Variations on the Slope Problem

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Acknowledgements

Abstract

The slope problem, in its most basic form, asks the question "Given n points in $\mathbb{R}^2$ which do not all lie on the same line, what is the smallest number of distinct slopes that they must determine?" The answer to this was settled in a 1980 paper by Peter Ungar, who used purely combinatorial methods to show that the answer was n if n was even, and $n-1$ if n was odd.

This paper considers several generalizations of this problem, such as:

1. Since the combinatorial methods invoked by Ungar essentially prove something about oriented matroids, can we find a similar matroid result if we ignore the orientation?

2. What if the slope(s) between certain predetermined points are ignored? If we construct a graph which records which slopes we are interested in, do properties of the graph tell us anything about the number of slopes required?

3. Having considered the more general slope problem involving graphs (i.e. of type A), is there any B analog to the slope problem?

As we will see, the answers to these questions are as follows.

1. There is a very strong analog of Ungar's result for matroids without orientation, and the corresponding smallest number of slopes turns out to grow much more slowly — more like $\sqrt{n}$ instead of n.

2. While the case of an arbitrary graph seems to be difficult, it appears that much can be said in the case of chordal graphs (and even more so for the class of threshold graphs, which are a subclass of chordal graphs). There are some initial general results, however, involving the chromatic number of a graph, as well as Crapo's β-invariant.

3. There is certainly a type B analog, which is essentially the problem of point configurations in $\mathbb{R}^2$ which are centrally symmetric.

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Chapter 1

Introduction

1.1 The Slope Problem

The vast majority of the work for this thesis grew out of the seemingly innocent question, "Given a certain number of points in the plane, what is the smallest number of slopes (i.e. parallel directions) that they must determine?" Since putting all of the points on a single line makes for the easy (and not particularly interesting) answer of "one", we rule out this possibility. As it turns out, adding the simple requirement that the points cannot all lie on the same line makes it a much more complicated question.

After playing around with point configurations which try to minimize the number of slopes, we might realize that it is fairly easy to have n points which determine only n slopes. This can be done by simply putting all of the points on the same line except for one.
[Picture]
Apparently, it was known for quite some time that it was possible to do slightly better than this if the number of points is odd. In that case, we can use symmetry to our advantage and obtain a configuration of n points (with n odd) which only determine $n - 1$ slopes. The configuration is similar to the first one we considered, only now all but two of the points are on the same line, and everything has to be centrally symmetric. Note that we need at least five points in order to this to work – three points will always determine three slopes (unless they are collinear).
[Picture]

While this shows us that it is possible to have a reasonably small number of slopes determined by a fixed number of points, we have so far left unanswered the question "Could we do even better?" Several attempts were made over the years by a variety of people to resolve this issue. However, it was not until rather recently (1980) that the answer to this question was discovered to be "No." Peter Ungar [21] was able to show that if there are an even number of points, say $2n$, then they *must* determine at least $2n$ slopes.

What we would like to consider in this paper are possible ways to generalize this result. We begin by looking at the methods employed by Ungar in his proof.

1.2 A Closer Look at Ungar's Method of Proof

Suppose we are given a configuration of n (non-collinear) points in the plane, labelled them 1 through n. Nearly every direction which we could pick in the plane will determine an order for the points, with the only problematic ones being those directions which are perpendicular to one of the slopes determined by the points (in which case we would have one or more ties). For example, using the points in the configuration

[Picture – MAKE SURE IT AGREES WITH THE TEXT]

and choosing our direction to be southwest-to-northeast, we see that the points are ordered as "3" first, then "5", "2" and "1", and finally "4" – or 35214 for short. In this way, given a particular point configuration, we can associate a permutation to (almost) any choice of direction.

Consequently, if we were to allow our direction in the plane to rotate a full 360°, we will generate a sequence of permutations, called an *allowable sequence*. Continuing with the point configuration above we have

[Picture]

As it turns out, sequences of permutations which arise in this fashion cannot be arbitrary; rather, they satisfy a few conditions which are not too difficult to see. For example,

- they must be periodic, and have an even period, since any time a permutation occurs in the sequence, its reverse must also occur,
- any two elements of the permutation change their relative position precisely twice per period, exactly half of a period apart, and
- each permutation can be obtained from the previous one by reversing one or more non-overlapping substrings of the permutation.

The last of these is perhaps the most difficult to see, but the reason for it is that the transition from one permutation to the next occurs when the direction which orders the points rotates through a slope which is perpendicular to one of the slopes determined by the point configuration.

[Picture]

Using only these properties, Ungar was able to find the minimal length of such a sequence of permutations, and consequently, the minimal number of slopes that a point configuration can determine (since each slope corresponds to the move from one permutation to the next). As it turned out, this lower bound agreed exactly with the previously known upper bounds ($2n$ slopes for $2n$ or $2n+1$ points), and the matter was brought to a close.

1.3 Attempts to Generalize Ungar's Proof

The slope problem deals with configurations of points in the plane. If we consider the point configuration along with all of the line segments joining different points, we essentially have a graph (i.e. a collection of vertices, along with edges which connect vertices in pairs). Depending on how we have place the points, some edges may overlap each other, but since we are considering *every* slope which the points determine, the graph in question is the so-called "complete graph" – a graph in which every pair of edges is connected by an edge.

Now we note that the direction which ordered the points provides us with a method of directing each edge of the graph under consideration. Furthermore, we couldn't possibly have any directed cycles arise in this manner, since the points are ordered by a straight line. In other words, we have an *acyclic orientation* of our graph. In fact, we have a sequence of acyclic orientations which obey properties very similar to the ones which we discussed for the allowable sequences of permutations.

The generalization that this seemed to suggest was to ask "What is the minimal number of slopes determined by n points in the plane if we *ignore* a certain predetermined set of slopes?", or equivalently "What is the least number of slopes with which a fixed graph G can be embedded in the plane?" Ideally, we would like to be able to determine a nice way of answering this question for any graph, but that seems to be a rather difficult question. However, there are certain classes of graphs for which we will be able to make some headway.

Additionally, the idea of acyclic orientations of graphs leads us (in a way we will describe later) to consider a similar problem dealing with *zonotopes*. A zonotope is a polytope which can be generated as the *Minkowski sum* of a (finite) set of line segments (the Minkowski sum of sets $A_1, A_2, \ldots A_n$ is just the set of all sums of the form $a_1 + a_2 + \ldots + a_n$, where a_i can be any element of A_i).

[Picture]

The corresponding problem for zonotopes deals with what are called *cellular strings*. For a zonotope, we can think of a cellular as a sequence of faces connecting two diametrically opposed vertices (it turns out that zonotopes always have central symmetry) so that the first face starts at the initial vertex, two adjacent faces share a vertex in common, and the last face ends at the final vertex. Furthermore, the faces are chosen in such a way that they continually move farther away from the start and closer to the finish, and the intermediate vertices (which link the faces together) always occur on opposite sides of the faces. While we will nail down the precise definition at a later point, at this time we let a few pictures convey the idea:

[Picture(s) – examples and counterexamples of cellular strings]

The question we want to consider here is "Given a particular zonotope, what is the length of its shortest possible (non-trivial) cellular string?" This turns out to be a more general question than our previous one about embedding graphs in the plane – for any graph, there is a corresponding zonotope, but not vice-versa. However, in the cases where there is a graph corresponding to a zonotope, the two questions are equivalent to each other, as we will later see.

Consideration of zonotopes leads in turn to one further level of generalization, the idea of *matroids* and *oriented matroids*. Matroids provide us with yet another level of abstraction beyond zonotopes (again, every zonotope has a corresponding matroid, but not vice-versa). They were originally studied as an abstraction of the properties of linear dependence in vector space, so much of the terminology and intuition about matroids have strong analogies to linear algebra. The matroid version of the problem we are considering deals with *strong maps* (or more specifically, non-annihilating strong maps), which are something of a matroid analog of linear maps. In order to deal with matroids at anything more than a superficial level, though, we need to introduce some specifics about their structure.

1.4 Outline of the Paper

Having introduced the problem on an informal level, the rest of this paper is dedicated to making the notions we have considered more precise, which will in turn enable us to prove some results (some of which have already been hinted at). In order to get a better overview of where this paper is headed, we now survey the material which we will cover.

The following chapter is dedicated to defining all of the objects which we will take under consideration — matroids and oriented matroids, hyperplane and pseudosphere arrangements, zonotopes, and graphic matroids. We will also define most of our notations here. The reader who is familiar with these subjects already can safely skim through these sections without missing anything new.

In Chapter 3, we introduce strong maps of matroids, cellular strings, and a notion of allowable sequences which has been generalized to acyclic orientations of a graph G, as opposed to only permutations. Furthermore, we explain why these three objects are equivalent, which will enable us to find generalizations to Ungar's theorem in a large variety of situations. Additionally, we prove the equivalence of the slope problem for graphs and the problem of shortest *coherent* cellular strings for the associated graphic zonotope.

After this, we present the proof of Ungar's theorem in Chapter 4. We discuss what this theorem says about matroids, and explain why it uses the oriented structure of a matroid in an essential way. We then go on to formulate and prove an analogous result for the underlying matroid, where we ignore the oriented structure.

Chapter 5 presents perhaps the most significant results in this paper. Here we consider the case of threshold graphs, and prove that the slope problem has a very nice solution in this case. Also, chordal graphs are considered, as they can be thought of as a generalization of threshold graphs.

Having explored the graphic case, Chapter 6 moves on to consider the corresponding type B_n analog, which involves signed graphs. Here we explain how the previously attained results can be extended to type B_n by using central symmetry.

The last chapter to establish new results is Chapter 7, where show that the chromatic number of a graph is always an upper bound for the length of its shortest allowable sequence of acyclic orientations. We also show, using a result of Oxley, that this in turn establishes that $\beta + 2$ is also an upper bound when β is small (here β is Crapo's β-invariant for the oriented matroid corresponding to the graph).

Finally, in Chapter 8, we summarize the results we have obtained throughout the paper, as well as provide several directions for further work.

Chapter 2

Preliminaries

2.1 Matroids

The concept of a *matroid* was introduced by Whitney in 1935 [24], motivated by an attempt to find axioms which described the combinatorial behavior of linear dependence among vectors in a vector spaces. As we have already mentioned, a matroid can be thought of as another level of abstraction after a vector space. More specifically, a matroid consists of a *ground set*, usually denoted by E, which is analogous to a set of vectors. Additionally, one specifies a collection of subsets of E which satisfy certain properties, modeling the structure of linear dependence. We might refer to the *independent sets* of a matroid, the *bases*, the *circuits* (i.e. minimal dependent sets), the *flats* (i.e. linearly closed sets) – for virtually any property that a subset of a vector space can have, there is a corresponding matroid property. This leads to several equivalent formulations, and we will use the following:

Definition 2.1.1. A collection $\mathcal{C}$ of subsets of a set E are the *circuits* of a matroid if they satisfy the following axioms:

(C1) $\emptyset \notin \mathcal{C}$

(C2) (incomparability) if X and Y are in $\mathcal{C}$ and $X \subseteq Y$, then $X = Y$

(C3) (weak elimination) if $X \neq Y$ are both in $\mathcal{C}$ and $e \in X \cap Y$, then there is some Z in $\mathcal{C}$ with $Z \subseteq (X \cup Y) \setminus \{e\}$.

