(67)

EXAMPLES of the geometry

for $\boxed{\begin{matrix} n=2 \\ m=2 \end{matrix}}$ $\quad A\bar{x}=\bar{b}$

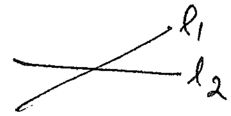
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \rightsquigarrow [A|\bar{b}] \rightsquigarrow \underbrace{[\tilde{A}|\tilde{b}]}_{\text{echelon form}}$$

Solving is intersecting lines in $\mathbb{R}^2$

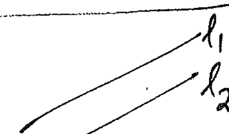
$$\begin{cases} l_1=\{a_{11}x+a_{12}y=b_1\} \\ l_2=\{a_{21}x+a_{22}y=b_2\} \end{cases}$$

# of sol'ns

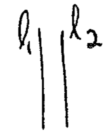
$= \Bigg\{$

$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} * \\ * \end{matrix}\right]$ 

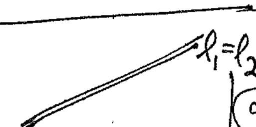
1

$\left[\begin{matrix} 1 & * \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 1 \end{matrix}\right]$ 

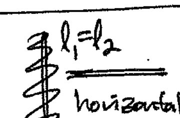
0

$\left[\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 1 \end{matrix}\right]$ 

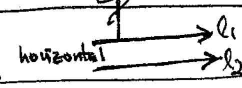
0

$\left[\begin{matrix} 1 & * \\ 0 & 0 \end{matrix}\middle|\begin{matrix} * \\ 0 \end{matrix}\right]$ 

∞

$\left[\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} * \\ 0 \end{matrix}\right]$ 

∞

$\left[\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 1 \end{matrix}\right]$ 

$\}$ 0

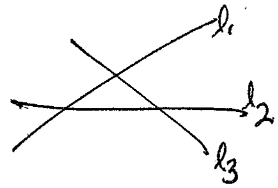
Q: What would $[\tilde{A}|\tilde{b}] = \left[\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 1 \\ 0 \end{matrix}\right]$ mean?

What about $[\tilde{A}|\tilde{b}] = \left[\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$?

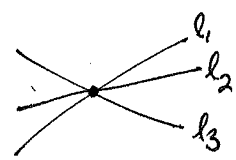
10/26/2016 >

(some) EXAMPLES for $n=2$, $m=3$

$[\tilde{A}|\tilde{b}] = \Bigg\{$

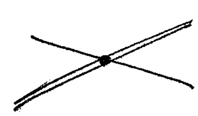
$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right]$ 

0

$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} * \\ * \\ 0 \end{matrix}\right]$ 

1

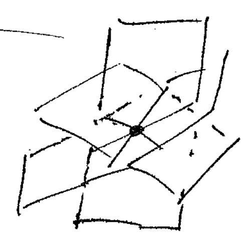
or

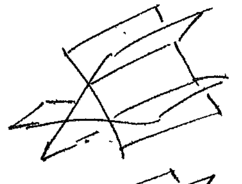


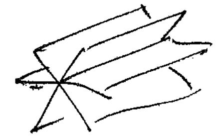
(some) EXAMPLES for $n=3$, $m=3$

Solving is intersecting planes in $\mathbb{R}^3$

$[\tilde{A}|\tilde{b}] = \Bigg\{$

$\left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\middle|\begin{matrix} * \\ * \\ * \end{matrix}\right]$  1

$\left[\begin{matrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right]$  0

$\left[\begin{matrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{matrix}\middle|\begin{matrix} * \\ * \\ 0 \end{matrix}\right]$  ∞

COR (THM 2.2.2) $A\bar{x}=\bar{b}$ has a unique solution $\forall \bar{b} \iff A$ row-reduces to $I=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}$

(in particular, A must be square $n\times n$).

proof: Note $[A|\bar{b}] \overset{\text{row-reduces}}{\rightsquigarrow} [\tilde{A}|\tilde{b}]$

$\Rightarrow A \overset{\text{row-reduces}}{\rightsquigarrow} \tilde{A}$

If some column of $\tilde{A}$ is non-pivotal, we can create $\bar{b}$ (e.g. $\bar{b}=\bar{0}$) with ∞ many sol'ns.

If some row of $\tilde{A}$ is zero, we can create $\bar{b}$ with no sol'ns (make $\tilde{b}_j \neq 0$ in the same row, and invert the row ops to get $\bar{b}$).

Hence ! sol'n $\forall \bar{b} \Rightarrow \tilde{A}=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}$.

Conversely $\tilde{A}=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix} \Rightarrow$ every $\bar{b}$ has ! sol'n, namely $\bar{x}=\tilde{b}$ ∎

How unique is $[\tilde{A}|\tilde{b}]$? Completely...

THM 2.1.7 (part 2): If $\tilde{A}_1, \tilde{A}_2$ are both in echelon form, and obtained from A by sequences of row operations, then $\tilde{A}_1=\tilde{A}_2$

proof: We'll use the fact that if $A=\begin{bmatrix}| & | & & |\\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n\\ | & | & & |\end{bmatrix}$

and $A[k] := \begin{bmatrix}| & | & & |\\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_k\\ | & | & & |\end{bmatrix}$ = leftmost k columns of A,

then $\tilde{A}_1[k], \tilde{A}_2[k]$ are still in ~~...~~ echelon form, and both row operation equivalent to $A[k]$.

