_{r-1}. For the moment, assume $1 > a_1 > a_2 > \cdots > a_{d-1} > 0$. The following is a complete list of rank 2 contractions:

$$\mathcal{R}_0 = \{(\mathbf{e}_p - \mathbf{e}_m) \vee (\mathbf{e}_p - \mathbf{e}_q) \vee (\mathbf{e}_m - \mathbf{e}_q) \,|\, p, q, m \in [d+2, d+r+1]\}$$

$$\mathcal{R}_1 = \{X_{1,j} \vee X_{a_1,j} \vee X_{a_2,j} \vee \cdots \vee X_{a_{d-1},j} \vee X_{0,j} \vee \overline{X} \,|\, 1 \le j \le r\}$$

$$\mathcal{R}_2 = \{(\mathbf{e}_p - \mathbf{e}_q) \vee X_{x,q} \vee X_{x,p} \,|\, x \in \{0, 1, a_1, \ldots, a_{d-1}\} \text{ and } p, q \in [d+2, d+r+1]\}.$$

In the cases where some a_i are negative, a similar collection of rank 2 contractions arises. As before, any localization σ, when restricted to the rank 2 contractions in $\mathcal{R}_0$, induces a linear order on the final r coordinates. Thus we again restrict attention to those localizations which have $\sigma(\mathbf{e}_p - \mathbf{e}_q) = +$ for all cocircuits corresponding to vectors in A'_{r-1}, and multiply the final count by $r!$. The linear order imposed on the coordinates by the rank 2 contractions in $\mathcal{R}_0$, together with the collection $\mathcal{R}_2$, implies $\sigma(X_{x,i}) \le \sigma(X_{x,j})$ if and only if $i < j$, within each B_x. Furthermore, assume $\sigma(\overline{X}) = +$. This requires doubling the final count.

The rank 2 contractions in $\mathcal{R}_1$ induce an order relation

$$\sigma(X_{1,j}) \le \sigma(X_{a_1,j}) \le \sigma(X_{a_2,j}) \cdots \le \sigma(X_{a_{d-1},j}) \le \sigma(X_{0,j})$$

among cocircuits in distinct blocks with the same nonzero entry in the final r coordinates. Thus all information about σ may be completely specified by a $(d+1) \times r$ tableau L with rows indexed by $1, a_1, a_2, \ldots, a_{d-1}, 0$ and columns indexed by $1, 2, \ldots, r$. Entry $L_{x,j}$ is $\sigma(X_{x,j})$. The conditions from the collections $\mathcal{R}_1$ and $\mathcal{R}_2$ imply that the entries of L must weakly increase along rows and down columns (see Figure 30, in which shaded boxes correspond to cocircuits X with $\sigma(X) = -$). Thus the number of localizations σ is given by the number of such arrays, which is $\binom{r+d+1}{r}$. When this number is multiplied by $2r!$ to allow for the possibility that $\sigma(\overline{X}) = -$ and for other orderings of

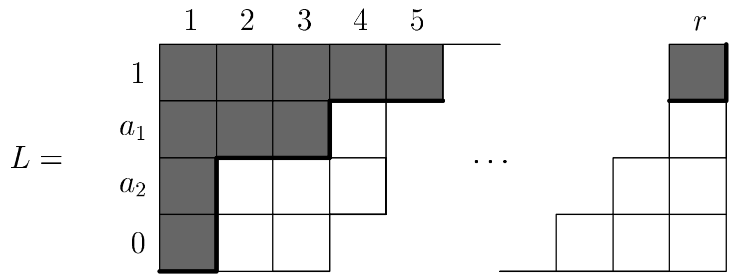


Figure 30: The tableau of cocircuit signatures for $d = 3$, $1 > a_1 > a_2 > 0$

the final r coordinates, we obtain the tiling count given in the statement of the theorem.

To count the number of coherent tilings, we will show that $\mathcal{D}(V)$ is free by showing that it is *supersolvable.* A d-arrangement $\mathcal{A}$ of rank r is *supersolvable* if it is possible to define an ordered partition $\pi = (\pi_1, \ldots, \pi_r)$ of its hyperplanes such that the subarrangement $\mathcal{A}_{r-1} = \cup_{k=1}^{r-1} \pi_k$ is supersolvable of rank $r-1$ and the intersection of any two hyperplanes in π_r is contained in some hyperplane in $\mathcal{A}_{r-1}$. The sequence $\mathcal{A} = \mathcal{A}_r \supset \mathcal{A}_{r-1} \supset \cdots \supset \mathcal{A}_0 = \emptyset$ is called an *M-chain.* Supersolvable arrangements are a proper subclass of free arrangements. Stanley [Sta1] showed that for a supersolvable arrangement $\mathcal{A}$ with $\pi = (\pi_1, \ldots, \pi_r)$, the roots of $\chi(\mathcal{A}, t)$ are given by $\{|\pi_i|\}$. Thus we may compute the number of coherent tilings of Z by exhibiting an M-chain for $\mathcal{D}(V)$.

Lemma 6.2 *Let $Z = Z(V)$ be a d-zonotope such that $\overline{V}$ consists of $d+2$ vectors in general position, and V is such that exactly one vector occurs with multiplicity $r \geq 1$, and all other vectors occur with multiplicity 1. Then $\mathcal{D}(V)$ is supersolvable with exponents*

$$\{1,\, d+1,\, d+2,\, \ldots,\, r+d\}.$$

Proof: After some row reduction, the discriminantal arrangement $\mathcal{D}(V)$ may be written as the $(r+1) \times (1 + r(d+1) + \binom{r}{2})$ matrix

$$\mathcal{D}(V) \sim \left(\ \mathbf{e}_1 \mid C_0 \mid C_1 \mid C_{a_1} \mid C_{a_2} \mid \ \cdots \ \mid C_{a_{d-1}} \mid A'_{r-1} \ \right)$$

where $\mathbf{e}_1$ is the standard basis vector $(1, 0, \ldots, 0) \in \mathbb{R}^{r+1}$, C_x is the set of all possible products $(x) \times -\mathbf{e}_i$ with $\mathbf{e}_i \in \mathbb{R}^r$, and here A'_{r-1} is identical to the matrix A'_{r-1} given above, with the first d rows truncated. For the presentation of the M-chain, we will view all column vectors of the above matrix as sums of standard basis vectors in $\mathbb{R}^{r+1}$, and identify each hyperplane with its defining normal vector.

The reader may verify that the following is an M-chain for the above matrix:

$$\begin{aligned}
\pi_1 &= \{\mathbf{e}_1\} \\
\pi_2 &= \{\mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2, a_1\mathbf{e}_1 - \mathbf{e}_2, a_2\mathbf{e}_1 - \mathbf{e}_2, \ldots, a_{d-1}\mathbf{e}_1 - \mathbf{e}_2\} \\
\pi_3 &= \{\mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_3, a_1\mathbf{e}_1 - \mathbf{e}_3, \ldots, a_{d-1}\mathbf{e}_1 - \mathbf{e}_3, \} \\
\pi_4 &= \{\mathbf{e}_4, \mathbf{e}_2 - \mathbf{e}_4, \mathbf{e}_3 - \mathbf{e}_4, \mathbf{e}_1 - \mathbf{e}_4, a_1\mathbf{e}_1 - \mathbf{e}_4, \ldots, a_{d-1}\mathbf{e}_1 - \mathbf{e}_4\} \\
&\ \vdots \\
\pi_k &= \{\mathbf{e}_k, \mathbf{e}_2 - \mathbf{e}_k, \mathbf{e}_3 - \mathbf{e}_k, \ldots, \mathbf{e}_{k-1} - \mathbf{e}_k, \mathbf{e}_1 - \mathbf{e}_k, a_1\mathbf{e}_1 - \mathbf{e}_k, \ldots, a_{d-1}\mathbf{e}_1 - \mathbf{e}_k\} \\
&\ \vdots \\
\pi_{r+1} &= \{\mathbf{e}_{r+1}, \mathbf{e}_2 - \mathbf{e}_{r+1}, \mathbf{e}_3 - \mathbf{e}_{r+1}, \ldots, \mathbf{e}_r - \mathbf{e}_{r+1}, \mathbf{e}_1 - \mathbf{e}_{r+1}\} \\
&\qquad \cup \{a_1\mathbf{e}_1 - \mathbf{e}_{r+1}, \ldots, a_{d-1}\mathbf{e}_1 - \mathbf{e}_{r+1}\}
\end{aligned}$$

This completes the proof of the lemma. $\square$

By Stanley's result, the roots of $\chi(\mathcal{D}(V), t)$ are $\{1, d+1, d+2, \ldots, r+d\}$, and so $\mathcal{D}(V)$ has

$$\frac{2(r+d+1)!}{(d+1)!}$$

chambers.

All that remains is to demonstrate that $Z = Z(V)$ has an incoherent tiling when $d = 3$ and exactly two of the generating vectors for Z have multiplicity two or greater. We know of no elegant proof of this fact. However, this is a sufficiently small obstruction that sets of tilings and coherent tilings may be computed using symbolic manipulation packages like MAPLE and GAP (code available from the author upon request). When $d = 3$ and exactly two vectors have multiplicity two, Z has 632 total tilings. The total number of coherent tilings is either 616, 620 or 624, depending on the choice of values for the parameters a_1 and a_2, but is always less than 632. By Lemma 5.4, this completes the proof. □

6.3 Five vectors in $\mathbb{R}^3$ containing a single 3-point line

Theorem 6.3 *Let $Z = Z(V)$ be a 3-zonotope such that the arrangement $\mathcal{A}(\overline{V})$ is projectively equivalent to the projectivized picture given in Figure 31. Then Z is coherent if and only if at most two of the generating vectors have multiplicities $r, s > 1$, and these vectors with multiplicity correspond to one of the pairs $\{(1,3),(1,5),(2,3),(2,5),(3,4),(4,5)\}$. If Z satisfies this condition, then the tilings of Z are enumerated by*

$$\frac{2(r+s+1)!(r+s+2)!}{(s+2)!(r+2)!}$$

The reason for the apparent asymmetry between hyperplanes 3, 4 and 5 is that hyperplanes 3 and 5 "separate" hyperplane 4 from the intersection of hyperplanes 1 and 2. More precisely, let H_i denote the i^{th} hyperplane for $i = 1, \ldots 5$, and let l denote the intersection of H_1 and H_2. Then any *path* (point set homeomorphic to the unit interval) originating at l and terminating at H_4 must also contain a point in either H_3 or H_5.

