We need to avoid this pathology.

DEF'N 1.9.6: A function $\bar{f}: \underset{\mathbb{R}^n}{\overset{\text{open}}{\mathcal{U}}} \longrightarrow \mathbb{R}^m$ is called continuously differentiable on $\mathcal{U}$ if all its partial derivatives $\frac{\partial f_i}{\partial x_j}$ $\begin{smallmatrix} i=1,\dots,m \\ j=1,\dots,n \end{smallmatrix}$ exist on $\mathcal{U}$, and are continuous on $\mathcal{U}$.

(NOTATION: $\bar{f}$ is C^1 on $\mathcal{U}$ or $\bar{f} \in C^1(\mathcal{U})$)

THM 1.9.8: If $\bar{f}$ is C^1 on $\mathcal{U}$, then it is differentiable at every $\bar{a} \in \mathcal{U}$

(and $D\bar{f}(\bar{a})$ has matrix $J\bar{f}(\bar{a}) = \left[\frac{\partial f_i}{\partial x_j}(\bar{a}) \right]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$, of course)

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proof: We are going to need the multivariate version of M.V.T. here (and again later):

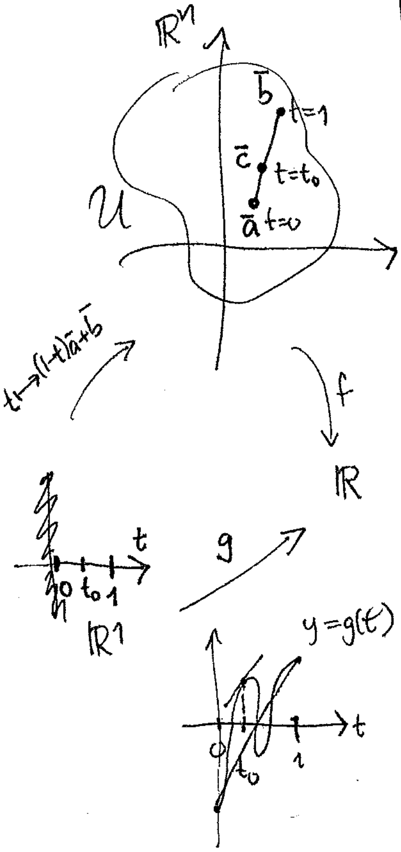
THM 1.9.1 (multivariate MVT)

If $f: \underset{\mathbb{R}^n}{\overset{\text{open}}{\mathcal{U}}} \longrightarrow \mathbb{R}$ contains a line segment $[\bar{a}, \bar{b}]$ and f is differentiable on $\mathcal{U}$, then $\exists$ some $\bar{c} \in [\bar{a}, \bar{b}]$ with $[Df(\bar{c})](\bar{b}-\bar{a}) = f(\bar{b}) - f(\bar{a})$.

UNNECESSARILY STRONG HYPOTHESIS — replace with: f having a directional derivative in direction $\bar{b}-\bar{a}$ at every point of $(\bar{a},\bar{b})$, continuous on $[\bar{a},\bar{b}]$

In particular (COR. 1.9.2), if $|[Df(\bar{c})]| \leq M \quad \forall \bar{c} \in [\bar{a}, \bar{b}]$

then $|f(\bar{b}) - f(\bar{a})| \leq M |\bar{b} - \bar{a}|$



proof of multivar. MVT:

Parametrize $[\bar{a}, \bar{b}] = \{(1-t)\bar{a} + t\bar{b} : 0 \leq t \leq 1\}$

and apply usual MVT to $g: [0,1] \longrightarrow \mathbb{R}$ (continuous on [0,1] - why? differentiable on (0,1) - why?)

$g(t) = f((1-t)\bar{a} + t\bar{b})$

$\left(= \begin{cases} f(\bar{a}) & \text{if } t=0 \\ f(\bar{b}) & \text{if } t=1 \end{cases} \right)$

to give $t_0 \in (0,1)$ with $g'(t_0) = \frac{g(1)-g(0)}{1-0} = f(\bar{b}) - f(\bar{a})$

$$g'(t_0) = \lim_{h \to 0} \frac{g(t_0+h) - g(t_0)}{h}$$

name [redacted] $\bar{c} := (1-t_0)\bar{a} + t_0\bar{b}$

$$= \lim_{h \to 0} \frac{f(\bar{c} + h(\bar{b}-\bar{a})) - f(\bar{c})}{h}$$

= dir. deriv. of f at $\bar{c}$ in dir. $\bar{b}-\bar{a}$

$= [Df(\bar{c})](\bar{b}-\bar{a})$ ■

Continuing the proof of THM 1.9.8,
assuming $\bar{f}: \underset{\mathbb{R}^n}{\overset{\cap}{U^{open}}} \to \mathbb{R}^m$ is C^1, and given $\bar{a} \in U$

we want to show $\lim_{h \to 0} \dfrac{\bar{f}(\bar{a}+\bar{h}) - f(\bar{a}) - [\mathbf{J}\bar{f}(\bar{a})]\bar{h}}{|\bar{h}|} = \bar{0}$.