We should remark that this is slightly different from (but nonetheless equivalent to) Whitney's characterization. We use this definition largely for the purpose of comparison with *oriented* matroids, which is to follow in Section 2.2.

Since the theory of matroids is intended to model the properties of vectors, it should not come as a surprise that many matroids can be represented by a collection of vectors E in some vector space, with the notion of independence (or bases, or circuits, etc.) being the usual one from linear algebra. What may be surprising is that there are some perfectly good matroids which cannot be represented by vectors in *any* vector space. This question of realizability is an important one, and we will return to it in Section 2.4.

For a more complete treatment of the subject of matroids, see [19].

2.2 Oriented Matroids

The main difference between an oriented matroid and the "non-oriented" variety that we have discussed already is that an attempt is made to keep track of direction, while still maintaining a level of abstraction over vector spaces. For example, knowing that a set of vectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ are linearly dependent is all we need to know as far as a matroid is concerned. For an oriented matroid, however, we want to know a little bit more. Specifically, there must be some linear dependence $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4} = 0$, and we might not be interested in *exactly* what the values of c_1, c_2, c_3 and c_4 are, but we would like to know whether or not they are positive or negative. This is the type of extra information that is carried by an oriented matroid.

This additional information is often expressed in the form of *sign vectors*, and so before proceeding directly to the definition of an oriented matroid, a few words are in order.

Definition 2.2.1. A *sign vector* X on a set E is an element of the set $\{+, 0, -\}^E$, i.e. a vector with $|E|$ coordinates, whose entries are either $+$, $-$, or 0.

When it comes to writing down sign vectors, we typically think of the set E as having some order associated to it, with the coordinates listed in the same order, much as one usually does with vectors. Consistent with this idea, if $e \in E$, we use X_e to denote the coordinate of X corresponding to e.

A few other comments about the standard notation are in order here. If X is a sign vector, we take $-X$ to have the expected meaning of the sign vector which has $-$'s where X has $+$'s and $+$'s where X has $-$'s, and zeros in exactly the same place. Also, we will often want to refer to the coordinates of X which have a particular value, so we let X^+ denote the coordinates which are all positive, and similarly for X^- and X^0 (note that these are *subsets* of E). The *support* of a sign vector, denoted $\underline{X}$, is the set of its nonzero coordinates, i.e. $\underline{X} = X^+ \cup X^-$.

There is a partial order associated with sign vectors, given by the component-wise extension of the partial order whose relations are $0 < +$ and $0 < -$. In other words, $X < Y$ if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$.

Finally, we define the *composition* of two sign vectors, denoted $X \circ Y$, as follows:

$$(X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0, \text{ and} \\ Y_e & \text{if } X_e = 0. \end{cases}$$

Intuitively, we think of this as "pulling X in the direction of Y", as $X \circ Y$ is obtained from X by filling in X's zero coordinates with the corresponding coordinate from Y. Hence $X \leq X \circ Y$. We should also remark that this operation is not symmetric in general.

We are now ready to define the circuit axioms of an oriented matroid. Note the similarities (and differences) with the axioms in definition 2.1.1.

Definition 2.2.2. A set of sign vectors $\mathcal{C}$ on a set E are the *signed circuits* of an oriented matroid if the following properties are satisfied:

(C0) (symmetry) $\mathcal{C} = -\mathcal{C}$

(C1) $\emptyset \notin \mathcal{C}$

(C2) (incomparability) if X and Y are in $\mathcal{C}$ and $\underline{X} \subseteq \underline{Y}$, then either $X = Y$ or $X = -Y$

(C3) (weak elimination) if X and Y are in $\mathcal{C}$ with $X \neq -Y$ and $e \in X^+ \cap Y^-$, then there is a Z in $\mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) \setminus e$ and $Z^- \subseteq (X^- \cup Y^-) \setminus e$.

For an extensive reference concerning oriented matroids, see [4]

2.3 Arrangements and Covectors

Hyperplane arrangements

One source of oriented matroids which will be of particular interest to us come from the theory of hyperplane arrangements, which brings us to our next definition.

Definition 2.3.1. A *hyperplane arrangement* $\mathcal{A}$ is a collection of hyperplanes (i.e. linear subspaces of codimension 1) in some vector space. If the underlying field of the vector space is $\mathbb{R}^n$ or $\mathbb{C}^n$, they are often referred to as *real* or *complex* hyperplane arrangements, respectively. A hyperplane arrangement is called *central* if all of the hyperplanes contain the origin, and *essential* if the intersection of all of the hyperplanes is precisely the origin.

By a *signed hyperplane arrangement*, we mean a hyperplane arrangement along with an assignment for each hyperplane of a positive and negative side.

Throughout the rest of this paper, all hyperplane arrangements are assumed to be real and central, unless noted otherwise.

One of the most straightforward types of oriented matroid information to extract from a hyperplane arrangement turns out not to be the circuits which we are already familiar with, but rather what are called *covectors* . Covectors are sign vectors which record which side of a hyperplane any given point is in. More precisely,

Definition 2.3.2. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be a signed hyperplane arrangement, with each hyperplane given by a linear form, i.e. $H_i = \{\vec{x} \mid \vec{a_i} \cdot \vec{x} = 0\}$, and the positive and negative sides of each hyperplane determined by the sign of $\vec{a_i} \cdot \vec{x}$. The set of *covectors* of this arrangement is the collection of sign vectors $\{V \in \{+, 0, -\}^n \mid V_i = \text{ sign}(\vec{a_i} \cdot \vec{x})\}$, where $\vec{x}$ ranges over the entire vector space.

[NOTE : good place for a picture (in the plane), with possibly a few words about why this doesn't just give you all 3^n possible sign vectors]

The covectors which arise from a hyperplane arrangement always satisfy the covector axioms for an oriented matroid. We refrain from stating these here, as they have a similar feel to the circuit axioms which we have already seen, and will not be used later on.

We may occasionally refer to the *cocircuits* of an oriented matroid. These are simply the non-zero covectors which are minimal with respect to the usual partial order on sign vectors. It turns out that they satisfy the same axioms for an oriented matroid as the circuits do.

Pseudosphere arrangements

In anticipation of the next section, we should remark that not every oriented matroid can be represented by a signed arrangement of hyperplanes (again, this is an issue of *realizability*, which will be dealt with soon). There is a slightly more general notion, however, which provides us with a method of geometrically representing *all* oriented matroids.

For a central hyperplane arrangement in $\mathbb{R}^d$, one way to simplify the geometric picture is to intersect the arrangement with S^{d-1}, the unit sphere centered at the origin. This gives us what we will call an *arrangement of spheres* – the hyperplanes "cut out" $(d-2)$-spheres on S^{d-1}. A *signed* arrangement of spheres is constructed in exactly the same way, since the positive side of each hyperplane will determine the positive side of the respective $(d-2)$-sphere.

[NOTE : Picture!]

With there being an obvious equivalence between arrangements of hyperplanes and arrangements of spheres, we should not expect anything miraculous to happen just yet. But if we allow these $(d-2)$-spheres to be slightly deformed, it turns out that we now have a means of representing *any* oriented matroid. This leads us to the notion of an arrangement of pseudospheres. By *pseudosphere*, we mean an object which is topologically equivalent to a sphere. An *arrangement of pseudospheres* is an embedding of (usually several) $(d-1)$-pseudospheres into a d-sphere, subject to certain restrictions which in essence require that they have the same types of intersections as an arrangement of (normal) $(d-1)$-spheres could have. In particular, any nonempty intersection should be topologically equivalent to a sphere of some dimension, and the arrangement should be reasonable enough that we can always tell on which side of any particular pseudosphere any given point happens to be (which should strongly suggest what a *signed* arrangement of pseudospheres should be.

[NOTE : Picture!]

The importance of pseudosphere arrangements is well attested to by the Topological Representation Theorem of Folkman and Lawrence [10]. One of the consequences of this theorem is the following fact:

Fact 2.3.3. There is a bijective correspondence between simple oriented matroids (up to reorientation) and arrangements of pseudospheres (up to topological equivalence).

By *simple* matroid, we mean one which has no loops or parallel elements. An element e of the ground set is called a *loop* if there is a covector which is nonzero at e and zero in every other coordinate. Two elements e and f of the ground set are *parallel* if neither are loops, but there is a covector which is nonzero at both e and f, and zero everywhere else. Often the term parallel is reserved for the case when the signs in this covector are the same, with *antiparallel* referring to the case when the signs are opposite. The addition of loops or parallel edges only adds a degree of redundancy to the original matroid, and does not change its structure in a fundamental way, so the above correspondence really is complete.

2.4 Zonotopes and Realizability

A oriented matroid which can be represented by honest spheres (and not topologically deformed ones) is called *realizable* . In such a situation there is a polytope which also represents the structure of the oriented matroid, called a zonotope.

Definition 2.4.1. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a (finite) set of vectors in $\mathbb{R}^n$. A *zonotope* is a polytope given by the Minkowski sum of the line segments $[-\vec{v}_1, \vec{v}_1], [-\vec{v}_2, \vec{v}_2], \ldots, [-\vec{v}_n, \vec{v}_n]$. Equivalently, it is the set of all points $\{\sum_{i=1}^{n} \lambda_i \vec{v}_i \mid -1 \leq \lambda_i \leq 1 \ \forall i\}$.

The vectors which generate a zonotope corresponding to a hyperplane arrangement are precisely the normal vectors of the hyperplanes. Just as the cones cut out by the hyperplane arrangement correspond to the covectors of the matroid, the faces of the zonotope are also in bijection with the covectors. The vertices of the zonotope correspond to using only $\lambda_i = \pm 1$ in the above sum, and the sign vector that each choice yields is the corresponding covector for the oriented matroid. Furthermore, faces of any dimension are given by choosing fixing certain coordinates at either $+1$ or -1, and allowing the rest to take on the full range of values. The corresponding covector has zeros for the coordinates which are allowed to vary, and the appropriate sign for the fixed values.

[NOTE : Picture(s)!]

Zonotopes can be equivalently defined as the image of an n-cube under an affine projection. Indeed, this is the definition given in many texts on the subject, for example [4] and [29]. For our purposes, however, the given definition will be more useful.