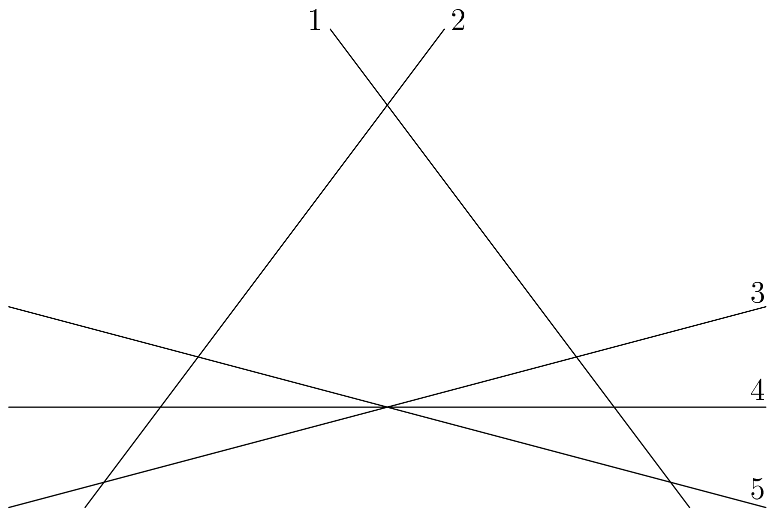


Figure 31: Five vectors in $\mathbb{R}^3$ with a single three-point line

Proof: Any arrangement in this class may be realized by the frame together with the vector $(a, 1, 1)$, where $a \neq 0, 1$, and so the arrangement in Figure 31 corresponds to

$$\overline{V} = ((0,0,1),(0,1,0),(1,0,0),(1,1,1),(a,1,1)).$$

For the rest of this section, we assume $a < 1$. The proof when $a > 1$ is similar (but be careful! When $a > 1$, H_4 and H_5 switch position). By symmetry, it is clear that there are two cases: where vectors 2 and 5 have multiplicity, and where vectors 4 and 5 have multiplicity. We present the proof of the second case; the proof of the first case is similar.

As before, we begin by computing $\mathcal{D}(V)$ and V^*. The vector $(1, 1, 1)$ occurs with multiplicity r and the vector $(a, 1, 1)$ occurs with multiplicity s. Then $\mathcal{D}(V)$ is given by the $(r + s + 3) \times [rs + (1/2)(r + s)(r + s + 1)]$ block matrix

$$\mathcal{D}(V) = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 & 0 & 0 \\ I_r & 0 & I_r & aI_r & A_{r-1} & 0 \\ 0 & -I_s & \overset{\times}{-} I_s & \overset{\times}{-} I_s & 0 & A_{s-1} \end{pmatrix}$$

where

$$B_1 = \begin{pmatrix} -1 & \cdots & -1 \\ -1 & \cdots & -1 \\ -1 & \cdots & -1 \end{pmatrix} \qquad B_2 = \begin{pmatrix} a & \cdots & a \\ 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} a-1 & \cdots & a-1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} \quad B_4 = \begin{pmatrix} 0 & \cdots & 0 \\ 1-a & \cdots & 1-a \\ 1-a & \cdots & 1-a \end{pmatrix}$$

with the necessary row lengths, A_{r-1} and A_{s-1} are the matrices for the braid arrangements of rank $r-1$ and $s-1$, respectively, and the block pairs $I_r \times -I_s$ and $aI_r \times -I_s$ denote all possible Cartesian products of basis vectors $\mathbf{e}_i$ or $a\mathbf{e}_i \in \mathbb{R}^r$ with basis vectors $-\mathbf{e}_j \in \mathbb{R}^s$, respectively.

The matrix V^* is given by

$$V^* = \begin{array}{r} r\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ s\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \end{array} \!\!\left(\begin{array}{ccc|c} 1 & 1 & 1 & \\ 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & I_{r+s} \\ a & 1 & 1 & \\ \vdots & \vdots & \vdots & \\ a & 1 & 1 & \end{array}\right).$$

The columns of $\mathcal{D}(V)$ may be partitioned into six blocks in the obvious way, from left to right. Let $\mathcal{C}_i^*$ denote the collection of cocircuits arising from the columns in the i^{th} block, and let $X_{j,k}^i$ denote the cocircuit in the i^{th} block with nonzero entries in positions $j+3$ and $k+r+3$, so that $j \in [1,r]$ and $k \in [1,s]$. Cocircuits in $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ will be denoted by $X_j^1 = X_{j,0}^1$ and $X_k^2 = X_{0,k}^2$, respectively.

The reader can verify that the following is a complete list of rank 2 contractions.

$$\mathcal{R}_0 = \{(\mathbf{e}_p - \mathbf{e}_m) \vee (\mathbf{e}_p - \mathbf{e}_q) \vee (\mathbf{e}_m - \mathbf{e}_q) \mid p,q,m \in [4, r+3] \text{ or } p,q,m \in [r+4, r+s+3]\}$$

$$\mathcal{R}_1^i = \{X_{j,p}^i \vee X_{j,q}^i \vee (\mathbf{e}_p - \mathbf{e}_q) \,|\, p,q \in [r+4, r+s+3] \text{ and } j \in [0,r]\} \quad \text{for } i = 2,3,4$$
$$\mathcal{R}_2^i = \{X_{q,k}^i \vee X_{p,k}^i \vee (\mathbf{e}_p - \mathbf{e}_q) \,|\, p,q \in [4, r+3] \text{ and } k \in [0,s]\} \quad \text{for } i = 1,3,4$$
$$\mathcal{R}_3 = \{X_j^1 \vee X_{j,k}^3 \vee X_{j,k}^4 \vee X_k^2\} \quad \text{if } 1 > a > 0.$$

A similar collection $\mathcal{R}_3$ arises for other possible values of a.

As usual, when a localization σ is restricted to $\mathcal{R}_0$, it corresponds to a permutation in $\mathcal{S}_r \times \mathcal{S}_s$. We will assume the ordering to be $\sigma(\mathbf{e}_p - \mathbf{e}_q) = +$ for $p < q$, and multiply the final count by $r!s!$. This ordering, together with the collections $\mathcal{R}_1^i$ and $\mathcal{R}_2^i$, implies

$$\sigma(X_{j,p}^i) \leq \sigma(X_{j,q}^i) \quad \text{for } r+4 \leq p < q \leq r+s+3, \quad \text{and}$$
$$\sigma(X_{p,k}^i) \geq \sigma(X_{q,k}^i) \quad \text{for } 4 \leq p < q \leq r+3.$$

The signatures of cocircuits in $\mathcal{C}_3^*$ and $\mathcal{C}_4^*$ may each be entered into an $r \times s$ tableau of the kind given in Figure 32, with certain restrictions. The collections $\mathcal{R}_1^3, \mathcal{R}_2^3, \mathcal{R}_1^4$, and $\mathcal{R}_2^4$ dictate that the signatures for each of $\mathcal{C}_3^*$ and $\mathcal{C}_4^*$ be weakly increasing along rows and weakly decreasing down columns. In Figure 32, the path running from the upper left corner to the lower right delineates the boundary between signatures $+$ and signatures $-$.

In particular, the above inequalities hold for the cocircuits in $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$. Let $\alpha \in [0,r]$ be the greatest index such that $\sigma(X_\alpha^1) = +$ (if $\sigma(X_j^1) = -$ for all j, then $\alpha = 0$), and let $\beta \in [0,s]$ be the greatest index such that $\sigma(X_\beta^2) = -$. The signatures which have $\alpha < j \leq r$ and $1 \leq k \leq \beta$ must all be $-$, by consideration of the collection $\mathcal{R}_3$, and similarly the signatures which have $1 \leq j \leq \alpha$ and $\beta < k \leq s$ must be $+$.

The pair of tableaux for $\mathcal{C}_3^*$ and $\mathcal{C}_4^*$ account for all of the information in the rank 2 contractions except for the relationship between $\sigma(X_{j,k}^3)$ and $\sigma(X_{j,k}^4)$ given by the collection $\mathcal{R}_3$. When $1 \leq j \leq \alpha$ and $1 \leq k \leq \beta$, the inequality

$$+ = \sigma(X_j^1) \geq \sigma(X_{j,k}^3) \geq \sigma(X_{j,k}^4) \geq \sigma(X_k^2) = -$$

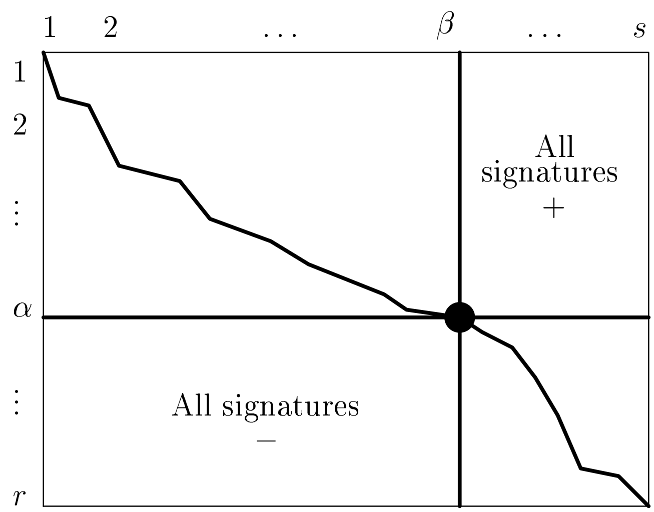


Figure 32: The tableau of cocircuit signatures for cocircuits in $\mathcal{C}_3^*$ or $\mathcal{C}_4^*$

holds, so in particular $\sigma(X_{j,k}^3) \geq \sigma(X_{j,k}^4)$. Similarly when $\alpha < j \leq r$ and $\beta < k \leq s$, the collection $\mathcal{R}_3$ implies $\sigma(X_{j,k}^3) \leq \sigma(X_{j,k}^4)$. Thus all information given by the rank 2 contractions may be encoded by superimposing the tableaux for $\mathcal{C}_3^*$ and $\mathcal{C}_4^*$ upon one another and enumerating the resulting pairs of paths in an $r \times s$ tableaux. That is, we must enumerate all $r \times s$ tableaux which contain a pair of monotonically decreasing paths from upper left to lower right, paths which may be concurrent with one another at points, but cross only once at a distinguished *root* defined by α and β (see Figure 33). Elnitsky [El] has enumerated the collection of such paths in his study of $(r, s, 1, 1)$ octagons. There are

$$\frac{2(r+s+1)!(r+s+2)!}{r!s!(r+2)!(s+2)!}$$

such tableaux. Multiplying this count by the factor $r!s!$ yields the count given in the statement of the theorem.

