Math 3592H Honors Math I
Final exam, Friday December 16, 2016

Name:

Instructions:
3 hours, closed book, no electronic devices, but a standard 8.5 by 11 page of notes (front and back) is allowed.
There are 8 problems, worth a total of 100 points.

1. (12 points; 6 points each part)
Consider the linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 8 & 12 \\ -1 & -2 & -3 \end{bmatrix}.$$

Write down a basis for ...

(a) the image of A.

(b) the kernel (nullspace) of A.

2. (12 points) Use Newton's method to approximately solve the system

$$\begin{aligned} x^3 + y^3 &= xy \\ x^4 + y^4 &= x + y \end{aligned}$$

starting with $\overline{\mathbf{a}}_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and finding the next approximation $\overline{\mathbf{a}}_1$.
(Make sure to clarify your procedure for the sake of partial credit.)

3. (12 points total; 6 points each part)

Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{x^3+y^2+z}$.

(a) Compute the Jacobian matrix $Jf(\overline{\mathbf{x}})$ at a general point $\overline{\mathbf{x}}$.

(b) For which unit vector $\overline{\mathbf{u}}$ in $\mathbb{R}^3$ will the directional derivative of $\overline{\mathbf{f}}$ at $\overline{\mathbf{x}} = \overline{\mathbf{0}}$ in the direction $\overline{\mathbf{u}}$ be largest? Explain.

4. (12 points) Describe, with explanation, the set of all points in $\mathbb{R}^2$ where this function is differentiable:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \frac{2xy}{x^2+5y^2} & \text{if } \overline{\mathbf{x}} \neq \overline{\mathbf{0}}, \\ 0 & \text{if } \overline{\mathbf{x}} = \overline{\mathbf{0}} \end{cases}$$

5. (13 points total)
Recall the *trace* of a matrix X in $\mathrm{Mat}(n,n)$ is $\mathrm{Tr}(X) := \sum_{i=1}^{n} x_{i,i}$.
Prove or disprove.

(a) (7 points) The function $f : \mathrm{Mat}(n,n) \to \mathbb{R}$ given by $f(X) = \mathrm{Tr}(X)$ is differentiable at every $X = A$ in $\mathrm{Mat}(n,n)$, with

$$Df(A)(H) = \mathrm{Tr}(H)$$

for all H in $\mathrm{Mat}(n,n)$.

(b) (6 points)The function $\overline{\mathbf{g}} : \mathrm{Mat}(n,n) \to \mathrm{Mat}(n,n)$ defined by

$$\overline{\mathbf{g}}(X) = \mathrm{Tr}(X)^5 \cdot X^2$$

is differentiable everywhere on $\mathrm{Mat}(n,n)$.

6. (13 points total) Let x in $\mathbb{R}$ be a constant, and $A(x) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$.

(a) (3 points) Compute $A(x)^2, A(x)^3$.

(b) (3 points) Give a formula for $A(x)^n$ as a function of $n = 1, 2, \ldots$, with proof.

(c) (3 points) Compute explicitly the entries of the 2×2 matrix

$$e^{A(x)} = I + A(x) + \frac{A(x)^2}{2!} + \frac{A(x)^3}{3!} + \frac{A(x)^4}{4!} + \cdots$$

leaving no summations in your answer.

(d) (4 points) For $\overline{\mathbf{f}} : \mathbb{R} \to \mathrm{Mat}(2,2)$ given by $\overline{\mathbf{f}}(x) = e^{A(x)}$, consider the linear map $D\overline{\mathbf{f}}(\pi) : \mathbb{R} \to \mathrm{Mat}(2,2)$, its derivative at $x = \pi = 3.14159....$

Write down the 2×2 matrix $D\overline{\mathbf{f}}(\pi)(h)$, that is, the linear map $D\overline{\mathbf{f}}(\pi)$ evaluated on h in $\mathbb{R}$.

7. (13 points total) For each θ in $[0, 2\pi)$, consider the linear transformation $A_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates about the x_3-axis and whose restriction to the (x_1, x_2)-plane rotates by an angle θ counterclockwise.

(a) (4 points) Write down the matrix A_θ representing this map with respect to the standard basis $\overline{\mathbf{e}}_1, \overline{\mathbf{e}}_2, \overline{\mathbf{e}}_3$.

(b) (5 points) Prove there are exactly two angles θ in $[0, 2\pi)$ for which A_θ is diagonalizable with eigenvalues in $\mathbb{R}$ and eigenvectors in $\mathbb{R}^3$.

(c) (4 points) For which angles θ in $[0, 2\pi)$ is A_θ diagonalizable if we allow eigenvalues in $\mathbb{C}$ and eigenvectors in $\mathbb{C}^3$?

8. (13 points total) Consider an $m \times n$ matrix A and $n \times m$ matrix B, so the product AB is well-defined and square $m \times m$. Recall that the *rank* of a matrix is the dimension of its image, considered as a linear transformation.

(a) (4 points) Prove that $\text{rank}(AB) \leq \text{rank}(A)$.

(b) (4 points) Prove that $\text{rank}(AB) \leq \text{rank}(B)$.

(c) (5 points) Prove that if AB is invertible then A is surjective, B is injective, and $m \leq n$.