Note that from definition, we can see that zonotopes must always be symmetric with respect to the origin. In fact, each face of a zonotope must be centrally symmetric. As it turns out, the converse of this is true in a very strong sense – McMullen [18] was able to prove that if all of the j-faces of a d-polytope are centrally symmetric, where $2 \leq j \leq d - 2$, then the polytope must be a zonotope. While the same is true for the 2-faces of a 3-polytope, McMullen also provides counterexamples which show that when $d \geq 4$, all of the $(d-1)$-faces of a d-polytope can be centrally symmetric without the polytope begin a zonotope.

2.5 Graphic Matroids

One special class of realizable oriented matroids is the class of graphic matroids. We begin our discussion with the following definition.

Definition 2.5.1. The *braid arrangement* is the arrangement of hyperplanes H_{ij} in $\mathbb{R}^n$, $1 \leq i < j \leq n$, where each hyperplane is given by $H_{ij} = \{\vec{x} \in \mathbb{R}^n | x_i - x_j = 0\}$. If we wish to have an orientation for this arrangement, we typically choose the positive side of H_{ij} to be the one for which $x_i - x_j$ is positive.

The braid arrangement, as well as its corresponding matroid, are often referred to as "type A_{n-1}", which comes from the theory of Coxeter groups (a good treatment of the subject can be found in [14]. The arrangement is hence often denoted $\mathcal{A}(A_{n-1})$ and the corresponding matroid by $\mathcal{M}(A_{n-1})$. The zonotope which corresponds to this arrangement is called the *permutahedron*, and is often denoted $\mathcal{Z}(A_{n-1})$.

We are now ready for the following definition.

Definition 2.5.2. A *graphic arrangement* is one which is a subarrangement of the braid arrangement. A *graphic matroid* (or *graphic zonotope*, etc.) is a matroid (zonotope, etc.) which corresponds to a graphic arrangement.

The motivation for this terminology comes from identifying an edge $\{i, j\}$ of a graph on $\{1, 2, \ldots, n\}$ with the hyperplane H_{ij}. Hence the braid arrangement $\mathcal{A}(A_{n-1})$ corresponds to the complete graph K_n — a graph having n vertices, with an edge between every pair — sometimes called a *clique*. Note the "off by one" phenomenon that occurs in the subscripts, which may seem unfortunate but is well established, as it makes for more uniformity throughout the theory of Coxeter groups.

2.6 A Remark About Notation

We have already introduced the notations $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{Z}$ for the braid arrangement, and their general meaning can be safely deduced from these usages. However, they will occasionally be used in slightly different ways, so a few remarks are in order to clear up any possible confusion.

In the following discussion, A, G and Z stand for a signed hyperplane arrangement, a graph, and a zonotope, respectively.

Generally speaking, $\mathcal{A}(\cdot)$ will always refer to the *signed* hyperplane arrangement associated to the given structure. Hence $\mathcal{A}(G)$ is the hyperplane arrangement corresponding to the graph G, while $\mathcal{A}(Z)$ denotes the hyperplane arrangement corresponding to the zonotope Z. We do not expect any confusion from this slight abuse of notation.

Continuing by analogy, $\mathcal{M}(\cdot)$ always denotes the *oriented* matroid corresponding to its argument. So $\mathcal{M}(A)$, $\mathcal{M}(G)$, and $\mathcal{M}(Z)$ all make sense. We could insist on usages such as $\mathcal{M}(\mathcal{A}(G))$ instead of $\mathcal{M}(G)$, but this only seems to complicate matters.

As we might expect, $\mathcal{Z}(A)$ denotes the zonotope corresponding to the arrangement A and $\mathcal{Z}(G)$ denotes the zonotope corresponding to the graph G. Similarly for anything else for which we might like to find the corresponding zonotope.

One final note, which is fairly important, is the distinction between oriented and non-oriented structures. This will mainly arise with matroids (where the distinction is the most important), and the lack of an orientation will always be denoted with an underscore. Thus $\mathcal{M}(G)$ is the *oriented* matroid corresponding to a graph G, while $\underline{\mathcal{M}}(G)$ is the matroid *without orientation* associated to the graph G.

Chapter 3

Equivalence of Basic Structures

The remainder of this paper will concentrate on finding minimal structures on matroids, zonotopes or graphs which generalize the "least number of slopes" sought after in the slope problem. In order to do this, we now introduce the three main objects we will be dealing with, and explain how they are essentially equivalent.

3.1 Strong Maps

The first notion we would like to introduce is that of a strong map of matroids. Strong maps are meant to be the matroid analog of the vector space notion of a linear map. Intuitively, the idea is that they should preserve linear dependence, and they are defined as follows.

Definition 3.1.1.

(i) Let M_1 and M_2 be matroids (without orientation) on the same ground set E. We say that the identity map on E induces a *strong map* $M_1 \twoheadrightarrow M_2$ if all of the flats (i.e. linearly closed sets) of M_2 are flats of M_1.

(ii) Let M_1 and M_2 be oriented matroids on the same ground set E. We say that the identity map on E induces a *strong map* $M_1 \twoheadrightarrow M_2$ if every covector of M_2 is a covector of M_1.

Note that while these definitions may appear to be different depending on whether or not the matroid is oriented, the differences are only superficial. With a little thought, we should be able to convince ourselves that the concept is the same for each case – the flats of a non-oriented matroid are precisely the underlying ground sets of the covectors of a corresponding oriented matroid (provide such an orientation exists).

Since every matroid must contain the origin in some fashion (e.g. the all zero vector, the empty set, etc.), a particularly trivial strong map would be the one that takes a matroid to the "zero matroid" which consists only of the origin. Less severe, but still of concern to us, is a strong map which takes some nonzero element to the origin. A *non-annihilating* strong map is one which never does this. In particular, using the terminology of definition 3.1.1, no maximal covector v of M_2 should have a zero entry, unless v was maximal in M_1 as well. For the case of a matroid without orientation, f cannot be a minimal flat of M_2 unless it is also a minimal flat of M_1.

The strong maps we will be most interested in will be those whose image (the M_2 in the definition) has rank two. As a result of the Topological Representation Theorem (Fact 2.3.3), a rank two oriented matroid is *always* realizable, since it corresponds to an arrangement of 0-spheres on a 1-sphere, and there is no real difference between a normal 0-sphere (which is just two points) and a topologically deformed one.

[Picture]

Furthermore, note that there is always a natural cyclic order on the nonzero covectors of a rank two image of a non-annihilating strong map (actually, two orders, since we can cycle in either direction). Any maximal covector in a rank two image will have precisely two maximal covectors which differ from it in a minimal number of coordinates, and between any two "adjacent" maximal covectors is a unique rank one covector. Note that covectors which are exactly one-half of a cycle away from each other will be opposite in sign. Additionally, we observe that the zero sets of the covectors at rank one partition the ground set, with each block of the partition occurring twice, since covectors always occur in pairs v and $-v$.

[Another picture or two?]

This "cyclic" notion of the nonzero covectors in a rank two image of a non-annihilating strong map gives us perhaps the most obvious connection with the other structures we would like to consider.

3.2 Cellular Strings

The next structure we would like to consider is that of a cellular string. While these can be defined for any polytope, we will be primarily interested in cellular strings on zonotopes.

Definition 3.2.1. Let P be a polytope in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear functional which is generic with respect to the vertices of P (i.e. f assumes a different value on each vertex of P). For any face F of P, let $\min(F)$ and $\max(F)$ denote the vertices of F for which f is minimal or maximal, respectively. A *cellular string* (sometimes called an *f-monotone cellular string*) on P is a sequence of faces $F_1, F_2, \ldots, F_k$ of P having dimension at least one such that $\min(F_1) = \min(P)$, $\max(F_k) = \max(P)$, and for each $i = 1, 2, \ldots, k-1$, $\max(F_i) = \min(F_{i+1})$.

[Picture]

We will occassionally refer to the *length* of a cellular string, and this will always mean the number of faces involved, not the number of vertices (i.e. the value of k in the definition).

Cellular strings were introduced by Billera, Kapranov and Sturmfels [2] in work related to the Baues problem.

One useful fact about cellular strings on zonotopes comes from comparing them to the corresponding hyperplane arrangement. The vertices $\min(F_1), \max(F_1) = \min(F_2), \ldots, \max(F_k)$ correspond to chambers of the hyperplane arrangement. The first and last chambers are directly opposite each other, so any path from one to the other must cross each hyperplane at least once. The path of chambers corresponding to the vertices of the cellular string goes between the two antipodal chambers, never crosses any hyperplane more than once, and moves from chamber to chamber across intersections of hyperplanes.

[Reference for equivalence in above paragraph? Is this even an established fact?]

[Picture?]

Recall that there is a zonotope that corresponds to an oriented matroid if and only if the matroid is realizable. However, in this case, we see that a cellular string is the essentially the same as a rank two image of a non-annihilating strong map.

Proposition 3.2.2. *Let M be a realizable oriented matroid with corresponding zonotope Z. The cellular strings of Z are in bijective correspondence with the rank two images of M under a non-annihilating strong map.*

Proof. Suppose $F_1, F_2, \ldots, F_k$ is a cellular string on Z. Using the central symmetry of Z, there is also a sequence of faces $F'_k, \ldots, F'_2, F'_1$ given by $F'_i = -F_i$ which is a cellular string on Z (if the F_i's form a cellular string which is f-monotone, the F'_i's will form a cellular string which is monotone with respect to $-f$). Let v_i and v'_i be the covectors corresponding to the faces F_i and F'_i, respectively. We claim that the set of covectors $\{v_1, v_2, \ldots, v_k, v'_1, v'_2, \ldots, v'_k\}$ are the rank one covectors of some rank two oriented matroid M_2. Furthermore, the maximal covectors of M_2 are precisely those covectors of M which correspond to the vertices $\min(F)$ and $\max(F)$ for each face F in the cellular string.

The reason these covectors (along with the zero vector) form a rank two oriented matroid is simply that they have the right structure. The covectors have a natural cyclic order arising from the order of the faces in the cellular strings, and the covectors exactly half of the cycle apart are negatives of each other (by central symmetry). This gives us natural way to realize this as a rank two oriented matroid. Finally, all of the covectors associated to the cellular string are covectors of the original matroid, so this is indeed the image of a strong map.

Conversely, suppose we have an oriented matroid M_2 which is a rank two image of M under a non-annihilating strong map. Let $v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_{2k}$ be the maximal covectors of M_2, taken in a natural cyclic order. Suppose that $P_1, P_2, \ldots, P_{2k}$ are the corresponding vertices of Z (these covectors must correspond to vertices since they are maximal, and the map is non-annihilating). Let $F_1, F_2, \ldots, F_{2k}$ be the faces of Z such that P_i and P_{i+1} are opposite vertices of F_i, for $1 \leq i \leq 2k-1$, and P_{2k} and P_1 are opposite vertices of F_{2k}. We claim that $F_1, F_2, \ldots, F_k$ is a cellular string on Z.