To count the coherent tilings, the reader can verify that $\mathcal{D}(V)$ is projectively equivalent to the $(r+s) \times [rs + (1/2)(r+s)(r+s+1)]$ block matrix

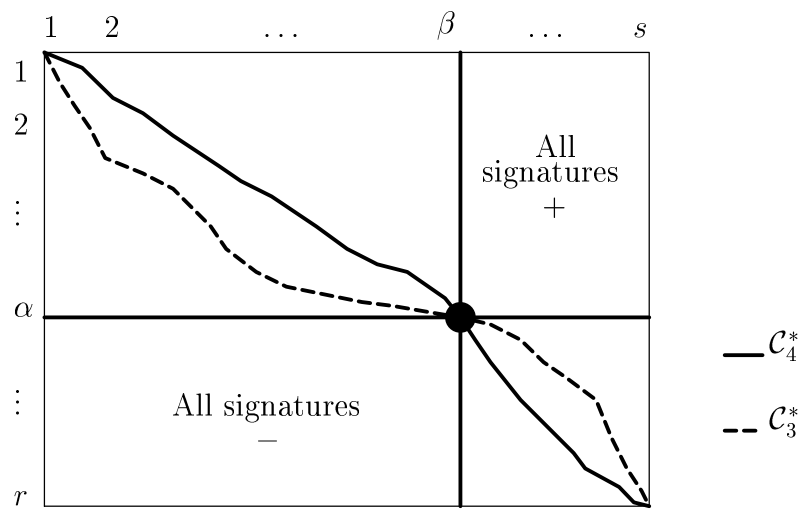


Figure 33: A tableau which encodes all information from the rank 2 contractions

$$\mathcal{D}(V) \sim \left(A_{r+s-1}|I_{r+s}|aI_r \times -I_s\right),$$

where, as above, the notation $aI_r \times -I_s$ denotes all possible Cartesian products of elements $a\mathbf{e}_i \in \mathbb{R}^r$ with elements $-\mathbf{e}_j \in \mathbb{R}^s$ (note that all columns of $\mathcal{D}(V)$ are distinct since $a \neq 0, 1$). This is a TG-graphic arrangement corresponding to a complete bipartite transitive gain graph on vertex sets K_r, K_s (see chapter 7). We demonstrate in chapter 7 (Corollary 7.6) that this arrangement is free with exponents

$$\{1, r+2, r+3, \ldots, r+s, s+2, s+3, \ldots, r+s, r+s+1\},$$

and so consequently, Z has the number of coherent tilings given in the statement of the theorem.

It only remains to demonstrate that Z has an incoherent tiling in those cases for which a pair of vectors with multiplicity $r, s > 1$ is not one of the pairs listed in the statement of the theorem. Again, we know of no elegant proof of this fact. However, by the use of the programs MAPLE and GAP, it is possible to show that if vectors 1 and 2 occur with multiplicity two, and all other vectors singleton, then Z has 400 total tilings, 384 of which are coherent. If the multiplicities are placed on any other forbidden pair, then Z has 304 total tilings, either 296 or 300 of which are coherent, depending on the choice of value for a. This fact, together with Lemma 5.4 finishes the proof. $\square$

It is interesting to note that when the multiset of multiplicities is $\{2, 2, 1, 1, 1\}$ and the vectors with multiplicity two are any forbidden pair other than $\{1, 2\}$, then $\mathcal{D}(V)$ is free with exponents $\{1, 4, 4, 5\}$ (unless $a = 1/2$). This is one of the few known counterexamples to the tempting but false conjecture that if $\mathcal{D}(V)$ is free, then Z is coherent.

6.4 Five vectors in $\mathbb{R}^3$ containing two 3-point lines

Finally, we consider the case in which the elements of $\overline{V}$ lie in two intersecting planes, P_1, P_2, with a single vector v, called the *common vector*, common to each. The remaining vectors, the *frame vectors* may naturally be partitioned into pairs, called *partnerships*, such that the two vectors v_1, v_2 of a partnership define a rank 2 space containing the common vector. For example, in Figure 34, 4 is the common vector, while the frame vectors $1, 2, 3, 5$ form the partnerships $\{1, 2\}$ and $\{3, 5\}$. This will complete the classification of coherent 3-zonotopes $Z = Z(V)$ with $|\overline{V}| \leq 5$.

Theorem 6.4 *Let $Z = Z(V)$ be a 3-zonotope such that $\overline{V}$ is as given in Figure 34.*

Z is coherent if and only if at most two frame vectors v_1, v_2 have multiplicity $r, s \geq 3$, some frame vector occurs with multiplicity one, and

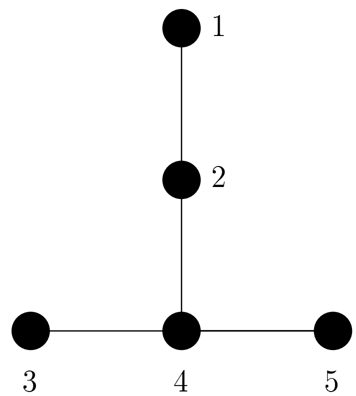


Figure 34: Five vectors lying on two three-point lines in $\mathbb{R}^3$

1) If v_1, v_2 form a partnership, then all other vectors, including the common vector, must have multiplicity 1. In this case, the tilings of Z are enumerated by

$$\frac{2(r+s)!(r+s+1)!}{(r+1)!(s+1)!}$$

2) If v_1, v_2 do not form a partnership, then the common vector may occur with arbitrary multiplicity t, and the multiplicities of the remaining frame vectors must be at most 2 and 1.

a) In the case that the multiplicities are $\{r, s, t, 1, 1\}$, the tilings of Z are enumerated by

$$(r+s+t)!$$

b) In the case that the multiplicities are $\{r, s, t, 2, 1\}$, such that the vector v_3 with multiplicity two forms a partnership with the vector v_1 with multiplicity r, the tilings of Z are enumerated by

$$\frac{2(r+s+t)!(r+s+t+1)!}{(r+1)!(s+t+1)!}$$

Proof: One advantage to the restricted position of the vectors in this case is that there is, up to projective equivalence, only one such vector configuration in $\mathbb{R}^3$. Thus for the remainder of this section, set

$$\overline{V} = ((0,0,1),(1,1,1),(1,0,0),(1,1,0),(0,1,0)).$$

For case (1) of the theorem, let $(0,0,1)$ occur with multiplicity r and $(1,1,1)$ occur with multiplicity s. Then $\mathcal{D}(V)$ is given by the $(r+s+3)\times[2rs+\binom{r}{2}+\binom{s}{2}+1]$ block matrix:

$$\mathcal{D}(V) = \begin{pmatrix} I_r & I_r & A_{r-1} & 0 & 0 \\ \overset{\times}{-}I_s & \overset{\times}{-}I_s & 0 & A_{s-1} & 0 \\ B_1 & B_2 & 0 & 0 & B_3 \end{pmatrix}$$

where

$$B_1 = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

and all other entries are as described previously. The dual configuration $Z(V^*)$ is given by

$$Z(V^*) = \left(\begin{array}{c|c|cc} 1 & & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & & 0 & 0 \\ 1 & I_{r+s} & 1 & 1 \\ \vdots & & \vdots & \vdots \\ 1 & & 1 & 1 \\ 0 & & 1 & 1 \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} r-1 \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} s \\ \\ \end{array}$$

As before, partition the cocircuits arising from $\mathcal{D}(V)$ into collections $\mathcal{C}_1^*$, $\mathcal{C}_2^*$, $\mathcal{C}_3^*$ $\mathcal{C}_4^*$ from left to right, and denote the cocircuit corresponding to the rightmost vector by $\overline{X}$. As before, let $X_{j,k}^i$ denote the cocircuit corresponding to the vector in $\mathcal{C}_i^*$ with nonzero entries in positions j and $r+k$, where $i=1$

or 2, $1 \leq j \leq r$ and $1 \leq k \leq s$. The reader may verify that the following is a complete list of rank 2 contractions:

$$\mathcal{R}_0 = \{(\mathbf{e}_p - \mathbf{e}_m) \vee (\mathbf{e}_p - \mathbf{e}_q) \vee (\mathbf{e}_m - \mathbf{e}_q) | p, q, m \in [1, r] \text{ or } p, q, m \in [r+1, r+s]\}$$

$$\mathcal{R}_1^i = \{X_{q,k}^i \vee X_{p,k}^i \vee (\mathbf{e}_p - \mathbf{e}_q) | 1 \leq p < q \leq r, k \in [1, s]\} \text{ for } i = 1, 2$$

$$\mathcal{R}_2^i = \{X_{j,p}^i \vee X_{j,q}^i \vee (\mathbf{e}_p - \mathbf{e}_q) | j \in [1, r], 1 \leq p < q \leq s\} \text{ for } i = 1, 2$$

$$\mathcal{R}_3 = \{X_{j,k}^2 \vee X_{j,k}^1 \vee \overline{X} | j \in [1, r], k \in [1, s]\}$$

As usual, the elements of $\mathcal{R}_0$ correspond to an element in $\mathcal{S}_r \times \mathcal{S}_s$. So again we enumerate all signatures σ which fix $\sigma(\mathbf{e}_p - \mathbf{e}_q) = +$ for $1 \leq p < q \leq r$ and for $r + 1 \leq p < q \leq r + s$, and also $\sigma(\overline{X}) = +$. This contributes a factor of $2r!s!$ to the final count.

As in the last section, the cocircuit signatures for cocircuits in $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ may be entered in $r \times s$ tableaux. The collections $\mathcal{R}_1^i$ require that the entries in each tableau must weakly decrease down columns, and the collections $\mathcal{R}_2^i$ require that the entries in each tableau must weakly increase along rows. Thus the cocircuit signatures for the collections $\mathcal{C}_i^*$, for $i = 1, 2$ are encoded by a tableau like the one in Figure 35.

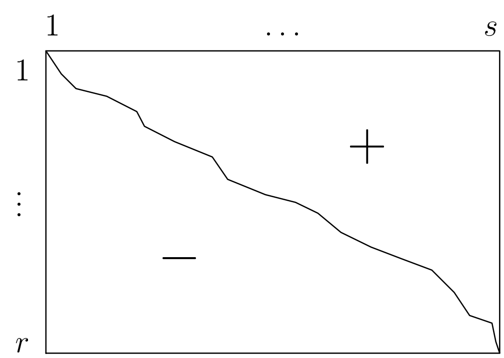


Figure 35: The tableau of cocircuit signatures for cocircuits in $\mathcal{C}_1^*$ or $\mathcal{C}_2^*$

All that remains is to take into account the elements of $\mathcal{R}_4$. Since $\sigma(\overline{X}) = +$, it follows that $\sigma(X^2_{j,k}) \leq \sigma(X^1_{j,k})$ for all pairs $\{j,k\}$. Thus by superimposing the tableau for $\mathcal{C}_1^*$ on the tableau for $\mathcal{C}_2^*$, all information given by the rank 2 contractions may be encoded in a single $r \times s$ array containing two non-crossing paths from upper left to lower right. The collection of all possible such non-crossing paths is given by

$$\frac{(r+s+1)!(r+s)!}{r!s!(r+1)!(s+1)!}$$

(see [Sta3], Section 2.7). Multiplying by $2r!s!$ gives the count in the statement of the theorem.

