Extra page with a few
REMARKS on spectral thm.

① Restated: $A = A^T$ real symm.

$\Longleftrightarrow$ $A = P^T \Lambda P$ for P orthogonal (i.e. $P^{-1} = P^T$)
and $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ real diagonal (so $\Lambda = \Lambda^T$)

$\Longrightarrow$ Thm proven

$\Longleftarrow$ note
$(P^T A P)^T = P^T \Lambda^T (P^T)^T$
$= P^T \Lambda P$

② Similar proof strategy via induction on n would have proven ...

THM (Hoffman & Kunze § 8.5) A $n \times n$ $\mathbb{C}$-matrix is normal, meaning $A(\bar{A}^T) = (\bar{A}^T)A$,

$\Longleftrightarrow$ $A = U^T \Lambda U$ for U unitary ($U^{-1} = \bar{U}^T$)
and $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ complex diagonal, $\lambda_i \in \mathbb{C}$

③ (Fairly) Similar strategy would prove ...

THM Every A $n \times n$ $\mathbb{C}$-matrix can be triangularized by an invertible matrix P over $\mathbb{C}$, i.e. $P^{-1}AP = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ & \lambda_2 & \ddots & * \\ & & \ddots & * \\ 0 & & & \lambda_n \end{bmatrix}$

... but in fact it's worth looking up a statement and/or quick proof of the Jordan canonical form for A, which is a much more precise triangular form for A.

④ Spectral thm is closely related to singular value decomposition (SVD) (important for principal component analysis – PCA)

of a rectangular real $\underset{m \times n}{X} = \underset{m \times m}{P} \, \underset{m \times n}{\Sigma} \, \underset{n \times n}{Q}$

P: orthogonal, $P^T = P^{-1}$; Σ: diagonal $= \left[\begin{array}{ccc|c} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right]$; Q: orthogonal, $Q^T = Q^{-1}$

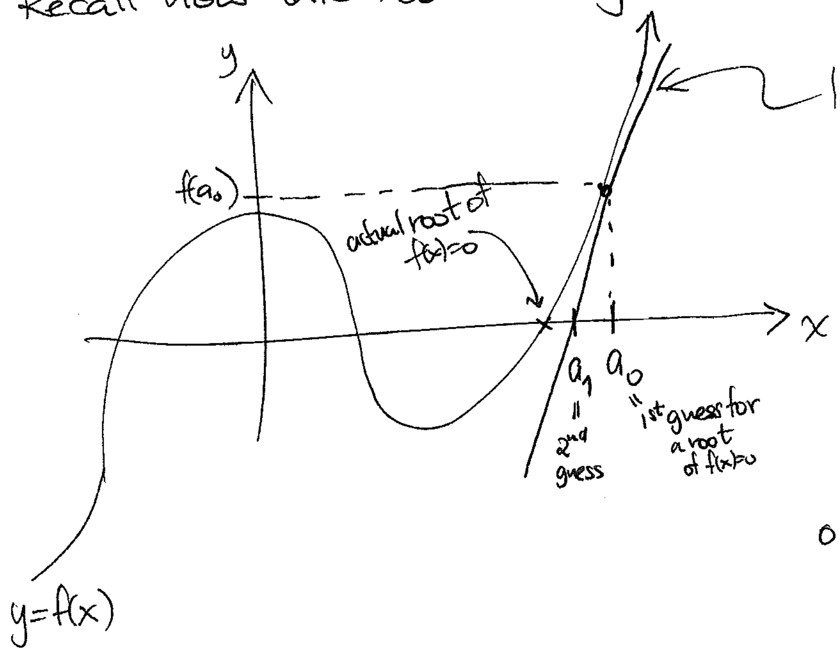
by applying spectral thm to
$A_1 = X^T X = Q^T \Sigma^T \Sigma Q$
or $A_2 = X X^T = P \Sigma \Sigma^T P$

11/28/2016 → Back to curvy (non-linear) things...

(108)

§2.8 Newton's method

Recall how this root-finding method works in one variable:



linear approximation to $y=f(x)$ at $x=a_0$ has equation

$$y-f(a_0)=f'(a_0)(x-a_0)$$

so solve for $y=0$ here to get the x-value a_1 for the approximate root:

$$0-f(a_0)=f'(a_0)(x-a_0)$$

$$x-a_0=-f'(a_0)^{-1}f(a_0)$$

$$x=\underbrace{-f'(a_0)^{-1}f(a_0)+a_0}$$

i.e. let a_1 be this.

Now repeat to get $a_2, a_3, \ldots$

Note that we needed $f'(a_0)\neq 0$ so that we could divide by it.

The multivariate version is analogous.