Again, the reason is simply the observation that it has the right structure. The vertices $P_1, P_2, \ldots, P_{k+1}$ of this cellular string determine a path between opposite chambers of $\mathcal{A}(Z)$ which crosses each hyperplane exactly once, and faces $F_1, F_2, \ldots, F_k$ correspond to the intersection of hyperplanes which is "moved across" at each step. This is precisely the structure of a cellular string on a zonotope.

[This proof could really use a good picture]

$\square$

Of particular interest to us will be a special type of cellular string, called a coherent cellular string . These are particularly well-behaved with respect to the geometry of $\mathbb{R}^n$.

Definition 3.2.3. Let P be a polytope in $\mathbb{R}^n$, and let f be a linear functional on $\mathbb{R}^n$ which is generic with respect to the vertices of P. Furthermore, suppose g is a different linear functional on $\mathbb{R}^n$. The *g-coherent cellular string* on P is the union

$$\cup_{x\in\mathbb{R}}\{y \in f^{-1}(x)|g(y) = \max g(f^{-1}(x))\},$$

i.e. the union of the portions of each fiber $f^{-1}(x)$ for on which g is maximized.

While it may not be immediately obvious, this definition does produce an f-monotone cellular string according to our original definition. For more details of this construction, see [3].

An equivalent way to think about coherent cellular strings (which can be read about in Chapter 9 of [29]) is that they are the inverse image of the upper hull of the projection $P \to \mathbb{R}^2$ given by $x \mapsto (f(x), g(x))$. This approach is perhaps conceptually more useful for us, and will be utilized in the next section.

[Picture]

3.3 Allowable Sequences

We now move on to the notion of an allowable sequence. These were introduced by Goodman and Pollack [13] as a means to study the combinatorial properties of planar point configurations.

Definition 3.3.1. An *allowable sequence* is a sequence of permutations $\pi_1, \pi_2, \ldots \in S_n$ which is periodic with period p and satisfies the following two properties:

(i) We can obtain π_{i+1} by reversing one or more non-overlapping substrings of π_i.

(ii) For any pair of elements $a, b \in [n]$, the moves $\pi_i \to \pi_{i+1}$ and $\pi_j \to \pi_{j+1}$ which reverse a and b are exactly $\frac{1}{2}p$ apart, i.e. $j = i + \frac{p}{2} \pmod p$.

Any configuration of distinct points $p_1, p_2, \ldots, p_n$ in $\mathbb{R}^2$ gives rise to an allowable sequence. To do this, suppose we pick a direction (i.e. linear functional) in the plane which is not parallel to any of the line segments $[v_i, v_j]$. This determines an ordering of the points (orthogonally project each point to the line in our chosen direction), and as this line is rotated, the order will change. The sequence of permutations obtained by the subscripts will then be an allowable sequence.

[Picture!]

We should note that not every allowable sequence can be obtained from a point configuration in this manner. Again, this is a realizabilty issue.

We would like to relate the concept of an allowable sequence to graphic matroids. To do this, we note that a permutation on $[n]$ can be thought of as an acyclic orientation of the complete graph, an hence an allowable sequence can be thought of as a sequence of acyclic orientations of K_n which obey certain rules. This motivates the following definition.

Definition 3.3.2. Let G be a graph. A *G-allowable sequence* is a sequence $\omega_1, \omega_2, \omega_3, \ldots$ of acyclic orientations of G which is periodic with period p and satisfies the following properties:

(i) We can obtain ω_{i+1} from ω_i as follows:

- Partition the vertices of G into sets $V_1, V_2, \ldots, V_k$ such that all edges from a vertex of V_i to a vertex of V_j are oriented in the same direction (i.e. either all edges go from V_i to V_j, or all edges go from V_j to V_i), for any $1 \leq i < j \leq k$.
- Reverse all edges whose endpoints are contained in the same block V_i, for each i.

(ii) For any edge of G, the moves $\omega_i \to \omega_{i+1}$ and $\omega_j \to \omega_{j+1}$ which reverse the orientation of this edge are exactly $\frac{1}{2}p$ apart, i.e. $j = i + \frac{p}{2} \pmod{p}$.

Note that we can obtain a G-allowable sequence from a point configuration in very much that same way that a regular allowable sequence is obtained. The only difference is that rather than having the points totally ordered by some linear functional, we ignore comparisons which correspond to "missing" edges of G.

[Picture?]

The acyclic orientations of a graph G correspond in a very natural way to the vertices of the corresponding zonotope $\mathcal{Z}(G)$, an observation of Greene and Zaslavsky [11][Picture?]. This correspondence motivates our next result, which, in light of the chapter thus far, may not come as much of a surprise.

Proposition 3.3.3. *Let G be a graph, with corresponding zonotope Z. The cellular strings of Z are in bijective correspondence with the G-allowable sequences of G.*

Proof. Suppose $\omega_1, \omega_2, \ldots, \omega_{k+1}$ is part of a G-allowable sequence, where $k = \frac{p}{2}$ is half of the period. Let $p_1, p_2, \ldots, p_{k+1}$ be the corresponding vertices of Z. Note that condition (ii) of definition 3.3.2 implies that ω_1 and ω_{k+1} are the reverse of each other (i.e. the orientation of ω_{k+1} is obtained from ω_1 by reversing the direction of every edge), so p_1 and p_{k+1} are opposite vertices of Z. Additionally, this condition guarantees us that the sequence of vertices $p_1, p_2, \ldots, p_{k+1}$ corresponds to a sequence of chambers of $\mathcal{A}(G)$ which crosses each hyperplane exactly once (and ending at the chamber which is opposite the initial chamber). Furthermore, condition (i) in the definition assures us that adjacent chambers in this path intersect at precisely some intersection of hyperplanes in $\mathcal{A}(G)$ — corresponding to partitions of the vertices of G. Again, this is precisely the structure of a cellular string on a zonotope.

Conversely, suppose $F_1, F_2, \ldots, F_k$ is a cellular string on Z. Let $p_1 = \min(F_1)$, $p_{k+1} = \max(F_k)$ and $p_i = \min(F_i) = \max(F_{i-1})$ for $2 \leq i \leq k$. If $\omega_1, \omega_2, \ldots, \omega_{k+1}$ are the corresponding acyclic orientations of G, we claim that these orientations determine a G-allowable sequence if we define ω_{i+k} to be the reverse of ω_i.

Once again, the verification is simply a matter of translating the properties of one structure to those of the other. Our requirement that ω_{i+k} is the reverse of ω_i, along with the facts that no hyperplane of $\mathcal{A}(G)$ is crossed twice in the path of chambers corresponding to $p_1, p_2, \ldots, p_{k+1}$ and that p_1 and p_{k+1} are antipodal points of Z translate to property (ii) in the definition of a G-allowable sequence. On the other hand, property (i) is simply the statement about graphs that the move from one chamber to the next crosses a lower-dimension intersection of the hyperplane arrangement. $\square$

Note that in light of Proposition 3.2.2, we see that in the case of a graphic matroid $\mathcal{M}(G)$, G-allowable sequences, cellular strings on $\mathcal{Z}(G)$ and strong maps $\mathcal{M}(G) \twoheadrightarrow M$ (where M is a rank two oriented matroid) are all equivalent to each other. For this reason, when refering to the *length* of a G-allowable sequence, we mean one half of the period (so that it coincides with the notion of length for a cellular string).

Perhaps a legitimate question to ask at this point is what anything in this chapter thus far has to do with the slope problem. To answer this, we introduce one last preliminary result, which was observed by Billera and Babson [1] in the case of the complete graph.

Proposition 3.3.4. *Let G be a graph, and label each vertex v of $\mathcal{Z}(G)$ with the acyclic orientation determined by $i \to j$ if $v_i < v_j$ [mention $v_i = v_j \Rightarrow \{i,j\} \notin E(G)$?].*

Let $f(\vec{x}) = \Sigma\alpha_i x_i$ and $g(\vec{x}) = \Sigma\beta_i x_i$ be linear functionals on $\mathbb{R}^n$, with f generic with respect to $\mathcal{Z}(G)$ (i.e. $\alpha_i \neq \alpha_j$ whenever $\{i,j\}$ is and edge of G).

Plot the points (α_i, β_i) in $\mathbb{R}^2$, and label each point by i. Then the G-allowable sequence for this point configuration is the same as the G-allowable sequence corresponding to the f-monotone cellular string on the $\mathcal{Z}(G)$ which is coherent with respect to g.

Proof. We begin with the simple observation that for a zonotope Z which is generated by the vectors $v_1, v_2, \ldots, v_k$, the directions of the edges (i.e. one-dimensional faces) are precisely the same as the generating vectors.

Next, consider the projection $\pi : Z \to \mathbb{R}^2$ by the mapping $\vec{x} \mapsto (f(\vec{x}), g(\vec{x}))$. The image of Z under this map is also a zonotope (since a projection of a zonotope is also a zonotope), and as we mentioned at the end of the previous section, the inverse image of the upper hull of this projection is an f-monotone, g-coherent celluar string. Furthermore, the edges in the upper hull of the projection occur in an order which agrees with their slope (beginning with a vertical slope and rotating clockwise).

[Picture]

According to our initial observaion, the slopes that occur on the upper hull are the projections of the generating vectors. In the case of a graphic matroid, these are vectors $e_i - e_j$, where $\{i,j\}$ is an edge of G. But $\pi(e_i - e_j) = (\alpha_i, \beta_i) - (\alpha_j, \beta_j)$, and these are the slope vectors for the point configuration under consideration. $\square$

The significance of this result is that for all graphic matroids, the problem of coherent cellular strings is now seen to be equivalent to the slope problem in $\mathbb{R}^2$. This fact, along with the equivalences that have already been discussed in this chapter, provide us with several approaches for generalizing the slope problem.

3.4 A Little More Notation

The rest of this paper will be concerned with finding the minimal value of structures corresponding to the slopes in the slope problem, such as

- The smallest number of maximal covectors in a rank two image of a non-annihilating strong map $M_1 \twoheadrightarrow M_2$
- The length of a shortest cellular string on the zonotope Z
- The smallest possible period of a G-allowable sequence.

To denote these, we will use the notation $\mathrm{Ungar}(\cdot)$, in honor of Ungar. Thus $\mathrm{Ungar}(M_1)$, $\mathrm{Ungar}(Z)$ and $\mathrm{Ungar}(G)$ denote the numbers described above.

