Math 3592H Honors Math I
Midterm exam 2, Thursday November 10, 2016

Instructions:
50 minutes, closed book and notes, no electronic devices.
There are four problems, worth a total of 100 points.

1. (30 points; 10 points each part)

Let A be a 3×5 matrix.

(i) Prove or disprove: there are no vectors $\overline{\mathbf{b}}$ in $\mathbb{R}^3$ for which $A\overline{\mathbf{x}} = \overline{\mathbf{b}}$ has exactly one solution $\overline{\mathbf{x}}$ in $\mathbb{R}^5$.

(ii) Now assume A can be row-reduced to $\tilde{A} = \begin{bmatrix} 0 & 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
Write down a basis for the subspace $V = \{\overline{\mathbf{x}} \in \mathbb{R}^5 : A\overline{\mathbf{x}} = \overline{\mathbf{0}}\}$.

(iii) Write down a matrix E having the following property:

if $A = \begin{bmatrix} \overline{\mathbf{r}}_1^\top \\ \overline{\mathbf{r}}_2^\top \\ \overline{\mathbf{r}}_3^\top \end{bmatrix}$ with $\overline{\mathbf{r}}_i$ in $\mathbb{R}^5$, then $EA = \begin{bmatrix} \overline{\mathbf{r}}_1^\top \\ \overline{\mathbf{r}}_2^\top \\ \overline{\mathbf{r}}_3^\top - 6\overline{\mathbf{r}}_1^\top \end{bmatrix}$

2. (20 points total) Prove or disprove: If $\overline{\mathbf{f}}, \overline{\mathbf{g}} : \mathbb{R}^4 \to \mathbb{R}^4$ are both differentiable everywhere, and $(\overline{\mathbf{f}} \circ \overline{\mathbf{g}}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}$ for all $\overline{\mathbf{x}}$ in $\mathbb{R}^4$, then the Jacobian matrix $[J\overline{\mathbf{f}}(\overline{\mathbf{a}})]$ is invertible for every[1] $\overline{\mathbf{a}}$ in $\mathrm{img}(\overline{\mathbf{g}})$.

[1]The exam had "for every $\overline{\mathbf{a}}$ in $\mathbb{R}^4$", which is not the assumption I intended!

3. (20 points total; 10 points each part) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & \alpha \end{bmatrix}$.

(i) Assuming that $A\overline{\mathbf{x}} = \overline{\mathbf{0}}$ has infinitely many solutions, what is α?

(ii) Assuming that α is chosen as in the answer to part (i), write down at least one explicit $\overline{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in $\mathbb{R}^3$ so that $A\overline{\mathbf{x}} = \overline{\mathbf{b}}$ has no solutions.

4. (30 points total; 10 points each part) Prove or disprove:

(a) If $\overline{\mathbf{v}}_1, \overline{\mathbf{v}}_2$ are nonzero, nonparallel vectors in $\mathbb{R}^3$, then $\{\overline{\mathbf{v}}_1, \overline{\mathbf{v}}_2, \overline{\mathbf{v}}_1 \times \overline{\mathbf{v}}_2\}$ are linearly independent.

(b) For any angle θ, the vectors $\overline{\mathbf{v}}_1 = \begin{bmatrix} -\cos(6\theta) \\ -\sin(6\theta) \end{bmatrix}, \overline{\mathbf{v}}_2 = \begin{bmatrix} \sin(6\theta) \\ -\cos(6\theta) \end{bmatrix}$ are orthonormal in $\mathbb{R}^2$.

(b) For any angle θ, the vectors $\overline{\mathbf{v}}_1 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \overline{\mathbf{v}}_2 = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$ are orthonormal in $\mathbb{R}^2$.