As for the coherent tilings, $\mathcal{D}(V)$ is again projectively equivalent to the Athanasiadis-type arrangement corresponding to a complete bipartite transitive gain graph on vertex sets K_r and K_s. Thus again by Corollary 7.10, $\mathcal{D}(V)$ is free with exponents

$$\{0, 1, r+1, r+2, \ldots, r+s-1, s+1, s+2, \ldots, r+s\}.$$

Finally, Theorem 5.1 completes the proof of case (1).

To prove case (2), assume the vector $(1,0,0)$ appears with multiplicity r, $(1,1,1)$ appears with multiplicity s, $(1,1,0)$ appears with multiplicity t, and $(0,1,0)$ appears with multiplicity two. Then $\mathcal{D}(V)$ is the $(r+s+t+3) \times (\binom{r}{2} + \binom{s}{2} + \binom{t}{2} + 2rt + 2rs + st + 1)$ block matrix

$$\mathcal{D}(V) = \begin{pmatrix} A_{r-1} & 0 & 0 & I_r & I_r & I_r & I_r & 0 & 0 \\ 0 & A_{s-1} & 0 & 0 & 0 & \overset{\times}{-}I_s & \overset{\times}{-}I_s & I_s & 0 \\ 0 & 0 & A_{t-1} & \overset{\times}{-}I_t & \overset{\times}{-}I_t & 0 & 0 & \overset{\times}{-}I_t & 0 \\ 0 & 0 & 0 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{pmatrix}$$

where

$$B_1 = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{pmatrix} \qquad B_3 = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix}$$

$$B_4 = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{pmatrix} \quad B_5 = \begin{pmatrix} -1 & \cdots & -1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} \qquad B_6 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

and all other entries are as described above.

The dual configuration $Z(V^*)$ is given by

$$Z(V^*) = \left(\begin{array}{c|ccc|ccc} 1 & & & & 0 & 0 & 0 \\ \vdots & & & & \vdots & \vdots & \vdots \\ 1 & & & & 0 & 0 & 0 \\ 1 & & & & 1 & 1 & 0 \\ \vdots & & I_{r+s+t-1} & & \vdots & \vdots & \vdots \\ 1 & & & & 1 & 1 & 0 \\ 1 & & & & 0 & 1 & 0 \\ \vdots & & & & \vdots & \vdots & \vdots \\ 1 & & & & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} r-1 \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} s \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} t \\ \\ \end{array}$$

As above, the cocircuits arising from the column vectors of $\mathcal{D}(V)$ may be partitioned in a natural way into eight classes $\mathcal{C}_1^*, \ldots, \mathcal{C}_8^*$, with the final, single cocircuit denoted by $\overline{X}$. Let $X^i_{j,k,l}$ denote the cocircuit vector in $\mathcal{C}_i^*$ with nonzero entries in positions j, $r+k$, and $r+s+l$, with $j \in [1,r]$, $k \in [1,s]$ and $l \in [1,t]$. If some cocircuit has a zero entry in all positions in the interval $[1,r]$, for example, then set $j = 0$. So all elements of $\mathcal{C}_7^*$ are written in the form $X^7_{0,k,l}$, and similarly for other $\mathcal{C}_i^*$. The reader may verify that the following is a complete list of rank 2 contractions:

$\mathcal{R}_0 = \{(\mathbf{e}_p - \mathbf{e}_m) \vee (\mathbf{e}_p - \mathbf{e}_q) \vee (\mathbf{e}_m - \mathbf{e}_q) | p, q, m \in [1,r] \text{ or } [r+1, r+s] \text{ or } [r+s+1, r+s+t]\}$

$$\mathcal{R}_1^i = \{X_{q,k,l}^i \vee X_{p,k,l}^i \vee (\mathbf{e}_p - \mathbf{e}_q) \,|\, p,q \in [1,r]\} \qquad \text{for } i = 4,5,6,7$$

$$\mathcal{R}_2^i = \{X_{j,k,p}^i \vee X_{j,k,q}^i \vee (\mathbf{e}_p - \mathbf{e}_q) \,|\, p,q \in [r+s+1, r+s+t]\} \qquad \text{for } i = 4,5,8$$

$$\mathcal{R}_3^i = \{X_{j,p,0}^i \vee X_{j,q,0}^i \vee (\mathbf{e}_p - \mathbf{e}_q) \,|\, p,q \in [r+1, r+s]\} \qquad \text{for } i = 6,7$$

$$\mathcal{R}_4 = \{X_{0,q,l}^8 \vee X_{0,p,l}^8 \vee (\mathbf{e}_p - \mathbf{e}_q) \,|\, p,q \in [r+1, r+s]\}$$

$$\mathcal{R}_5^i = \{X_{j,k,l}^{i+1} \vee X_{j,k,l}^i \vee \overline{X}\} \qquad \text{for } i = 4,6$$

$$\mathcal{R}_6^i = \{X_{j,k,0}^i \vee X_{j,0,l}^{i-2} \vee X_{0,k,l}^8\} \qquad \text{for } i = 6,7$$

Again, the elements of $\mathcal{R}_0$ define a permutation in $\mathcal{S}_r \times \mathcal{S}_s \times \mathcal{S}_t$. So we will enumerate those localizations σ which fix $\sigma(\mathbf{e}_p - \mathbf{e}_q) = +$ for $p < q$ and $\sigma(\overline{X}) = +$, and multiply this count by $2r!s!t!$.

Once again, the cocircuit signatures for the cocircuits in the remaining classes $\mathcal{C}_4^*, \ldots, \mathcal{C}_8^*$ may be entered in tableaux with entries weakly increasing along rows and weakly decreasing down columns. Furthermore, as in earlier cases, certain similar cocircuit classes can be paired off, with their tableaux superimposed upon one another. The reader can verify that the collection of tableaux in Figure 36 encodes the information from all rank 2 contractions except $\mathcal{R}_6^6$ and $\mathcal{R}_6^7$.

If $L_{r,t}$, $L_{r,s}$ and $L_{s,t}$ encoded the information from **all** rank 2 contractions, then the final count would be obtained by simply enumerating all possible tableaux of each type and taking the product. However, the information from the rank 2 contractions $\mathcal{R}_6^6$ and $\mathcal{R}_6^7$ still has to be taken into account. It turns out that these rank 2 contractions may be used to define a bijection between the collection of localizations and a somewhat simpler collection of tableaux.

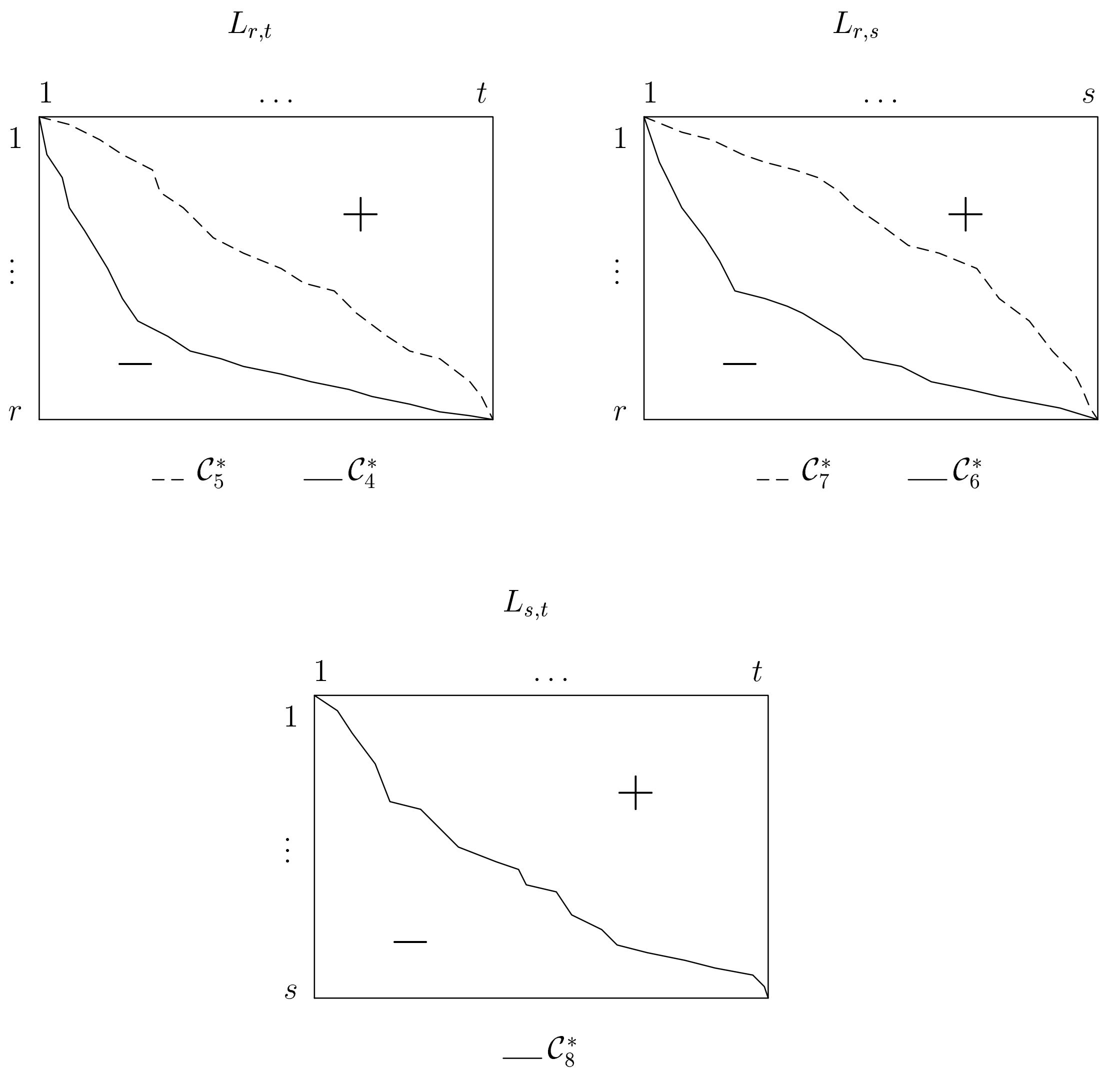


Figure 36: The tableaux of cocircuit signatures for Z

Each of the tableaux $L_{r,t}$ and $L_{r,s}$ may be thought of as an collection of columns, ordered from left to right. Specifically, one may index each column h of each tableau with an ordered pair (j_1, j_2), where j_1 is the greatest row index of a cell in h lying above the dotted path, and j_2 is the greatest row index of a cell lying above the solid path. One may then partially order a collection of columns of the same size by the product partial order on pairs (j_1, j_2), namely, $(j_1, j_2) \leq (j'_1, j'_2)$ if and only if $j_1 \leq j'_1$ and $j_2 \leq j'_2$. It is clear that a tableau L contains two noncrossing, monotonically decreasing paths if and only if the columns of L define some linear extension of this partial order. Let the columns of $L_{r,t}$ be indexed by $(\alpha_1, \alpha_2, \ldots, \alpha_t)$, and the columns of $L_{r,s}$ by $(\beta_1, \beta_2, \ldots, \beta_s)$.