Since the limits go componentwise for $\bar{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$, we may as well assume $m=1$,

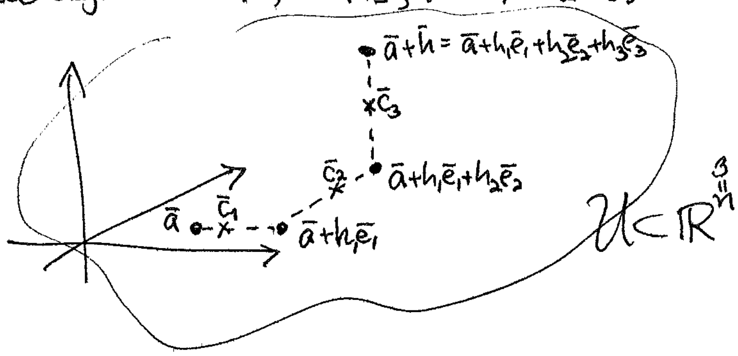
i.e. $f: \underset{\mathbb{R}^n}{\overset{\cap}{U^{open}}} \to \mathbb{R}$, and we want to show

$$\lim_{\bar{h} \to 0} \frac{f(\bar{a}+\bar{h}) - f(\bar{a}) - [\mathbf{J}f(\bar{a})]\bar{h}}{|\bar{h}|} = 0$$

where here $[\mathbf{J}f(\bar{a})] = \left[\frac{\partial f}{\partial x_1}(\bar{a}) \cdots \frac{\partial f}{\partial x_n}(\bar{a})\right]$

Make sure $|\bar{h}|$ is small enough so that when we write $\bar{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} = h_1\bar{e}_1 + \dots + h_n\bar{e}_n$,

all these segments $[\bar{a}, \bar{a}+h_1\bar{e}_1]$, $[\bar{a}+h_1\bar{e}_1 + h_2\bar{e}_2]$, ----, $[\bar{a}+h_1\bar{e}_1+\dots+h_{n-1}\bar{e}_n, \bar{a}+\bar{h}]$ lie in U:



Then by telescoping,

$$f(\bar{a}+\bar{h}) - f(\bar{a}) = \sum_{i=1}^{n} f(\bar{a}+h_1\bar{e}_1+\dots+h_i\bar{e}_i) - f(\bar{a}+h_1\bar{e}_1+\dots+h_{i-1}\bar{e}_{i-1})$$

Use the multivariate MVT on each segment to find a $\bar{c}_i$ on the segment $[\bar{a}+h_1\bar{e}_1+\dots+h_{i-1}\bar{e}_{i-1}, \bar{a}+h_1\bar{e}_1+\dots+h_{i-1}\bar{e}_{i-1}+h_i\bar{e}_i]$ for $i=1,\dots,n$

$$\text{with } f(\bar{a}+h_1\bar{e}+\dots+h_i\bar{e}_i) - f(\bar{a}+h_1\bar{e}_1+\dots+h_{i-1}\bar{e}_{i-1}) = \mathbf{D}f(\bar{c}_i)(h\bar{e}_i) = h[\mathbf{J}f(\bar{c}_i)](\bar{e}_i)$$
$$= h \frac{\partial f_i}{\partial x_i}(\bar{c}_i)$$

Thus
$$\left|\frac{f(\bar{a}+\bar{h}) - f(\bar{a}) - [\mathbf{J}f(\bar{a})]\bar{h}}{|\bar{h}|}\right| = \left|\sum_{i=1}^{n} h_i \frac{\partial f_i}{\partial x_i}(\bar{c}_i) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(\bar{a}) h_i \right| \Big/ |\bar{h}|$$
$$= \left|\sum_{i=1}^{n} \frac{h_i}{|\bar{h}|}\left(\frac{\partial f_i}{\partial x_i}(\bar{c}_i) - \frac{\partial f_i}{\partial x_i}(\bar{a})\right)\right| \le \sum_{i=1}^{n} \underbrace{\frac{|h_i|}{|\bar{h}|}}_{\text{all} \le 1} \left|\frac{\partial f_i}{\partial x_i}(\bar{c}) - \frac{\partial f_i}{\partial x_i}(\bar{a})\right|$$

Now use C^1: as $\bar{h} \to 0$, each $h_i \to 0$, and each $\bar{c}_i \to \bar{a}$, so continuity of $\frac{\partial f_i}{\partial x_i}$

implies $\frac{\partial f_i}{\partial x_i}(\bar{c}_i) \to \frac{\partial f_i}{\partial x_i}(\bar{a})$. Hence the above expression $\to 0$ ∎

Chapter 2 Solving equations (i.e. back to more linear algebra!)

How would you solve the system

$$\begin{aligned} x+y+z+w &= 1 \\ 2x+2y+6z+8w &= 0 \\ 4x+4y+8z+10w &= 1 \end{aligned}$$?

In matrix form, it's $A\bar{x} = \bar{b}$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 8 \\ 4 & 4 & 8 & 10 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\bar{b}}$$

Adding/scaling equations is easiest to perform/mimic on the augmented matrix $[A|\bar{b}] =$

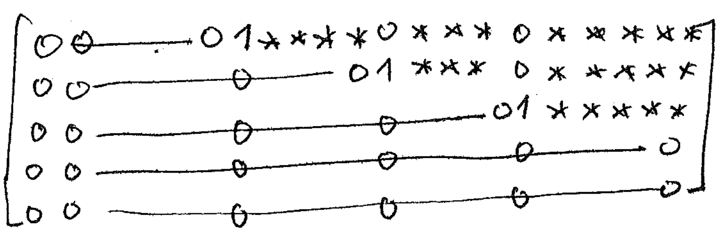
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 8 & 0 \\ 4 & 4 & 8 & 10 & 1 \end{array}\right]$$

by performing (invertible) row operations: (DEFN 2.1.1)

1. scale a row by some $c \in \mathbb{R} - \{0\}$
2. add a multiple of a row to another
3. exchange rows

One systematic way to do this brings $[A|\bar{b}]$ into what is called (row-reduced) echelon form (DEF'N 2.1.4):

schematically:



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rules:
1. every row has leftmost nonzero entry scaled to 1 (called a pivotal 1)
2. pivot 1's go left-to-right as row index increases
3. pivot 1's are the only nonzero entry in their columns
4. zero rows can only go at the bottom