Furthermore, it will turn out that in some cases, the minimal value that we are looking for will depend on the starting point – either the initial vertex T for the cellular string of a zonotope, or the initial acyclic orientation ω of a G-allowable sequence. If we want to specify a particular starting point, we will use the notation $\mathrm{Ungar}(Z, T)$ or $\mathrm{Ungar}(G, \omega)$.

We will occasionally want to single out cellular strings (or corresponding G-allowable sequences) which are *coherent*. When this is the case, we will use $\mathrm{Ungar}_{\mathrm{coh}}(\cdot)$ (also allowing either one or two arguments here).

Finally, as the next chapter will show, the presence or absence of an orientation can profoundly affect the size of the minimal value that we seek. Without an orientation, we can only use the language of matroids, and the meaning of $\mathrm{Ungar}(M)$ could be discerned simply from knowing whether or not M was oriented or not. To avoid any possible confusion, however, we will use $\underline{\mathrm{U}}\mathrm{ngar}(M)$ when we are talking about a matroid without an orientation (at the cost of the occasionally redundant expression of the form $\underline{\mathrm{U}}\mathrm{ngar}(\underline{M})$).

[NOTE: We will most likely change the $\mathrm{Ungar}(\cdot)$ notation to the Fraktur $\mathfrak{U}(\cdot)$.]

Chapter 4

The complete graph K_n

4.1 Ungar's Theorem

We are now ready to address the main question of this paper in its most basic form, namely, "Given n points in the plane, not all of which lie on a line, how many slopes must these points determine?" As we have seen from the previous chapter, there are several other ways we can look at this problem, such as

- What is the smallest number of maximal covectors that a non-annihilating strong map from $\mathcal{M}(A_{n-1})$ to a rank two oriented matroid can have?
- What is the length of the shortest cellular string on the n-permutahedron (other than the trivial one)?
- What is the length of the shortest allowable sequence taking the permutation $1\ 2\ \ldots\ n$ to its reverse (provided we don't reverse the entire permutation in a single step)?

The answer to this question was ultimately resolved by considering allowable sequences. The following theorem of Ungar [21] established a lower bound for the number of slopes, and this lower bound was already known to be attainable using examples which will soon follow.

Theorem 4.1.1. *(Ungar) Let $n = 2m$ be even. Consider an allowable sequence starting with the permutation $12\ldots n$ in which the moves from one permutation to the next involve only reversing increasing substrings during the first half period. If we do not reverse the entire permutation at one step, the number of steps required to get from $12\ldots n$ to $n(n-1)\ldots 1$ is at least n.*

Proof. We visualize a barrier in the middle of the permutation, between positions m and $m+1$:

$$1\ 2\ \ldots m \,\Big|\, m+1 \ldots n.$$

This barrier must be crossed at least n times (once by each letter) in order to reverse the permutation. Consider the moves in two separate cases – *crossing* moves, where some letters are taken across this center barrier, and *non-crossing* moves, where this does not occur.

Suppose there are t crossing moves, and that at least d_i letters are moved across the barrier from either side at the i^{th} crossing move. Then we have $2d_1 + \ldots + 2d_t \geq n$.

For the non-crossing moves, we note that before the i^{th} crossing move, we have an increasing string extending at least d_i positions on either side of the barrier, and after the crossing move is made, there is a decreasing string extending at least d_i positions on either side of the barrier. We can only take one letter off each end of the decreasing string in the center per move (since only increasing strings can be inverted), and similarly, we can only add one letter to each end of the increasing string in the center for each move. Hence in order to get from crossing move i to crossing move $i+1$, we need at least $d_i + d_{i+1} - 1$ moves.

So far, this requires t crossing moves and $(d_1 + d_2 - 1) + \ldots + (d_{t-1} + d_t - 1)$ non-crossing moves, for a total of $d_1 + 2d_2 + \ldots + 2d_{t-1} + d_t + 1$. The remaining moves come from setting up the first crossing move and finishing off after the last one. The number of moves required is $d_1 + d_t - 1$, which is due to the cyclic nature of the permutations – there is essentially no difference between going from crossing move 1 to 2 and going from crossing move t to 1. So the total number of moves is $2d_1 + \ldots + 2d_t$, which we have already seen to be at least n. □

As we have said, this theorem establishes the fact that any configuration of $2m$ points in the plane must have *at least* $2m$ slopes, but to finish the job, we need to show that this number really is attainable. Any example will do, but the two standard configurations of points to consider are

- All points on a line, except for one
- The vertices of a regular $2m$-gon

[NOTE: picture!] For the case of an odd number of points, say $2m+1$, we are again guaranteed to have at least $2m$ slopes, since we certainly have $2m$ points. Adding the last point carefully, however, will *not* add any new slopes. For example, if we have the vertices of a regular $2m$-gon, we can add the center of the $2m$-gon and the number of slopes will be unchanged. Or for the other configuration above, we may put all of the points on the same line except for *two*, only now we also require that the configuration have central symmetry.

Many other point configurations are known which achieve the lower bound for the required number of slopes, including two other infinite families and over 100 sporadic examples. For further information, see [16] and [17].

4.2 Transitive Partitions of $\binom{[n]}{2}$

The rest of this paper will deal with different variations and generalizations of the above result. For the first, we note that from the matroid perspective, using allowable sequences necessitates that we are talking about *oriented* matroids – it would not make much sense to talk about a sequence of permutations where we made no distinction between i coming before j and i coming after j. So in order to find a similar result for matroids without orientation, we need to rephrase Ungar's result in different terms.

Perhaps the most natural "different terms" to consider is that of strong maps. Recall that in an oriented matroid, the covectors always occur in pairs v and $-v$. If we call such a pair of covectors a *parallel class*, we can rephrase Ungar's theorem as follows.

Fact 4.2.1 (Theorem 4.1.1, rephrased)**.** Let $\mathcal{M}(A_{n-1}) \twoheadrightarrow M$ be a non-annihilating strong map from the oriented matroid $\mathcal{M}(A_{n-1})$ to a rank two oriented matroid M. The number of parallel classes of maximal covectors of M must be at least n if n is even and $n-1$ if n is odd.

The corresponding question for non-oriented matroids that we would like to consider is "What is the smallest number of maximal covectors that a non-annihilating strong map from $\mathcal{M}(A_{n-1})$ to a rank two matroid (*not* oriented) must have?" Perhaps surprisingly, ignoring the orientation allows us to do much better, as the number grows more like $\sqrt{n}$ than n.

In order to deal with this question, we need to understand how rank two images of $\mathcal{M}(A_{n-1})$ under a non-annihilating strong map behave. This is the motivation for the next definition, as we will see shortly.

Definition 4.2.2. A *transitive partition* of $\binom{[n]}{2}$ is a partition of the pairs $\{i, j\}$ such that whenever any two of the three pairs $\{i, j\}, \{i, k\}$ or $\{j, k\}$ are in the same block, the third pair must also be in that same block.

Note that putting every pair into the same block results in a transitive partition, which we will refer to as the *trivial transitive partition*. Predictably, by *nontrivial transitive partition*, we mean any one having more than one (nonempty) block. Also, since a transitive partition can be thought of as a decomposition of K_n into cliques, we will usually refer to the elements of the index set $[n]$ as *vertices*.

The importance of transitive partitions is seen in the following theorem.

Proposition 4.2.3. *There is a bijective correspondence between non-annihilating strong maps from $\underline{\mathcal{M}}(\underline{A}_{n-1})$ to rank two matroids and nontrivial transitive partitions of $\binom{[n]}{2}$.*

Proof. Suppose M_2 is a rank two image of a non-annihilating strong map $\underline{\mathcal{M}}(\underline{A}_{n-1}) \twoheadrightarrow M_2$. By definition, every flat of M_2 must also be a flat of $\underline{\mathcal{M}}(\underline{A}_{n-1})$. But flats of of a graphic matroid are obtained by partitioning the vertex set and choosing precisely those edges whose endpoints belong to the same block of the partition. This is precisely the type of closure that occurs for transitive partitions.

Conversely, suppose we are given a transitive partition of $\binom{[n]}{2}$. One candidate for a rank two matroid corresponding to this can be constructed as follows.

Rank=0 : The empty set.

Rank=1 : The ground set elements, i.e. pairs $\{i, j\}$.

Rank=2 : Each block of the transitive partition.

A little thought verifies that this is indeed a matroid. Once again, the two types of closure (for transitive partitions and graphic matroids) are the same, so this is indeed the image of $\underline{\mathcal{M}}(\underline{A}_{n-1})$ under a non-annihilating strong map. $\square$

Now our question of rank two images of $\mathcal{M}(A_{n-1})$ under a strong map has become "What is the smallest number of blocks that a nontrivial transitive partition of $\binom{[n]}{2}$ must have?", and it is this question that we are now prepared to answer.

Theorem 4.2.4. *Any nontrivial transitive partition of $\binom{[n]}{2}$ must have at least $\lceil \sqrt{n}\, \rceil + 1$ blocks. Furthermore, whenever $\lceil \sqrt{n}\, \rceil$ is a prime power, this lower bound is sharp.*

Proof. We begin by considering the largest clique which occurs in any block of the transitive partition; suppose such a clique has m vertices. Without loss of generality, we may assume that this is a clique on the vertices $1, 2, \ldots, m$. Since this partition is nontrivial, $m < n$, and we have m pairs $\{1, n\}, \{2, n\}, \ldots, \{m, n\}$ which are not part of this clique (since it was maximal). Additionally, no two of these pairs, say $\{i, n\}$ and $\{j, n\}$, can be in the same block, as transitivity would then require that the pair $\{i, j\}$ also be in that block, and hence they would all belong to the clique. So the existence of a clique on m vertices will require at least $m + 1$ different blocks.

On the other hand, if we consider all of the $n - 1$ pairs of the form $\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}$, we note that if any k of these pairs are in the same block, they form a clique on at least $k + 1$ vertices. So if the largest clique has m vertices, we must have at least $\lceil \frac{n-1}{m-1} \rceil$ blocks.

[NOTE: include pictures for these two arguments]

At this point, we have established that the smallest number of blocks is at least $\max\{m + 1, \lceil \frac{n-1}{m-1} \rceil\}$, where $2 \le m \le n - 1$. To find the smallest value of this maximum, we consider three cases:

Case 1 : $m > \sqrt{n}$

$$\max\{m + 1, \lceil \tfrac{n-1}{m-1} \rceil\} \ge m + 1 > \sqrt{n} + 1 \ge \lceil \sqrt{n}\, \rceil + 1 \text{ (since } m \text{ is an integer)}$$

Case 2 : $m < \sqrt{n}$

$$\max\{m + 1, \lceil \tfrac{n-1}{m-1} \rceil\} \ge \lceil \tfrac{n-1}{m-1} \rceil \ \ge \lceil \tfrac{n-1}{\sqrt{n}-1} \rceil \ = \lceil \sqrt{n} + 1 \rceil = \lceil \sqrt{n}\, \rceil + 1$$

Case 3 : $m = \sqrt{n}$

$$\max\{m + 1, \lceil \tfrac{n-1}{m-1} \rceil\} = \max\{\sqrt{n} + 1, \lceil \sqrt{n} + 1\ \rceil\} = \lceil \sqrt{n}\, \rceil + 1$$

This establishes $\lceil \sqrt{n}\, \rceil + 1$ as a lower bound.