The single path P in $L_{s,t}$ defines an *interweaving* of the columns $\alpha_a \in L_{r,t}$ and $\beta_b \in L_{r,s}$ in the following manner. For each unit segment z of P, give z the label α_l, if z is a horizontal segment adjacent to cells with column index l, and give z the label β_k if s a vertical segment adjacent to cells with row index k. By following P from the upper left corner of $L_{s,t}$ to the lower right and reading off the labels, one obtains an interweaving ω of the α_a with the β_b which preserves the original linear order for each collection of columns.

Lemma 6.5 *Let σ be a cocircuit signature on the cocircuits of $Z(V^*)$ respecting the restrictions imposed by $\mathcal{R}_0, \mathcal{R}_1^i, \mathcal{R}_2^i, \mathcal{R}_3^i, \mathcal{R}_4$ and $\mathcal{R}_5^i$, and furthermore satisfying $\sigma(\mathbf{e}_p - \mathbf{e}_q) = +$ for all possible p, q and $\sigma(\overline{X}) = +$.*

Let $L_{r,t}$, $L_{r,s}$ and $L_{s,t}$ be as described above. The cocircuit signature σ respects the rank 2 contractions of $\mathcal{R}_6^i$ (and thus is a localization) if and only if the interweaving ω orders the columns of $L_{r,t}$ and $L_{r,s}$ in a manner consistent with the partial order on columns.

Proof: Suppose the path P in $L_{s,t}$ is such that one encounters the adjacent labels α_l and β_k in order in a walk from the upper left corner to the lower right corner (they form a "northeast corner" in P). In particular, this implies that $\sigma(X^8_{0,k,l}) = -$. The interweaving ω implies $\alpha_l \leq \beta_k$.

Suppose instead that either $\alpha_l > \beta_k$ or the two are incomparable under the partial order on columns. This will happen if and only if the following statement:

$$\text{There exists an index } j \text{ such that } \sigma(X^{i-2}_{j,0,l}) > \sigma(X^i_{j,k,0})$$

holds for at least one of $i = 6$ or $i = 7$. However, if σ respects $\mathcal{R}^i_6$, then this statement implies that $\sigma(X^8_{0,k,l}) = +$, which is a contradiction. This demonstrates the necessity of the condition in the lemma.

To demonstrate sufficiency, suppose there exist indices j, k, l such that

$$(*) \qquad \sigma(X^i_{j,k,0}) = - \qquad \sigma(X^{i-2}_{j,0,l}) = + \qquad \sigma(X^8_{0,k,l}) = -$$

for $i = 6$ or $i = 7$. If the cell (k, l) of $L_{s,t}$ is bordered by P above and on the right, then ω implies $\alpha_l \leq \beta_k$. If the columns α_l and β_k satisfy this relation, then necessarily $\sigma(X^{i-2}_{j,0,l}) \leq \sigma(X^i_{j,k,0})$ for all j and $i = 6, 7$. But this already contradicts the assumption (*).

If the cell (k, l) is not bordered by P in the manner described above, then it is still possible to move from the cell (k, l) to a cell (k', l') which is bordered by P and satisfies a condition like the one given in $(*)$. Since σ satisfies all rank 2 contractions except $\mathcal{R}^i_6$ for $i = 6, 7$, moving from (k, l) in the direction of decreasing k and increasing l preserves the signatures in $(*)$. Then the condition $(*)$ for the cell (k', l') yields a contradiction also. This demonstrates that if σ is not a localization, then the ordering ω will not agree with the natural ordering on the columns of $L_{r,t}$ and $L_{r,s}$.$\square$

As a result of Lemma 6.5, we now see that for a given localization σ, **all** necessary information from the rank 2 contractions may be encoded by taking a collection of tableaux as given in Figure 36 and interweaving the columns of $L_{r,t}$ in $L_{r,s}$ to obtain a single $r \times (s+t)$ tableau L such that the columns of L define a linear extension of the partial order on columns. In particular, L must contain two noncrossing, monotonically decreasing paths. An example of this interweaving is given in Figure 37.

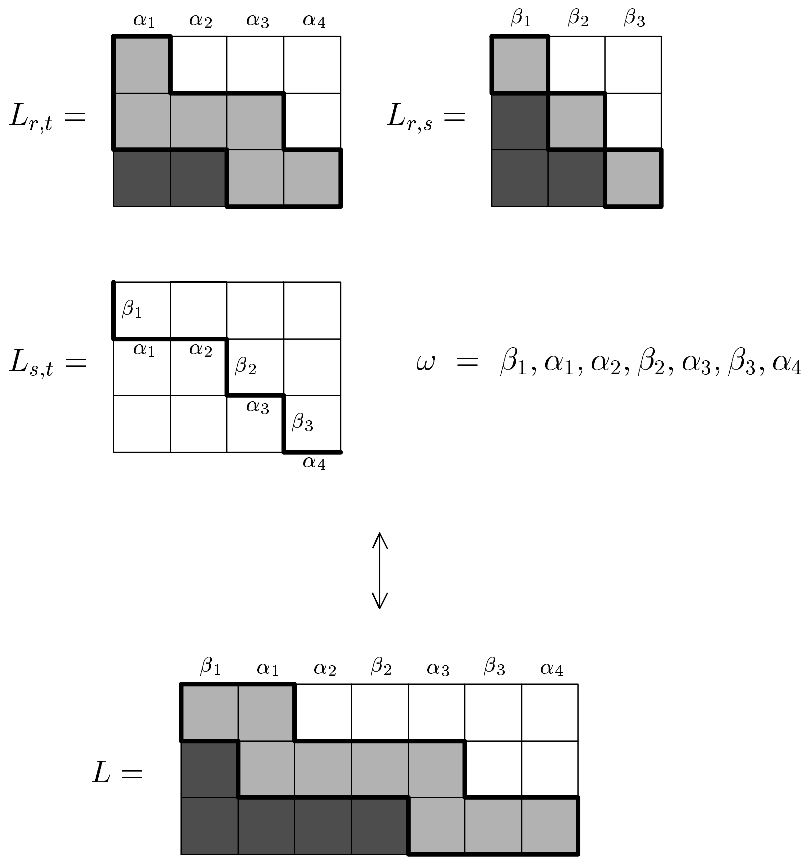


Figure 37: The rank 2 contractions $\mathcal{R}_6^i$ define an interweaving of the columns of $L_{r,t}$ and $L_{r,s}$

Thus the total number of localizations is counted by multiplying the number of $r \times (s+t)$ tableaux L containing two noncrossing paths by the number of ways of partitioning the columns of such a tableau L into sets of sizes s and t. Again using the result of Stanley [Sta3], Section 2.7, this number is

$$\frac{(r+s+t)!(r+s+t+1)!}{r!(r+1)!(s+t)!(s+t+1)!} \cdot \frac{(s+t)!}{s!t!}.$$

Multiplying this last number by $2r!s!t!$ gives the result in the statement of the theorem.

To enumerate the coherent tilings of Z, the $\mathcal{D}(V)$ corresponds to an Athanasiadis-type complete bipartite transitive gain graph on vertex sets K_r and K_{s+t}. Again, $\mathcal{D}(V)$ is free with exponents

$$\{0, 1, r+1, r+2, \ldots, r+s+t-1, s+t+1, s+t+2, \ldots, r+s+t\},$$

by Corollary 7.10, and so Z has the desired number of coherent tilings as well.

Next consider the case where two frame vectors have multiplicity one, and the other multiplicities are r, s, t as in the statement of the theorem. By Lemma 5.4, we already know that Z is coherent. Here $\mathcal{D}(V)$ may be written in block form as

$$\mathcal{D}(V) = \begin{pmatrix} A_{r-1} & 0 & 0 & I_r & I_r & 0 \\ 0 & A_{s-1} & 0 & \overset{\times}{-} I_s & 0 & I_s \\ 0 & 0 & A_{t-1} & 0 & \overset{\times}{-} I_t & \overset{\times}{-} I_t \\ 0 & 0 & 0 & B_1 & B_2 & B_3 \end{pmatrix}$$

where

$$B_1 = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{pmatrix} \qquad B_3 = \begin{pmatrix} -1 & \cdots & -1 \\ 0 & \cdots & 0 \end{pmatrix}.$$

The reader can verify that this arrangement is projectively equivalent to the braid arrangement $A_{r+s+t-1}$, which is known to be free with exponents $\{1, 2, \ldots, r+s+t-1\}$.

Finally, we must show that Z is incoherent in the case that the multiplicities on its vectors do not satisfy the hypotheses of the theorem. Again, we resort to brute force computation in GAP to show that this is the case.

First, if the vector multiplicities are $\{3, 3, 2, 1, 1\}$ where the vectors with multiplicity three form a partnership, then Z has 211680 tilings, 210816 of which are coherent. This gives the necessity of the condition in part (1) of the theorem. Furthermore, if all frame vectors have multiplicity two and the common vector has multiplicity one, then Z has 25408 tilings, 23136 of which are coherent. This demonstrates the necessity of the condition that one frame vector have multiplicity one. Together, these conditions show the necessity of the statement that at most two frame vectors may have multiplicities of three or greater. This completes the proof of the theorem. $\square$

7 TG-graphic Arrangements

7.1 Introduction

A long-standing question in the theory of hyperplane arrangements has been that of determining sufficient conditions under which the characteristic polynomial $\chi(\mathcal{A}, t)$ of a hyperplane arrangement $\mathcal{A}$ factors with positive integer roots. Several such conditions have been discovered — specifically, when $\mathcal{A}$ is *free*, *factored*, *inductively factored*, or *supersolvable* (See [Sta1], [Te], [JP],[OT]).