DEF'N 2.8.1 (multivariate Newton's method)

When looking for a solution to n equations in n unknowns $\bar{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$

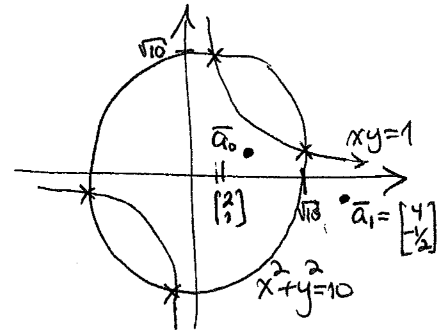
$$\left\{\begin{array}{c} f_1(\bar{x})=0\\ f_2(\bar{x})=0\\ \vdots\\ f_n(\bar{x})=0\end{array}\right\}$$

if we regard $\bar{f}=\begin{bmatrix}f_1\\ \vdots\\ f_n\end{bmatrix}$ as a map $f: U \longrightarrow \mathbb{R}^n$, $U\subset\mathbb{R}^n$, and it is differentiable

at some $\bar{a}_0\in U$ with $D\bar{f}(\bar{a}_0)$ invertible, we can try to approximate a solution by instead solving the linear system $\bar{y}-\bar{f}(\bar{a}_0)=D\bar{f}(\bar{a}_0)(\bar{x}-\bar{a}_0)$ for $\bar{y}=\bar{0}$, i.e. $\boxed{\text{find } \underset{\bar{a}_1}{\bar{x}} \text{ such that } D\bar{f}(\bar{a}_0)(\bar{x}-\bar{a}_0)=-f(\bar{a}_0)}$

(or equivalently $\bar{a}_1=-D\bar{f}(\bar{a}_0)^{-1}f(\bar{a}_0)+\bar{a}_0$)

<u>EXAMPLE</u>: Where do the circle $x^2+y^2=10$ and hyperbola $xy=1$ intersect?



(We could solve this <u>directly</u> via $y=\frac{1}{x}$

and substituting $x^2+\left(\frac{1}{x}\right)^2=10$ mult. by x^2

$$x^4+1=10x^2$$

$$x^4-10x^2+1=0$$

$$x^2=\frac{10\pm\sqrt{10^2-4}}{2}$$

$$x=\pm\sqrt{\frac{10\pm\sqrt{96}}{2}} \approx \pm(3.14626,\, 0.317837),\ \pm(0.317837,\, 3.14626)\ )$$

from my computer

Now let's try Newton's method for solving

$$\begin{cases} 0=f_1(x,y)=x^2+y^2-10 \\ 0=f_2(x,y)=xy-1 \end{cases}$$

so we have $\mathbb{R}^2 \xrightarrow{\bar{f}} \mathbb{R}^2$

$$\bar{x}=\begin{pmatrix}x\\y\end{pmatrix} \longmapsto \bar{f}(\bar{x})=\begin{pmatrix}f_1(\bar{x})\\f_2(\bar{x})\end{pmatrix}=\begin{pmatrix}x^2+y^2-10\\xy-1\end{pmatrix}$$

with derivative $\mathbb{R}^2 \xrightarrow{D\bar{f}(\bar{a})} \mathbb{R}^2$

$$\begin{bmatrix}x\\y\end{bmatrix} \longmapsto \underbrace{\overset{\text{Jacobian}}{\big(J\bar{f}(\bar{a})\big)}}_{\begin{bmatrix}2x & 2y\\ y & x\end{bmatrix}\Big|_{\binom{x}{y}=\binom{a_1}{a_2}}}\begin{bmatrix}x\\y\end{bmatrix}$$

As an initial guess, if we try $\bar{a}_0=\begin{pmatrix}1\\1\end{pmatrix}$, we have $D\bar{f}(\bar{a}_0)=\begin{bmatrix}2&2\\1&1\end{bmatrix}$ <u>not</u> invertible; can't use Newton!

If we try $\bar{a}_0=\begin{pmatrix}2\\1\end{pmatrix}$, we have $D\bar{f}(\bar{a}_0)=\begin{bmatrix}4&2\\1&2\end{bmatrix}$ invertible,

and next guess $\bar{a}_1=-D\bar{f}(\bar{a}_0)^{-1}\,\bar{f}(\bar{a}_0)+\bar{a}_0=-\begin{bmatrix}4&2\\1&2\end{bmatrix}^{-1}\begin{bmatrix}2^2+1^2-10\\2\cdot1-1\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}4\\-\frac{1}{2}\end{bmatrix}$

then $\bar{a}_2\approx\begin{bmatrix}3.3\\0.16\end{bmatrix}$, $\bar{a}_3\approx\begin{bmatrix}3.154\\0.31\end{bmatrix}$, $\bar{a}_4\approx\begin{bmatrix}3.14629\\0.317817\end{bmatrix}$, $\bar{a}_5\approx\begin{bmatrix}3.14626\\0.317837\end{bmatrix}$

(same as above!)