To see that this lower bound is sharp whenever $q = \sqrt{n}$ is a prime power, consider the finite affine plane over $GF(q)$. Any pair of points in this plane must have one of $q + 1$ possible slopes If we label the points of this plane 1 through n, we can partition $\binom{[n]}{2}$ according to slope, and this is in fact a transitive partition.

Finally, if $q = \lceil \sqrt{n}\, \rceil$ is a prime power (even though $\sqrt{n}$ may not be an integer), we can obtain a transitive partition by simply eliminating some of the indices which occurred in the previous partition, since this will not create any new blocks or interfere with transitivity. $\square$

A few remarks are in order. First, the appearance of finite fields is hardly unprecedented, as vector spaces over finite fields provide examples of matroids which are realizable but not orientable. The geometry of finite affine spaces is a perfectly good one, but the lack of a complete order on $GF(q)$ more or less ruins our chances of being able to orient it in some reasonable fashion.

The existence of transitive partitions of $\binom{[m^2]}{2}$ having precisely m blocks can be expressed as a problem in the theory of *block designs*. If V is a set of cardinality v, a $t-(v,k,\lambda)$ design is a collection of subsets (called *blocks* of V, all having k elements. Furthermore, if T is any subset of V having cardinality t, there are precisely λ blocks which contain all of the elements of T. An introduction to the theory can be found in [22], Chapter 19.

Proposition 4.2.5. *Let $\sqrt{n} \in \mathbb{Z}$. The following are equivalent:*

(i) There exists a transitive partition of $\binom{[n]}{2}$ having $\sqrt{n}+1$ blocks.

(ii) There exists an affine plane of order n (i.e. a 2-$(n^2, n, 1)$ design).

(iii) There exists a projective plane of order n (i.e. a 2-$(n^2+n+1, n+1, 1)$ design).

Proof. For the equivalence of (i) and (ii), we first remark that a $2-(n^2,n,1)$ design can always be *resolved*, i.e. the $n(n+1)$ blocks needed for such a design can be grouped into $n+1$ equivalence classes, with each class containing n blocks. Furthermore, each element of V occurs precisely once in each equivalence class. It is straightforward to verify that such a resolution is achieved by the (equivalence) relation $B_1 \sim B_2$ if and only if $B_1 = B_2$ or $B_1 \cap B_2 = \emptyset$. Also, we can obtain a transitive partition directly from this resolution by having each equivalence class correspond to a block of the transitive partition. The set of pairs in a block of the partition is the collection of all pairs within the same block of the corresponding equivalence class in the design. [NOTE : it would be great to avoid using “block” in two contexts]

Conversely, if we have a nontrivial transitive partition on n vertices with precisely $\sqrt{n}+1$ blocks, we claim that the cliques within each block form a $2-(n^2,n,1)$ design. To see this, suppose the largest clique in the partition is on m vertices. As we saw in the proof of theorem 4.2.4, this forces us to have at least $m+1$ blocks in the partition, so we must have $m = \sqrt{n}$ if we are to have any hope of attaining the lower bound. Hence the most pairs we can have in a single block is attained by having $\frac{n}{\sqrt{n}} = \sqrt{n}$ cliques of size $\sqrt{n}$ in each block. So altogether in the $\sqrt{n}+1$ blocks, we can account for $(\sqrt{n}+1)(\sqrt{n})\binom{\sqrt{n}}{2} = \binom{n}{2}$ pairs, i.e. all of them, with no room to spare. We then have a total of $n(n+1)$ cliques of size n, and no pair of vertices are in two different cliques. But this is precisely a $2-(n^2,n,1)$ design.

The equivalence of (ii) and (iii) is a straightforward exercise in the theory of block designs. For more details, see [5], Section 6.4. $\square$

The existence of projective planes of different orders is a long-standing open problem in the theory of block designs. Projective planes of order n have long been known to exist when n is a prime power, but little is known for other values. It has also been shown (by exhaustive computer searching) that there are *no* projective planes of order 6 or 10, but the question of existence for all other orders (except prime powers, of course) is one of the open problems in the theory of block designs. For a survey, see [6].

In some of the cases where $\sqrt{n}$ is not an integer, we can still say a few things about the minimal number of blocks in a transitive partition of $\binom{[n]}{2}$. Since the number of blocks is at least $\lceil \sqrt{n}\ \rceil + 1$, we can obtain a transitive partition with a minimal number of blocks by restricting any minimal transitive partition on $\binom{\lceil \sqrt{n}\ \rceil^2}{2}$ – removing vertices will certainly not add new blocks, nor will it interfere with transitivity. Hence we can achieve the lower bound of $\lceil \sqrt{n}\ \rceil + 1$ whenever $\sqrt{n}$ is a prime power. Also, if $\lceil \sqrt{n}\ \rceil$ is 6 or 10, a transitive partition which attains the $\lceil \sqrt{n}\ \rceil + 1$ lower bound is impossible, so the best we can do is $(\lceil \sqrt{n}\ \rceil + 1) + 1$ blocks, which can be obtained by restricting the partitions corresponding to the primes $\sqrt{n}$=7 or $\sqrt{n} = 11$. There is unfortunately little to say about any other non-prime power.

Asymptotically, we can see that the number of blocks required is significantly less if the orientation is ignored. Recall that in the case of the oriented matroid $\mathcal{M}(A_{n-1})$, Ungar's theorem showed that a rank two image of a nonannihilating strong map must have at least $2\lceil \frac{n}{2} \rceil$ maximal covectors – growing essentially like n. However, even just rounding up to the next power of two (instead of the nearest prime power), we see that the case where orientation is ignored requires at most $2^{\lceil \frac{1}{2}\log_2 n \rceil} + 1$ maximal covectors. The quick estimate

$$\sqrt{n} + 1 \leq \underline{\mathrm{U}}\mathrm{ngar}(\underline{K_n}) \leq 2^{\lceil \frac{1}{2}\log_2 n \rceil} + 1 \leq 2^{(\frac{1}{2}\log_2 n)+1} + 1 = 2\sqrt{n} + 1$$

reveals how much better we can do if the orientation is ignored. In the non-oriented case, we have $\sqrt{n} + 1 \leq \underline{\mathrm{U}}\mathrm{ngar}(\underline{K_n}) \leq 2\sqrt{n} + 1$, with the upper bound being only a rough estimate. Contrast this with Ungar's theorem, which states that $n - 1 \leq \mathrm{Ungar}(K_n) \leq n$.

Chapter 5

Threshold and Chordal Graphs

Now we return to the case of oriented matroids begin to generalize Ungar's theorem to graphic matroids. The first step focuses on the class of threshold graphs – because of the rich structure of these graphs, we can obtain fairly complete results for them. We then move on to consider chordal graphs, a class of graphs which includes threshold graphs.

5.1 Threshold Graphs

One particularly well behaved class of graphs is the class of threshold graphs. They were originally introduced by Chvátal and Hammer [8], and have applications in the theory of parallel processing.

Definition 5.1.1. Let $G(V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Identify the subsets of V with the vertices of the n-dimensional hypercube (i.e. for a subset S of V, choose the vertex of the hypercube with a zero in each coordinate corresponding to any $v_i \notin S$ and a one in each coordinate corresponding to any $v_j \in S$. G is a *threshold graph* if there is a hyperplane in $\mathbb{R}^n$ such that the set of vertices on one side of this hyperplane corresponds precisely to the subsets of V which are totally disconnected (i.e. have no edges between any pair of vertices in the subset).

This definition is admittedly somewhat complicated, but it turns out that there are many other ways to characterize threshold graphs, as the following proposition states.

Proposition 5.1.2. *For a simple graph $G(V, E)$, the following are equivalent:*

(i) G is a threshold graph.

(ii) G can be constructed by starting with a single vertex and adding vertices one at a time which are either completely isolated from the existing vertices or are connected by an edge to every existing vertex.

(iii) G contains no vertex induced subgraphs which are isomorphic to $2K_2$, P_4 or C_4. [Pictures would be better than new notation here.]

(iv) *For all pairs x and y of distinct vertices, $\{x, y\}$ is an edge of G if and only if* $\deg(x)+\deg(y) > m$*, where m is the number of distinct (nonzero) vertex degrees of G.*

(v) *There is a partition of the vertex set into two sets $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$ such that $\{x_i, y_j\} \in E \Rightarrow \{x_{i'}, y_{j'}\} \in E$, where $i' \geq i$ and $j' \geq j$. Furthermore, the subgraphs induced by the sets X and Y are the empty graph on s vertices and the complete graph on t vertices, respectively.*

For a proof of this equivalence, see Chapter 10 of [12] (although the reader should be warned that some of the equivalences are left as exercises).

The above list could be extended much further; these characterizations are only given as a sample of the many characterizations of threshold graphs. We will be most interested in the second property in the above proposition, which can safely be used for our working definition of threshold graph, while the other properties are ignored. In such a construction, we classify each vertex as either a *isolated vertex* or a *cone vertex*, depending on if it is added to the graph isolated from all previous vertices or connected to all previous vertices, respectively.

[Picture]

Because threshold graphs are such a well-behaved class of graphs, the slope problem for threshold graphs turns out to have a straightforward resolution. Before introducing our first result, however, we make the following two observations.

- Any embedding of the complete graph K_n in $\mathbb{R}^2$ determines at least n slopes if n is even or $n-1$ slopes if n is odd, unless all of the vertices are collinear (this is just theorem 4.1.1).
- Let G be the graph obtained by deleting a single edge from the complete graph K_n. Then any non-collinear embedding of G in $\mathbb{R}^2$ must determine at least $n-1$ slopes if n is even or $n-2$ slopes if n is odd (this is just one less slope than the complete graph required, since the deleted edge may have been the only one determining a particular slope).

Theorem 5.1.3. *Let G be a (connected) threshold graph other than the complete graph, and suppose its construction by adding cone or isolated vertices ends with adding n consecutive cone vertices. Then for any acyclic orientation ω of G,* $\mathrm{Ungar}_{\mathrm{coh}}(G, \omega) = n + 1$.

Proof. Suppose G is a threshold graph on m vertices, and that its construction by adding isolated and cone vertices ends with n consecutive cone vertices. We consider two separate cases. Recall that since we are in the coherent case, this is exactly the same as the slope problem for graphs.