One particularly fruitful method for characterizing these properties for large classes of arrangements is to consider families of arrangements which correspond to (directed or undirected) graphs in a natural way (see [Za2]). It turns out that some of the discriminantal arrangements arising in the zonotope classification correspond in a natural way to a particular class of gain-graphic arrangements which we call *transitive-gain-graphic* arrangements, or *TG-graphic* arrangements for short. They correspond to a subclass of gain graphs which we call *simple* gain graphs. In this chapter we provide a complete gain-graphic characterization of those simple gain-graphic arrangements which are free. This will complete the proof of Theorem 6.3. It turns out that it takes very little additional work to determine that the simple gain-graphic arrangements which are factored, inductively factored and supersolvable all coincide, and to give a gain-graphic characterization of this subclass as well.

7.2 Definitions and terminology

The definitions here follow Zaslavsky's terminology (see [Za3]). A *graph* is a pair $\Gamma = \Gamma(N, E)$, where $N = N(\Gamma)$ is a collection of nodes, and $E = E(\Gamma)$ is a set of edges, together with an endpoint mapping ν_Γ, which assigns to each edge e a multiset of at most two nodes, not necessarily distinct. Note that this definition allows for the possibility that Γ may have multiple edges or loops. Here we restrict our attention to those graphs Γ with $|\nu_\Gamma(e)| = 2$ for all $e \in E$. Zaslavsky calls such a graph an *ordinary graph.* An edge is a *link* if it has two distinct endpoints, and a *loop* if it has two coincident endpoints.

Let $X \subseteq N$. Then the *induced subgraph* $\Gamma : X$ of Γ is $\Gamma : X = (X, E : X)$, where

$$E : X = \{e \in E \mid \nu_\Gamma(e) \subseteq X\}.$$

A *gain graph* (also known as a *voltage graph*) $\Phi = (\Gamma, \phi)$ consists of an underlying graph $\|\Phi\| = \Gamma = (N, E)$ and a *gain mapping* $\phi : E \to \mathcal{G}$ from the set of edges into a *gain group* $\mathcal{G}$. In our case, set $\mathcal{G} = \mathbb{Z}$ under addition. Also, some links and all loops e of Γ will be directed, and it is understood that $\phi(e^{-1}) = \phi(e)^{-1}$ for directed edges e, where e^{-1} means e with its direction reversed. Figure 38 shows an example of a gain graph. A labeled edge $e : v \to w$ carries the label $\phi(e)$; for unlabeled edges e, it is understood that $\phi(e) = 1$.

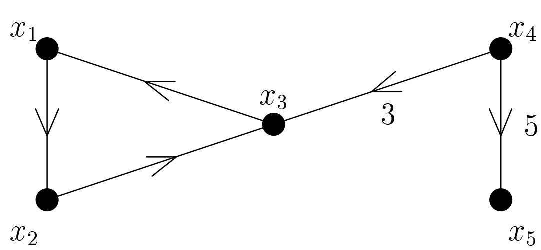


Figure 38: A (loopless) gain graph

Let $\mathbf{k}$ be a field and $a \in \mathbf{k}$ an element of infinite multiplicative order. Given a gain graph Φ on n nodes, the arrangement of hyperplanes $\mathcal{A}(\Phi)$ in

$\mathbf{k}^n$ associated to Φ is defined by

$$\mathcal{A}(\Phi) := \{x_i = 0\}_{i=1}^n \cup \{a^k x_i = x_j \mid \text{there exists } e : x_i \to x_j \in E \text{ with } \phi(e) = k\}$$

where $a \in \mathbf{k}$, $a \neq 0, 1$. All gain graphs we will consider implicitly contain all possible loops $e : x_i \to x_i$ with $\phi(e) = 1$ and all possible links $e : x_i \to x_j$ with $\phi(e) = 0$, although these edges will be suppressed in the diagrams. In particular, each $\mathcal{A}(\Phi) \subseteq \mathbf{k}^n$ will contain the braid arrangement of type A_{n-1}. Also, since each loop corresponds to the hyperplane $ax_i = x_i$, $\mathcal{A}(\Phi)$ will always contain the *Boolean arrangement* of type n. For example, the gain graph in Figure 38 corresponds to the arrangement of hyperplanes

$$\{x_i = 0\}_{i=1}^5 \cup \{x_i = x_j\}_{1 \leq i < j \leq 5}$$
$$\cup \{ax_1 = x_2, ax_2 = x_3, ax_3 = x_1, a^3 x_4 = x_3, a^5 x_4 = x_5\}.$$

For the remainder of the paper, we will identify a gain graph with its corresponding arrangement of hyperplanes, saying that Φ is *free* or *supersolvable* if and only if $\mathcal{A}(\Phi)$ is. The principal focus of this section will be arrangements corresponding to gain graphs with the properties that

- $\phi(e) = 0$ or ± 1 for all edges e.
- For each pair $\{v_1, v_2\}$ of (not necessarily distinct nodes) in X, there is at most one edge $e \in E$ with the property that $\nu_\Gamma(e) = \{v_1, v_2\}$ and $\phi(e) \neq 0$.

If a gain graph Φ satisfies these conditions, then we call Φ *simple*.

One advantage to considering (gain-)graphic arrangements of hyperplanes is the ease with which one can apply the Localization Theorem of Orlik and Terao. For an arrangement $\mathcal{A}$, the hyperplanes in the arrangement intersect in a collection $L = L(\mathcal{A})$ of subspaces of various dimension. The elements of L may be partially ordered by reverse inclusion, with the ambient vector space $\mathbf{k}^n$ as the unique minimal element. Orlik and Terao's theorem is

Theorem 7.1 *[OT Theorem 4.37] If $\mathcal{A}$ is free then $\mathcal{A}_Y$ is free for all $Y \in L(\mathcal{A})$, where*

$$\mathcal{A}_Y := \{H \in \mathcal{A} | Y \subseteq H\}.$$

The arrangement $\mathcal{A}_Y$ is called the *localization* of $\mathcal{A}$ to the subspace Y. If Φ' is an induced subgraph of a gain graph Φ on node set X, then the arrangement $\mathcal{A}(\Phi')$ is simply the localization of $\mathcal{A}(\Phi)$ to the subspace of $\mathbf{k}^n$ which has all coordinates $x_j = 0$ for $x_j \in X$. Thus by considering induced subgraphs of Φ, one may use Theorem 7.1 and an obstruction argument to determine necessary conditions for $\mathcal{A}(\Phi)$ to be free.

For an arrangement $\mathcal{A}$ and a subspace $Y \in L(\mathcal{A})$, define the *restriction* of $\mathcal{A}$ to Y to be

$$\mathcal{A}^Y := \{H \cap Y | H \in \mathcal{A} - \mathcal{A}_Y\}.$$

The sequence of arrangements $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is called a *triple* if there exists a hyperplane $H \in \mathcal{A}$ such that $\mathcal{A}' = \mathcal{A} - H$ and $\mathcal{A}'' = \mathcal{A}^H$. Orlik and Terao's Addition-Deletion Theorem states

Theorem 7.2 *[OT Theorem 4.51] Suppose $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a triple. Any two of the following statements imply the third:*

$\mathcal{A}$ *is free with exponents* $\{b_1, \ldots, b_{l-1}, b_l\}$,

$\mathcal{A}'$ *is free with exponents* $\{b_1, \ldots, b_{l-1}, b_l - 1\}$,

$\mathcal{A}''$ *is free with exponents* $\{b_1, \ldots, b_{l-1}\}$.

Once a class of forbidden subgraphs has been determined, an induction argument together with Theorem 7.2 finishes the proof that any gain graph which avoids the forbidden subgraphs is in fact free.

Figure 39: Two nonfree induced subgraphs which are an obstruction to freeness

7.3 TG-graphic arrangements

We now determine necessary and sufficient conditions for a simple gain graph to be free. Terao [Te] showed that if an arrangement of hyperplanes $\mathcal{A}$ is free, then its *characteristic polynomial*

$$\chi(\mathcal{A}, t) = \sum_{Y \in L} \mu(Y) t^{\mathrm{rank}(Y)}$$

factors with positive integer roots, where μ is the *Möbius function* for $L(\mathcal{A})$ with the first variable held equal to $\mathbf{k}^n$. Using the fundamental recursion ([OT], section 2.3)

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t)$$

to compute the characteristic polynomial of the two gain graphs shown in Figure 39, it follows immediately that neither of the gain graphs in Figure 39 are free. The gain graph (i) has characteristic polynomial $(t-1)(t^2-7t+13)$, and the gain graph (ii) has characteristic polynomial $(t-1)(t^2-8t+18)$.

It therefore follows that a necessary condition for a simple gain graph to be free is that it be *transitive*. A transitive gain graph Φ satisfies the property that if $e_1 : x \to y \in \Gamma$ and $e_2 : y \to z \in \Gamma$ with $\phi(e_1) = \phi(e_2) = 1$, then there exists $e_3 : x \to z \in \Gamma$ with $\phi(e_3) = 1$.

To continue the characterization of free simple gain graphs requires consideration of a slightly larger family of gain graphs, called *augmented transi-*

tive gain graphs. We define an *augmented* transitive gain graph to be a gain graph $\Phi^* = (\Gamma^*, \phi^*)$ with distinguished node v^* such that

a) The gain graph $\Phi = (\Gamma, \phi)$ is a simple transitive gain graph, where Γ is the induced subgraph $\Gamma^* : N(\Gamma^*) \setminus \{v^*\}$ and $\phi = \phi^*|_{E(\Gamma)}$.

b) If $e : v^* \to x \in \Gamma^*$ with $\phi^*(e) = k$ and $e' : x \to y \in \Gamma$ with $\phi(e') = 1$, then there exists $e'' : v^* \to y \in \Gamma^*$ with $\phi^*(e'') = k$.

c) For all $x \in \Gamma$, if there exists $e : v^* \to x \in \Gamma^*$ with $\phi^*(e) = k$, then there exists a link $e' : v^* \to x \in \Gamma^*$ with $\phi^*(e') = k - 1$, for all $k \geq 1$.

d) There exists some $q \in \mathbb{Z}$ such that for each $x \in \Gamma$, the maximum value of $\phi^*(e)$ occuring for any link $e : v^* \to x$ is either q or $q - 1$.