Thus if we regard $A[k]=\left[A[k-1]\,\middle|\,\begin{matrix}|\\ \bar{a}_k\\ |\end{matrix}\right]$ as an augmented matrix for a system of equations $A[k]\bar{x}=\bar{a}_k$ (*)

then $\left[\tilde{A}_1[k-1]\,\middle|\,\begin{matrix}|\\ \tilde{a}_k\\ |\end{matrix}\right], \left[\tilde{A}_2[k-1]\,\middle|\,\begin{matrix}|\\ \tilde{a}_k\\ |\end{matrix}\right]$ have the same solutions as that system (*).

In particular, they have no sol'n $\iff k$ is a pivotal column, so their pivot columns are the same.

(69)

e.g. $\tilde{A}_1 = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \left[\begin{matrix} 0 & 1 & a & b & 0 & c & 0 & e & h \\ & & & & 1 & d & 0 & f & i \\ & & & & & & 1 & g & j \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right] \end{array}$ (and $\tilde{A}_2$ has similar echelon form)

But now we can get the nonpivotal column entries in $\tilde{A}_1$ as follows:

To get c, d, solve the system $\underbrace{\left[A[5] \,\middle\|\, \bar{a}_6\right]}$ uniquely (i.e. $A[5]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \bar{a}_6$)

depend only on A

with $x_3 = x_4 = 0$,

and then $x_2 = c$
$x_5 = d$

To get h, i, j, solve the system $\left[A[8] \,\middle|\, \bar{a}_9\right]$ uniquely

$A[8]\bar{x} = \bar{a}_9$ where $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_8 \end{bmatrix}$

with $x_3 = x_4 = x_6 = x_8 = 0$

and then $x_2 = h$
$x_5 = i$
$x_7 = j$

Thus they must agree in $\tilde{A}_1, \tilde{A}_2$ ∎

We can now understand invertibility a bit better.

~~PROP 2.3.1, 2.3.2: A has a (2-sided) matrix inverse $\iff A\bar{x} = \bar{b}$ has a unique sol'n $\forall \bar{b}$~~
~~$\overset{(*)}{\iff}$ A row reduces to $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$~~

~~and in particular this forces A to be square.~~

~~proof: We already saw the $\overset{(*)}{\iff}$.~~
~~If A has a 2-sided inverse A^{-1}, then $A\bar{x} = \bar{b}$ has a solution $\bar{x} = A^{-1}\bar{b}$~~
~~since $AA^{-1}\bar{b} = I\bar{b} = \bar{b}$~~
~~used $AA^{-1} = I$~~

~~and $\bar{x}$ is unique since $A\bar{x} = \bar{b}$~~
~~mult. on left by A^{-1}: $A^{-1}A\bar{x} = A^{-1}\bar{b}$~~
~~used $A^{-1}A = I$: $I\bar{x} = A^{-1}\bar{b}$~~
~~$\bar{x} = A^{-1}\bar{b}$~~

~~Note that we used both $AA^{-1} = I$~~
~~$A^{-1}A = I$~~

REPLACE WITH NEXT PAGE

~~THM~~ THM (2.2.2, 2.3.1, 2.3.2) For an $m \times n$ matrix A, the following are equivalent:
("T.F.A.E.")

(1) $m=n$, i.e. A is square, and A row-reduces to $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

(2) $A\bar{x} = \bar{b}$ has a unique solution $\bar{x} \in \mathbb{R}^n$ $\forall \bar{b} \in \mathbb{R}^m$

(3) The linear transformation $\begin{array}{rcl} T: \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ \bar{x} & \longmapsto & A\bar{x} \end{array}$

is a bijection of sets
(1-1, onto)
injective, surjective

(4) $\begin{array}{rcl} T: \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ \bar{x} & \longmapsto & A\bar{x} \end{array}$ has a 2-sided inverse (set) map $T^{-1}: \mathbb{R}^m \longrightarrow \mathbb{R}^n$

(meaning $T \circ T^{-1} = 1_{\mathbb{R}^m}$
$T^{-1} \circ T = 1_{\mathbb{R}^n}$)

(5) $T: \mathbb{R}^n \to \mathbb{R}^m$ has a linear 2-sided inverse $T^{-1}: \mathbb{R}^m \to \mathbb{R}^n$

(6) A is invertible, i.e. A has a 2-sided inverse matrix A^{-1}, i.e. $AA^{-1} = I_m$
$A^{-1}A = I_n$

In particular, (2) or (3) or (4) or (5) or (6) forces $m=n$ (!).

proof: We already showed (1) $\Leftrightarrow$ (2). Condition (3) is just a rephrasing of (2), since having a solution $\bar{x}$ $\forall \bar{b}$ is surjectivity
having a unique solution $\bar{x}$ is injectivity.

(3) $\Leftrightarrow$ (4) is an easy statement about set bijections.

(4) $\Leftrightarrow$ (5) we proved in PROP. 1.3.14 (which asserts (4) $\Rightarrow$ (5)).

(5) $\Leftrightarrow$ (6) follows since $\underbrace{[T \circ T^{-1}]}_{= \underbrace{[T]}_{A}\underbrace{[T^{-1}]}_{A^{-1}}} = \underbrace{[1_{\mathbb{R}^m}]}_{= I_m}$, $\underbrace{[T^{-1} \circ T]}_{= \underbrace{[T^{-1}]}_{A^{-1}}\underbrace{[T]}_{A}} = \underbrace{[1_{\mathbb{R}^n}]}_{= I_n}$ ∎