Case 1 : n is odd

First, note that the case $n = 1$ is trivial, as such a threshold graph has an articulation point and can be easily embedded in $\mathbb{R}^2$ using two slopes.

[Picture?]

For other odd values of n, we observe that any $n + 1$ vertices of the graph which include the first n induce a subgraph which is isomorphic to K_{n+1}. Hence G contains $m - n$ subgraphs, all of which are isomorphic to K_{n+1} and whose common intersection is isomorphic to K_n. Recall that if K_{n+1} is embedded in $\mathbb{R}^2$, either all of the vertices must be collinear or we will require at least $n+1$ slopes. Since we cannot have all of the vertices of G on the same line, we necessarily have one of the K_{n+1}'s embedded with non-collinear vertices, and hence we require at least $n+1$ slopes. Furthermore, this bound can be attained by embedding G in $\mathbb{R}^2$ so that all of its vertices are collinear *except* the last isolated vertex in its construction.

[Picture]

Case 2 : n is even

Again, we deal with the smallest case, when $n = 2$, separately. We would then have $m - 2$ triangles which all share a common edge. Since we cannot put all of the vertices on the same line, there must be at least one non-collinear embedding of a triangle, which will require at least three slopes.

For larger even values of n, we now observe that any set of $n + 2$ vertices which includes the n terminal cone vertices must induce a subgraph which is isomorphic to either a normal K_{n+2} or a K_{n+2} with a single edge removed. The former requires at least $n + 2$ slopes for a noncollinear embedding, while the latter requires $n + 1$, as noted above. Just as before, we cannot have every point lie on the same line, so one of these subgraphs is embedded in such a way as to determine at least $n + 1$ slopes. Additionally, the same configuration used in the odd case shows that this bound is attainable.

Since the same configuration shows that the bound is attainable in both the even and odd cases, we note that there is no restriction on the left-to-right order of the points. Thus we can find a G-allowable sequence of length $n + 1$ for any initial orientation ω. □

Conjecture 5.1.4. *Let G be a (connected) threshold graph, and suppose its construction by adding cone or isolated vertices ends with adding n consecutive cone vertices. Then*

$$\operatorname{Ungar}(G) = \begin{cases} 2 & \text{if } n = 1, \\ 3 & \text{if } n = 2, \\ n & \text{if } n \geq 4 \text{ is even,} \\ n+1 & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

In other words, for $n \geq 2$, we have $\operatorname{Ungar}(G) = \operatorname{Ungar}(K_{n+1})$.

As we have already noted, such a threshold graph contains several complete subgraphs isomorphic to K_{n+1}, all of which overlap on a subgraph isomorphic to K_n, so we must have $\operatorname{Ungar}(G) \geq \operatorname{Ungar}(K_{n+1})$. Also, $\operatorname{Ungar}(G) \leq \operatorname{Ungar}_{\mathrm{coh}}(G) = n + 1$. Combining these two inequalities, we see that this conjecture is true for all odd values of n, and the case with $n = 2$ is easily verified. So if this conjecture is wrong, it is not by much – the only room for change is in the case where n is even and at least four, and the only other possible choice for $\operatorname{Ungar}(G)$ is $n + 1$ instead of n.

5.2 Chordal Graphs

Proposition 5.2.1. *For a simple graph G, the following are equivalent:*

(i) G has a simplicial elimination ordering, i.e. one can order the vertices $v_n, \dots, v_1$ in such a way that in the graph induced by $v_i, \dots, v_1$, v_i is a simplicial vertex (i.e. its neighbors form a clique).

(ii) Every cycle of G (of length at least four) has a chord

(iii) Every minimal separating set of two non-adjacent vertices of G induces a clique.

A proof of this theorem can be found in many introductory books on graph theory, such as [23].

Definition 5.2.2. A *chordal graph* is a graph with one (and hence all) of the above properties.

We make the remark that all threshold graphs are chordal graphs (but not the other way around). This is not too difficult to see by comparing (iii) in proposition 5.1.2 (the forbidden subgraph characterization for threshold graphs) with (ii) in proposition 5.2.1 (the lack of chordless cycles). However, the order of the vertices in the cone/isolated vertex construction of a threshold graph *cannot* typically be reversed to obtain a simplicial elimination ordering.

Just as the case was the case with threshold graphs, it turns out that there is one particular characterization that will be of the most interest to us. For chordal graphs, it turns out to be the first property in proposition 5.2.1 – the existence of a simplicial elimination ordering, which says that we can deconstruct the graph by removing one vertex at a time, in such a way that the neighbors of the vertex to be removed induce a clique.

Definition 5.2.3. Let G be a chordal graph with simplicial elimination ordering $v_n, v_{n-1}, \dots, v_1$. The *exponents* e_i, $2 \leq i \leq n$ of G are the size of the cliques induced by the vertices v_i in the graph induced by $v_1, v_2, \dots, v_i$. In other words, e_i is the degree of the simplicial vertex v_i in the graph obtained by removing vertices $v_n, v_{n-1}, \dots, v_{i+1}$ from G.

[Picture]

Note that the order of the exponents is not strictly determined by the graph itself, but may depend upon a choice of a simplicial elimination ordering.

[Picture?]

However, taken as a *set*, the exponents are completely determined by the graph, and this set turns out to be an important invariant of chordal graphs.

Conjecture 5.2.4. *Suppose G is a (connected) chordal graph which is not the complete graph, and let n be the smallest value which can be the first exponent for G (i.e. the smallest degree of a simplicial vertex). Then* $\mathrm{Ungar}_{\mathrm{coh}}(G) = n + 1$.

Chapter 6

Type B Analogs

6.1 Signed Graphs and Central Symmetry

One approach to generalizing graphs which arises from the theory of Coxeter groups is the idea of a signed graph. Recall that the braid arrangement (type A_{n-1}) corresponded to the arrangement of hyperplanes $H_{ij} = \{\vec{x} \in \mathbb{R}^n | x_i - x_j = 0\}$, where $1 \leq i < j \leq n$. Another hyperplane arrangment of interest to us is the "type B_n" (sometimes called type BC_n) consists of all three of the following families of hyperplanes:

- $H_{ij} = \{\vec{x} \in \mathbb{R}^n | x_i - x_j = 0\}$, $1 \leq i < j \leq n$
- $H'_{ij} = \{\vec{x} \in \mathbb{R}^n | x_i + x_j = 0\}$, $1 \leq i < j \leq n$
- $H''_i = \{\vec{x} \in \mathbb{R}^n | x_i = 0\}$, $1 \leq i \leq n$

In other words, the set of hyperplanes has normal vectors $e_i \pm e_j$, $1 \leq i < j \leq n$ and e_i, $1 \leq i \leq n$. Note that this includes all of the hyperplanes from type A_{n-1}, where the normal vectors were $e_i - e_j$, $1 \leq i < j \leq n$.

There is also an extension of the analogy between graphs and subarrangements of the braid arrangement, using Zaslavsky's theory of signed graphs (see [26] for an introduction). A *signed graph* is a graph which can have three kinds of edges — positive, negative and (unsigned) loops.
[Picture]
Comparing the edges to the hyperplanes in the type B_n arrangement, we have positive edges corresponding to the hyperplanes H_{ij}, negative edges corresponding to H'_{ij}, and loops corresponding to H''_i.

An equivalent way to think of signed graphs (and one which will more easily extend our previous results) is as a regular graph on twice the number of vertices, and which obeys some symmetry properties. In particular, a signed graph G on n vertices (labelled $1, 2, \ldots, n$) is equivalent to a standard graph $\tilde{G}$ [is there a standard notation? same question for $\pm$ edges in the table] on $2n$ vertices (labelled $\pm 1, \pm 2, \ldots, \pm n$), such that whenever $\{i, j\}$ is an edge of $\tilde{G}$, so is $\{-i, -j\}$. The equivalence is obtained by making the following identifications (here i and j are always positive integers):

Edges of G	Edges of $\tilde{G}$
Positive edge $\{i,j\}^+$	Edges $\{i,j\}$ and $\{-i,-j\}$
Negative edge $\{i,j\}^-$	Edges $\{i,-j\}$ and $\{-i,j\}$
Loop at vertex i	Edge $\{i,-i\}$

[Picture?]

6.2 Ungar(G) for Signed Graphs

One straightforward result that this leads us to concerns the complete signed graph. Just as in the case of unsigned graphs, everything is in agreement here – the high degree of symmetry assures us that every chamber of $\mathcal{Z}(B_n)$ looks essentially the same, so Ungar$(\mathcal{Z}(B_n), T)$ is independent of T, and hence all are equal to Ungar$(\mathcal{Z}(B_n))$.

Considering the corresponding complete graph (unsigned) on $2n$ vertices, we know that Ungar$(K_{2n}) = 2n$, and hence Ungar$(\mathcal{Z}(B_n)) = 2n$. Furthermore, we have Ungar$_{\text{coh}}(\mathcal{Z}(B_n)) = 2n$, since the coherent case, which corresponds to the slope problem, now concerns *centrally symmetric* configurations of $2n$ (noncollinear) points in the plane. We have already seen that such configurations exist which only determine $2n$ slopes – indeed, all of the examples we presented of $2n+1$ points which determined only $2n$ slopes were centrally symmetric, and the removal of the center point will be the type of configuration for which we are looking.

Another immediate consequence can be seen in the case of the complete signed graph *without* orientation, although the result is one-sided. The closure operations for the corresponding non-oriented matroid are slightly strong than those for (unsigned) graphs – we need to add the additional restriction that if any two edges from the set $\{ij^+, ij^-, ii, jj\}$ are present, then all four must be (Here we have abbreviated edges $\{i,j\}$ as ij and loops on i or j as ii or jj, respectively). Hence we need at least as many blocks in a partition of $\binom{[2n]}{2}$ which is to be a non-annihilating strong map image of $\underline{\mathcal{M}}(\underline{B}_n)$ as we did in the case of transitive partitions of $\binom{[2n]}{2}$. Hence $\underline{\text{U}}\text{ngar}(\underline{\mathcal{M}}(\underline{B}_n)) \geq \lceil\sqrt{2n}+1\rceil$.

Chapter 7

Relations to the Chromatic number and Crapo's β invariant

7.1 The Chromatic Number

Recall that a proper coloring of a graph G is an assignment of colors to the vertices of G such that two vertices of the same color are never connected by an edge of G. The minimal number of colors required to do this is called the *chromatic number* of G, and is often denoted by $\chi(G)$.