We make the further requirement that no two edges $e_1 \neq e_2$ have the property that $(\nu_\Gamma(e_1), \phi(e_1)) = (\nu_\Gamma(e_2), \phi(e_2))$.

An example of an augmented transitive gain graph is given in Figure 40. It is convenient to use the label $[q]$ to indicate that for vertices v^* and x, and for every $j \in [0, q]$, there exists a link $e : v^* \to x$ with $\phi^*(e) = j$.

If Φ^* is an augmented transitive gain graph, then for $x \in \Gamma$, define a_x to be the number of links of Γ^* directed toward x. For example, in Figure 40, $a_x = q + 2$. The collection $\{a_x\}_{x\in\Gamma}$ is a multiset of cardinality n. Let $\{a_i\}_{i=1}^n$ be any ordering of the multiset such that $a_i \geq a_j$ for $i < j$. The following theorem completely characterizes the free augmented transitive gain graphs.

Theorem 7.3 *Let $\Phi^* = (\Gamma^*, \phi^*)$ be an augmented transitive gain graph. Then $\mathcal{A}(\Phi^*)$ is free if and only if Φ^* does not contain an induced subgraph of the kind shown in Figure 41 for any positive integer k. Furthermore, the exponents of $\mathcal{A}(\Phi^*)$ are*

$$\{1\} \cup \{a_i + i + 1\}_{i=1}^n.$$

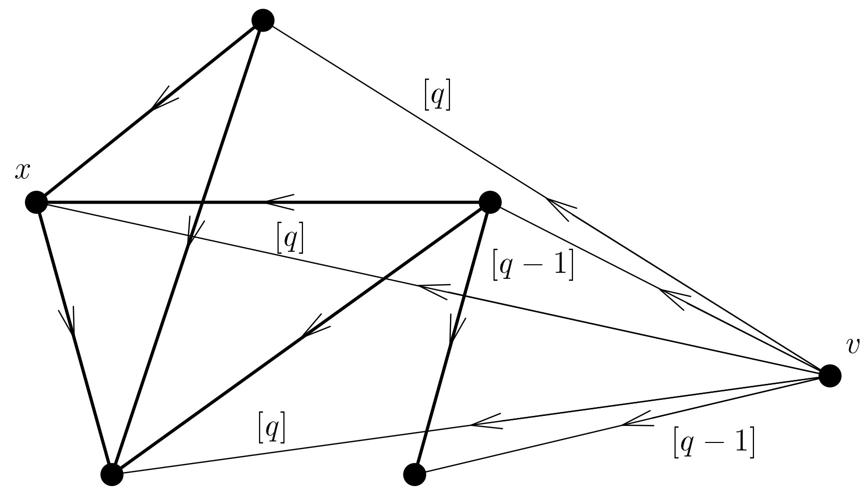


Figure 40: An augmented transitive gain graph

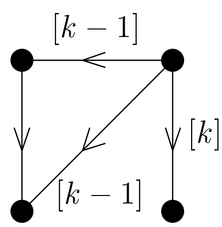


Figure 41: The induced subgraph which is the obstruction to freeness of an augmented transitive gain graph

Note that any simple gain graph Φ on n vertices may be considered an augmented transitive gain graph on $n-1$ vertices (since Γ is transitive, there must exist some node v^* with indegree zero). Thus the theorem immediately implies

Corollary 7.4 *Let $\Phi = (\Gamma, \phi)$ be a simple transitive gain graph on n vertices such that the node v^* has indegree zero. For nodes $x \neq v^*$, define a_x and the ordering $\{a_i\}_{i=1}^{n-1}$ as above. Then $\mathcal{A}(\Phi)$ is free if and only if Φ does not contain an induced subgraph of the kind shown in Figure 41 for $k=1$. In this case,* *the exponents of* $\mathcal{A}(\Phi)$ *are*

$$\{1\} \cup \{a_i + i + 1\}_{i=1}^{n-1}.$$

Proof (of Theorem 7.3): To prove the necessity of the theorem, let H be the augmented transitive gain graph shown in Figure 41. One may use the fundamental recursion for the characteristic polynomial to compute

$$\chi(\mathcal{A}(H), t) = (t-1)(t-(k+3))\left[(t-(k+2))^2 - (t-(k+2)) + 1\right]$$

Setting $X = t-(k+2)$, it is clear that $\chi(\mathcal{A}(H), t)$ does not factor over $\mathbb{Z}$, and so $\mathcal{A}(H)$ is not free. By Theorem 7.1, no free augmented transitive gain graph Φ^* contains H as an induced subgraph.

To show sufficiency, we first need a lemma.

Lemma 7.5 *Suppose* Φ *is a simple transitive gain graph which does not contain a forbidden subgraph. For* $x \in \Gamma$, *define*

$$U_x := \{z \in \Gamma \mid \textit{there exists } e : z \to x \in E(\Gamma) \textit{ with gain 1 }\}.$$

Then for any pair $x \neq y$ *of elements in* Γ, *either* $U_x \subseteq U_y$ *or* $U_y \subseteq U_x$.

Proof: First, note that since the lemma does not consider *augmented* transitive gain graphs, all links e have $\phi(e) = 0$ or 1. Now suppose $x \neq y$ are elements of Γ, and suppose there exist $z_1, z_2 \in \Gamma$ such that $z_1 \in U_x \cap U_y^c$ and $z_2 \in U_x^c \cap U_y$. Then the induced subgraph on x, y, z_1, z_2 is a forbidden subgraph. $\square$

We now proceed to show the sufficiency of the characterization in the theorem. Let Φ^* be an augmented transitive gain graph, and let q be the maximum value of ϕ^* for any edge e. Define K to be the induced subgraph on the node set $\{v_j \mid \text{there exists } e : v^* \to v_j \in E(\Gamma^*) \text{ with gain } q\}$. Lemma

7.5 guarantees the existence of a node $x^* \in K$ such that $U_{x^*} \subseteq U_y$ for all $y \in K$. Note that in particular, this implies that there is no link $e : y \to x^*$ of gain 1 in $E(\Gamma)$ for any $y \in K$.

The proof proceeds by use of induction, applying the Addition-Deletion theorem to the link $\overline{e} : v^* \to x^*$ with $\phi(\overline{e}) = q$. It is first necessary to verify that the gain graphs Φ', Φ'' satisfy the hypotheses of the theorem, where Φ' is the gain graph which results from the deletion of the link $\overline{e}$ and Φ'' is the gain graph which corresponds to restriction to the link $\overline{e}$.

Deletion: Clearly removal of $\overline{e}$ does not affect the underlying simple gain graph Γ. Also, since $\phi(\overline{e}) = q$ is maximal, and since there is no link $e : y \to x^*$ of gain 1 in $E(\Gamma)$ for $y \in K$, the deletion Φ' is again an augmented transitive gain graph. All that remains is to check that Φ' avoids the forbidden subgraph.

Removal of $\overline{e}$ can yield a forbidden subgraph in two possible ways:

(iii) (iv)

If (iii) occurs as an induced subgraph of Φ', then Φ^* was not an augmented transitive gain graph since $\overline{e} \in \Phi^*$ and $e : x^* \to x_2$ of gain 1 is in Φ^* but $e' : v^* \to x_2$ of gain q is not in Φ^*. If (iv) occurs, then $x_1 \in K$, but $x_2 \in U_{x^*} \cap U^c_{x_1}$, contradicting the hypothesis $U_{x^*} \subseteq U_y$ for all $y \in K$.

Therefore we may conclude that Φ' satisfies the hypotheses of the theorem.

Restriction: To determine the result of restriction to the link $\overline{e}$, we partition the node set of Γ into three classes:

Class 1) The set of nodes $x \notin K$,

Class 2) The set of nodes $x \in K$ such that no edge $e : x^* \to x$ with $\phi(e) = 1$ exists,

Class 3) The set of nodes $x \in K$ such that $e : x^* \to x$ with $\phi(e) = 1$ does exist.

To verify that Φ'' is an augmented transitive gain graph, note that the underlying simple transitive gain graph of Φ'' is the induced subgraph $\Gamma - x^*$ of Γ. For the links incident to v^*, it is a routine matter to verify that the maximum gain for a link from v^* to any node in Class 1 increases from $q-1$ to q, the maximum gain for a link from v^* to any node in Class 2 remains q, and the maximum gain for a link from v^* to any node in Class 3 increases from q to $q+1$. Since the definition of augmented transitive gain graph precludes the possibility of an link $e : x \to x' \in E(\Gamma)$ of gain 1 with x in Class 3 and x' in Classes 1 or 2, Φ'' is an augmented transitive gain graph. Moreover, since the underlying simple graph $\Gamma - x^*$ does not contain a forbidden subgraph by hypothesis, any forbidden subgraph in Φ'' must involve v^*.

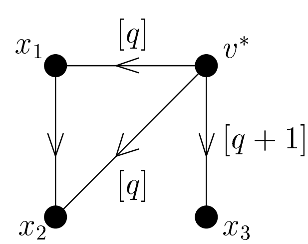


Figure 42: The possible obstruction to freeness for Φ''

What properties do the nodes x_1, x_2, and x_3 have in Φ^*? Clearly $x_3 \in K$, since the link $e_3 : v^* \to x_3$ with $\phi(e_3) = q + 1$ exists in Φ''. If both $x_1 \notin K$ and $x_2 \notin K$, then the induced subgraph on nodes v^*, x_1, x_2, x_3 in Φ^* is of the forbidden type. Thus it must be the case that at least one of x_1, x_2 is in K, and since Φ^* is an augmented transitive gain graph and $e_0 : x_1 \to x_2$ with $\phi(e_0) = 1$ is in Φ^*, it follows that $x_2 \in K$.

However, it follows from Lemma 7.5 that either $U_{x_3} \subseteq U_{x_2}$ or $U_{x_2} \subseteq U_{x_3}$ in Φ^*. If the former case holds, note that since $e_3 \in \Phi''$ it follows that $x^* \in U_{x_3} \subseteq U_{x_2}$. But if $x^* \in U_{x_2}$, then $e : v^* \to x_2$ with $\phi(e) = q + 1$ is in Φ'' and there is no obstruction. If the latter case holds, then $x_1 \in U_{x_3}$ and again there is no obstruction. So Φ'' is again an augmented transitive gain graph which avoids the forbidden induced subgraph, and thus satisfies the hypotheses of the theorem.

So suppose $\{a_i\}_{i=1}^n$ is the ordered multiset described on page 95, and suppose a_j is the indegree of x^*. There is no loss of generality in assuming that $a_j > a_{j+1}$, or that $a_j = a_n$ in the case that a_j is a minimum. Therefore by induction, $\mathcal{A}(\Phi')$ is free with exponents

$$\{1, a_1 + 2, \ldots, a_{j-1} + (j - 1) + 1, a_j + j, a_{j+1} + (j + 1) + 1, \ldots, a_n + n + 1\}.$$

To determine the exponents of $\mathcal{A}(\Phi'')$, consider how restriction on $\overline{e}$ affects the indegree of each node of Γ^*. If x is in Class 1, it is clear that a_x is replaced by $a_x + 1$, with the addition of the an link $e : v^* \to x$ with gain q. If x is in Class 2, it is clear that a_x is unchanged. Finally, if x is in Class 3, then the link $e : v^* \to x$ with gain $q + 1$ *replaces* the link $e' : x^* \to x$ with gain 1, and so a_x remains unchanged.

Before we can write down the exponents of Φ'', however, we must consider how the new a_x are related to a_j. Since restriction does not change a_x for x in Classes 2 or 3, it is enough to consider only x in Class 1.

If x is in Class 1, so $x \notin K$, then in particular, no link $e : x^* \to x$ with gain 1 is in Γ^*, for otherwise Γ^* violates condition (b) in the definition of an augmented transitive gain graph. Suppose U_{x^*} is properly contained in U_x. Then there can be no link $e : x \to x^*$ with gain 1 in Φ^*, for the existence of such a link implies that U_x is properly contained in U_{x^*}. Furthermore, there must exist some $z \in \Gamma^*$ such that $e' : z \to x$ with gain 1 is in Γ^* but no $e'' : z \to x^*$ with gain 1 is in Γ^*. Since $e' \in \Gamma^*$ but $x \notin K$, then $z \notin K$. But then the induced subgraph of Γ^* on vertices v^*, x^*, x, z is the forbidden subgraph shown in Figure 41. Consequently, for x in Class 1, the containment $U_x \subseteq U_{x^*}$ must hold. Furthermore, since $x^* \in K$ but $x \notin K$, one may conclude that $a_j > a_x$. Therefore replacing a_x with $a_x + 1$ upon restriction does not alter the order on the multiset $\{a_i\}$.

So $\mathcal{A}(\Phi'')$ is free with exponents

$$\{1, a_1 + 2, \ldots, a_{j-1} + (j-1) + 1, (a_{j+1} + 1) + j + 1, (a_{j+2} + 1) + (j+1) + 1, \\ \ldots, (a_n + 1) + (n-1) + 1\}$$

or equivalently

$$\{1, a_1{+}2, \ldots, a_{j-1}{+}(j{-}1){+}1, a_{j+1}{+}(j{+}1){+}1, a_{j+2}{+}(j{+}2){+}1, \ldots, a_n{+}n{+}1\}$$

Thus by the Addition-Deletion theorem, $\mathcal{A}(\Phi^*)$ is free with exponents

$$\{1, a_1 + 2, \ldots, a_{j-1} + (j-1) + 1, a_j + j + 1, a_{j+1} + (j+1) + 1, \ldots a_n + n + 1\}.$$

This completes the proof of Theorem 7.3 and thus proves Corollary 7.4. $\square$

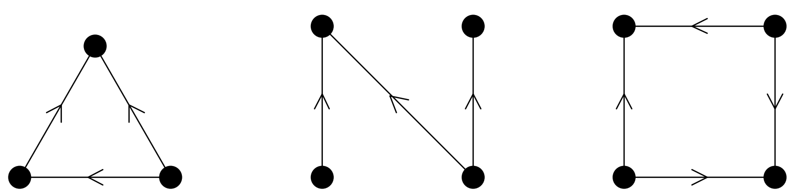
A further corollary, which follows from Corollary 7.4, is

Corollary 7.6 *Let Φ be a complete, transitive m-partite simple gain graph, with parts $K_1, \ldots, K_m$ and part sizes $b_i = |K_i|$ for all $1 \le i \le m$. Then $\mathcal{A}(\Phi)$ is free with exponents*

$$\{1, [2 + \hat{S}_1, S], [2 + \hat{S}_2, 1 + S], \cdots, [2 + \hat{S_m}, 1 + S]\}$$

where
$$S = \sum_{j=1}^{m} b_j \qquad \textit{and} \qquad \hat{S}_i = \sum_{j \neq i} b_j$$
and $[p, q]$ *denotes the interval of integers* $\{p, p+1, p+2, \ldots, q\}$.

Finally, we characterize those simple gain-graphic arrangements which are factored, inductively factored and supersolvable. Since an arrangement in any of these classes has a characteristic polynomial with integer roots, the graphs in Figure 39 and Figure 41 are obstructions in these classes as well. Note that in particular, this implies that if a simple gain-graphic arrangement is factored, it is also free (supersolvable and inductively factored arrangements are free in general). Furthermore, consider the two gain graphs pictured in Figure 43.



(v) (vi) (vii)

Figure 43: Three obstructions to supersolvability

By Corollary 7.4 the arrangement corresponding to (v) is free with exponents $\{1, 4, 4\}$, the arrangement corresponding to (vi) is free with exponents $\{1, 4, 4, 4\}$, and the arrangement corresponding to (vii) is free with exponents $\{1, 4, 4, 5\}$. However, it is easy to check by trial and error that none of these arrangements have an M-chain (see page 69) corresponding to these exponents, and so none is supersolvable. Stanley [Sta2] showed that induced subgraphs act as obstructions to supersolvability just as they do for freeness.

Theorem 7.7 *Let $\mathcal{A}$ be supersolvable. Then A^X and A_X are each supersolvable for all $X \in L(\mathcal{A})$.*

A similar result holds for factored arrangements (see [JP]). Since the TG-graphic arrangements corresponding to the graphs in Figure 43 are not factored either (which also implies that they are not supersolvable), no factored, inductively factored or supersolvable simple transitive gain graph may contain any of the graphs shown in Figure 43 as an induced subgraph. The necessity of the characterization in the following theorem is immediate:

Theorem 7.8 *The classes of factored, inductively factored and supersolvable simple gain-graphic arrangements coincide. These classes are characterized by those free simple gain-graphic arrangements whose gain graph Φ has the property that all links e with gain 1 are directed toward a "star" node v.*

Proof: To show sufficiency, let $(v_1, v_2, \ldots, v_m)$ be an arbitrary ordering of those nodes distinct from v. Identify each hyperplane in $\mathcal{A}(\Phi)$ with the corresponding edge in Φ A moment's thought reveals that the following is an M-chain for $\mathcal{A}(\Phi)$.

$$\begin{aligned}
&\{\text{Those edges } e \text{ such that } v_m \in \nu_\Gamma(e)\},\\
&\{\text{Those edges } e \text{ such that } v_{m-1} \in \nu_\Gamma(e)\},\\
&\vdots\\
&\{\text{Those edges } e \text{ such that } v_1 \in \nu_\Gamma(e)\},\\
&\{\text{The loop at } v\}.
\end{aligned}$$

Since all supersolvable arrangements are also inductively factored and factored, this completes the proof. $\square$

Remarkably, Christos Athanasiadis [Ath] uses the same class of gain graphs to prove the freeness of a *different* class of hyperplane arrangements, a class $\mathbf{c}\hat{\mathcal{A}}_{n,E}$ which interpolates between the cone over the braid arrangement $\mathbf{c}A_{n-1}$ and the cone over the *Shi arrangement* $\mathbf{c}\hat{\mathcal{A}}_n$ in $\mathbb{R}^{n+1}$. The cone over the Shi arrangement is defined by

$$\begin{aligned} x_{n+1} &= 0 \\ x_i - x_j &= 0 \quad \text{for} \quad 1 \leq i < j \leq n, \\ (*) \qquad x_i - x_j - x_{n+1} &= 0 \quad \text{for} \quad 1 \leq i < j \leq n. \end{aligned}$$

By choosing a particular subset of the hyperplanes listed in $(*)$, Athanasiadis defines the class $\mathbf{c}\hat{\mathcal{A}}_{n,E}$ of arrangements, where E corresponds to the edge set of a particular gain graph. In particular, given a simple gain graph Φ on n nodes with gain group $\mathbb{Z}$, and edge set E with no loops, define $\mathbf{c}\hat{\mathcal{A}}_{n,E}$ by

$$\begin{aligned} x_{n+1} &= 0 \\ x_i - x_j &= 0 \quad \text{for} \quad 1 \leq i < j \leq n, \\ x_i - x_j - x_{n+1} &= 0 \quad \text{for } e : x_j \to x_i \in E, \phi(e) = 1 \end{aligned}$$

Using techniques similar to those above, Athanasiadis shows ([Ath], Theorem 4.1)

Theorem 7.9 *If Φ is a simple gain graph with $\Gamma = (N, E)$, where $|N| = n$, then $\mathbf{c}\hat{\mathcal{A}}_{n,E}$ is free if and only if Φ is transitive and does not contain a forbidden subgraph of the kind in Figure 41 for $k = 1$. In this case, $\mathbf{c}\hat{\mathcal{A}}_{n,E}$ has exponents*

$$\{0, 1\} \cup \{a_i + i\}_{i=1}^{n-1}$$

where $\{a_i\}_{i=1}^{n-1}$ are as described above.

As above, the following corollary is immediate:

Corollary 7.10 *Let Φ be a complete, transitive m-partite simple gain graph on n vertices with link set E, parts $K_1, \ldots, K_m$ and part sizes $b_i = |K_i|$ for all $1 \le i \le m$. Then $\mathbf{c}\hat{\mathcal{A}}_{n,E}$ is free with exponents*

$$\{0, 1, [1 + \hat{S}_1, S - 1], [1 + \hat{S}_2, S], \cdots, [1 + \hat{S_m}, S]\}$$

where

$$S = \sum_{j=1}^{m} b_j \qquad \textit{and} \qquad \hat{S}_i = \sum_{j \ne i} b_j.$$

The characterization of supersolvability for these arrangements also uses the same gain graphs as in Theorem 7.8. Interestingly, Athanasiadis first proved a statement about freeness for the affine arrangement $\hat{\mathcal{A}}_{n,E}$ corresponding to a gain graph Φ, obtained by restricting the arrangement $\mathbf{c}\hat{\mathcal{A}}_{n,E}$ to the hyperplane $x_{n+1} = 1$. In this case, the exponents are $\{0\} \cup \{a_i + i\}_{i=1}^{n-1}$. In particular, they are obtained by subtracting 1 from each exponent for the TG-graphic arrangement corresponding to the same gain graph Φ. It is unclear whether any more meaningful relationship exists between these two classes of arrangements.

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