Definition 7.1.1. Let G be a graph whose vertices have been properly colored. An orientation of G is said to be *compatible* with this coloring if the orientation of any edge is completely determined by the colors of its endpoints. In other words, suppose we have two edges $\{v_1, v_2\}$ and $\{v_1', v_2'\}$ in G, with $\text{color}(v_1) = \text{color}(v_1')$ and $\text{color}(v_2) = \text{color}(v_2')$. Then if the first edge is oriented as $v_1 \to v_2$, the second edge must be oriented as $v_1' \to v_2'$.

Note that any proper coloring of a graph G can be used to induce an acyclic orientation by choosing an order for the colors and always directing edges toward the endpoint whose color is the greatest. An acyclic orientation is said to be *compatible* with a particular coloring (or class of colorings) if it can be attained in this manner.

Proposition 7.1.2. *Let G be a graph. If ω is acyclic orientation of G which is compatible with some n-coloring of G, then* $\text{Ungar}(G, \omega) \leq \text{Ungar}(K_n)$.

Proof. Essentially what we do is to map the vertices of G to the vertices of K_n via the proper coloring, and then show that any K_n-allowable sequence will translate into a G-allowable sequence using this map.

Partition the vertices into n blocks $V_1, V_2, \ldots, V_n$ according to color. Since ω is a color-compatible acyclic orientation, all of the edges between any two particular blocks will all have the same orientation, i.e. they will all start in the same block and end in the other block. Furthermore, since we are using a proper coloring, there will be no edges between vertices which are in the same block.

Identify each block V_i to a (distinct) vertex of K_n. Note that this identification induces an acyclic orientation on K_n since the edge orientations between any two blocks are in agreement.

Suppose we have some K_n-allowable sequence. Each acyclic orientation in this sequence corresponds in the same way to some (color-compatible) acyclic orientation of G. A brief check verifies that this sequence of orientations of G is in fact a G-allowable sequence. $\square$

In light of this theorem, we can immediately conclude that $\mathrm{Ungar}(G) \leq \mathrm{Ungar}(K_{\chi(G)}) \leq \chi(G)$. In fact, $\mathrm{Ungar}(G, \omega) \leq \chi(G)$ for any acyclic orientation ω of G compatible with a $\chi(G)$-coloring.

7.2 The β Invariant

The β-invariant was introduced by Crapo in 1967 as an invariant for matroids [9]. This invariant turned out to have several nice properties, for example, $\beta(M)$ is always at least zero, and is strictly positive precisely when M is a *connected* matroid without any loops. Also, with the exception of the minimally small matroids consisting either of a single *loop* or single *isthmus*, the β-invariant takes on the same values for a matroid and its dual. It can be defined in terms of the non-oriented matroid structure as

$$\beta(M) = (-1)^{\mathrm{rank}(M)} \sum_{X \subseteq E} (-1)^{|X|} \mathrm{rank}(X),$$

where E is the ground set of the matroid. For a more complete treatment of the β-invariant, we defer to [28].

One combinatorial interpretation of the β-invariant involves taking an arrangement of hyperplanes, distinguishing one of the hyperplanes, and then intersecting all of the remaining hyperplanes with one which is parallel to (but displaced from) the distinguished plane. This creates an arrangement of *affine* subspaces in one lower dimension, and this affine arrangement will typically have both bounded and unbounded chambers. The number of bounded chambers is exactly the β-invariant. Remarkably, this value is independent of the choice of the distinguished hyperplane.

[Picture]

We might expect that it is possible to find minimal cellular strings which mainly intersect chambers which would be bounded for some choice of a distinguished hyperplane, and in the worst possible case, we would need to move through every such chamber. We would also require a move at the beginning and end of the cellular string to move us into and out of the set of such chambers. This leads us to the following conjecture.

Conjecture 7.2.1. *Let M be an oriented matroid and let $\beta(M)$ be its β-invariant. Then* $\mathrm{Ungar}(M) \leq \beta(M) + 2$.

Unfortunately, the inspiration for the conjecture does not directly inspire a proof. However, this conjecture seems to be strongly supported by numerical evidence – typically $\beta(M)$ is much larger than $\mathrm{Ungar}(M)$. Additionally, it is at least true for $1 \leq \beta \leq 4$, which we explain as follows.

We have already introduced the chromatic number of a graph, but it is possible to define the chromatic number in purely matroid-theoretic terms. We begin with the *chromatic polynomial*. For a connected matroid with ground set E, this is given by

$$P(M;\lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{\text{rank}(M) - \text{rank}(X) + 1},$$

and represents (in the graphic case), the number of proper colorings of a graph using λ colors. The *chromatic number* is then the smallest value of λ for which $P(M;\lambda)$ is positive, i.e. $\chi(M) = \min\{\lambda \in \mathbb{N} | P(M;\lambda) > 0\}$. Again, for a more complete treatment, we refer the reader to either [28] or [7].

Oxley [20] was able to show that for small values of β (in particular, $1 \leq \beta \leq 4$), $\beta(M) \leq \chi(M)$. In light of Proposition 7.1.2, we see that the conjecture is true for small values of β. Furthermore, the author is not aware of the existence of counterexamples to either $\chi(G) \leq \beta(M(G)) + 2$ or $\text{Ungar}(G) \leq \beta(M(G)) + 2$ for larger values of β.

Chapter 8

Closing Remarks

8.1 Summary of results

We now present a brief re-cap of what we have establish in this paper.

First, while most of the results in Chapter 3 were observational in nature, we did establish the equivalence of G-allowable sequences, cellular strings on zonotopes, and rank two images of non-annihilating strong maps of certain matroids. These observations were key to opening the road to the many generalizations which we considered.

The first substantial result was to establish a non-oriented analog of Ungar's theorem. We showed that the corresponding lower bound (for K_n) in this case was $\lceil \sqrt{n} \rceil + 1$, and sharpness of this bound was equivalent to a well-known and long-standing open problem in the theory of block designs.

Perhaps the most substantial results which we were able to obtain were for the well-behaved class of threshold graphs. The slope problem for threshold graphs was completely resolved, and the general question of Ungar(G) for a threshold graph has been narrowed down to at most two possible values. This in turn suggested many promising conjectures for the chordal graphs.

There were also considerations of signed graphs. Again, while the result for signed graphs were largely observational, they do suggest new directions to take.

Finally, we established that the chromatic number is *always* an upper bound on Ungar(G), regardless of the graph under consideration. This in turn gave us initial results relating Ungar(G) to the β-invariant, with a general result which would follow from the extension of a result of Oxley concerning small values of β.

8.2 Other Possible Generalizations

In many ways, we have only begun to scratch the surface of the potential results in this area. Several different directions of generalization suggest themselves at this point.

Other classes of graphs

By restricting ourselves to threshold or chordal graphs, we may have made the problem easier, but we have also left many questions unanswered. It would be nice to have results about Ungar(G) for any graph, although this might turn out to be a rather difficult question.

"Slopes" in higher dimensions

One completely different direction to attempt to generalize the slope problem is to try to determine the least number of parallel directions that a configuration of n points in $\mathbb{R}^d$ must determine, given that their affine span is $\mathbb{R}^d$ (i.e. they do not all lie in the same $(d-1)$-dimensional affine subspace). Since Ungar's method relied on an allowable sequence generated by a rotating linear functional, we meet a new difficulty if we hope to use the same approach in higher dimensions – how should the linear functional rotate? In the plane there is essentially one choice, but once we leave the plane there is no longer such a clear choice for how this linear functional should reverse itself. The analog of the slope problem in higher dimensions appears to be a difficult question, but there is a conjecture of Jamison which seems to be worth investigation.

Conjecture 8.2.1. *(Jamison [15]) The number of "slopes" determined by* $2n$ *points whose affine span is* $\mathbb{R}^d$ *is at least* $(d-1)n - \binom{d}{2}$*, where* $n \geq 2d$.

This is particularly interesting in the case $d = 3$, as there are a large number of point configurations which attain this bound, and while several of them arise by simple extensions of two dimensional configurations with a minimal number of slopes, several of them do not. In fact, even the vertices of the regular icosahedron achieve this bound. However, there are no known configurations (to the best of the knowledge of the author) which can do better.

In matroid-theoretic terms, we consider a rank *three* image M of a non-annihilating strong map $\mathcal{M}(A_{n-1}) \twoheadrightarrow M$, and what we desire to minimize is the number of rank two covectors.

Some initial approaches using a sort of "doubly-indexed" allowable sequences may provide some headway with this result, but more work needs to be done.

Other affine subspaces in higher dimensions

Another possibility which arises in higher dimensions is counting the number of parallel classes of other affine subspaces determined by a configuration of points. For example, in $\mathbb{R}^3$, rather than counting parallel classes of lines, we could choose to count parallel classes of planes. Here there is yet another decision to make, which regards how the planes are determined.

Choosing a more geometrically inspired approach, we might want to consider a plane as being generated by three points. Here we run into another decision to make if this is to be a meaningful generalization. Any two distinct points determines a line, but three points will only determine a plane if they are noncollinear. Consequently, we might need to consider point configurations in $\mathbb{R}^3$ which do *not* have any collinear triple of points, rather than just looking at general point configurations in $\mathbb{R}^3$.

This geometric approach, however, diverges from the matroid analogy which we were able to appeal to in $\mathbb{R}^2$. A more matroid-friendly method of determining planes is two pick any two lines determined by the points, and finding the plane parallel both. This will certainly determine more parallel classes of plains, but it turns out that this is the approach which corresponds to considering the number of maximal covectors in a rank three image M under a non-annihilating strong map $\mathcal{M}(A_{n-1}) \twoheadrightarrow M$. Again, it is not clear if we should only consider point configurations for which any triple of points is noncollinear.

More non-oriented results

Our results for the non-oriented case were restricted to the case of complete graphs. However, it makes perfect sense to ask the same sorts of questions for other classes of graphs.

More results for signed graphs

We could also ask more or less the same questions for signed graphs that we did for graphs, since many of the notions can be carried over in a reasonable way. For example, there are notions of colorings for signed graphs [27][?], as well as characterizations of supersolvable signed graphs (recall that supersolvable unsigned graphs were precisely the chordal ones) [25][?].

More non-graphic results

Further expanding our vision, we can ask many of these questions for any matroid, even if it is not graphic or signed-graphic. For example, it makes sense to talk about exponents for any supersolvable arrangement of hyperplanes, so can the chordal conjecture be generalized to include any supersolvable matroid? Similarly, the chromatic number and β-invariant can be defined purely in matroid-theoretic terms, so we could ask similar questions for those bounds.

Minimizing $\mathrm{Ungar}(G)$ according to the number of edges

Finally, we present one last question which may be more of a curiosity than anything else. Suppose we wanted to minimize the number of slopes generated by embedding a graph with n vertices and k edges in the plane, but the structure of the graph was up to us. Are there certain types of graphs which always do better than others having the same number of edges?

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