EXAMPLES ①

$$\begin{bmatrix}1&2&3\\6&5&4\end{bmatrix}\begin{bmatrix}2\\0\\3\end{bmatrix} = \begin{bmatrix}1\cdot2+2\cdot0+3\cdot3\\6\cdot2+5\cdot0+4\cdot3\end{bmatrix} = \begin{bmatrix}11\\24\end{bmatrix}$$

$$\underset{2\times3}{A} \quad \underset{3\times1}{B} \quad = \quad \underset{2\times1}{C}$$

$$\begin{bmatrix}— \bar{r}_1^T —\\— \bar{r}_2^T —\end{bmatrix}\begin{bmatrix}|\\ \bar{c}_1\\ |\end{bmatrix} = \begin{bmatrix}\bar{r}_1^T\bar{c}_1\\ \bar{r}_2^T\bar{c}_2\end{bmatrix} = \begin{bmatrix}\bar{r}_1\cdot\bar{c}_1\\ \bar{r}_2\cdot\bar{c}_2\end{bmatrix}$$

②

$$\bar{r}_2^T \rightarrow \begin{bmatrix}1&6\\2&5\\3&4\end{bmatrix}\begin{bmatrix}-1&0&1&\overset{\bar{c}_3\,\downarrow}{2}\\1&0&1&4\end{bmatrix} = \begin{bmatrix}5&0&7&26\\3&0&7&24\\1&0&7&22\end{bmatrix}$$

$$\underset{3\times2}{A} \quad \underset{2\times4}{B} \quad = \quad \underset{3\times4}{C}$$

$$\bar{r}_2^T\bar{c}_3 = \bar{r}_2\cdot\bar{c}_3 = \begin{bmatrix}2\\5\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} = 2\cdot2+5\cdot4=24$$

Matrix multiplication has lots of good properties that are not hard to verify, such as its interactions with

scaling matrices ~~componentwise~~ (entrywise) $cA := \begin{bmatrix}ca_{11} & ca_{12} & \cdots\\ ca_{21} & \ddots & \\ \vdots & & \end{bmatrix}$

adding matrices (entrywise) $A+B := \begin{bmatrix}a_{11}+b_{11} & a_{12}+b_{12} & \\ a_{21}+b_{21} & \ddots & \\ & & \end{bmatrix}$

like

$$(cA)\cdot B = A(cB) = cAB$$

$$(A_1+A_2)B = A_1B + A_2B$$

$$A(B_1+B_2) = AB_1 + AB_2$$

all pretty easy

associativity: (PROP 1.2.9 in book) $(AB)C = A(BC)$ for $\overset{n\times m}{A}, \overset{m\times p}{B}, \overset{p\times q}{C}$

9/14/2016

Proof: Let's calculate (i,j) entry on both sides for $i=1,\dots,m$, $j=1,\dots,q$:

$$\big[(AB)C\big]_{i,j} = \sum_{l=1}^{p}(AB)_{il}c_{lj} = \sum_{l=1}^{p}\Big(\sum_{k=1}^{m}a_{ik}b_{kl}\Big)c_{lj} = \sum_{l=1}^{p}\sum_{k=1}^{m}a_{ik}b_{kl}c_{lj}$$

same!

$$\big[A(BC)\big]_{i,j} = \sum_{k=1}^{m}a_{ik}(BC)_{kj} = \sum_{k=1}^{m}a_{ik}\Big(\sum_{l=1}^{p}b_{kp}c_{pj}\Big) = \sum_{k=1}^{m}\sum_{l=1}^{p}a_{ik}b_{kl}c_{lj} \quad ■$$

$$(AB)^T = B^T A^T$$ (THEOREM 1.2.17 in book , EXER. 1.2.14 on HW)

Perhaps disappointingly, but interestingly,

$$AB \neq BA \text{ in general !}$$

not commutative, even when both are square of same dimensions

e.g. $$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}}_{AB}$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}}_{BA}$$

not equal

(Why is there no hope for $AB = BA$ if A is $m \times n$, B is $n \times m$ with $m \neq n$?)

Try it with $m = 1$, $n > 1$.

The special case of $\underbrace{A}_{m\times n}\underbrace{B}_{n\times 1}$ where $B = \bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is a column vector

gives us all linear transformations...

THM 1.3.4:

(1) Every $m\times n$ matrix A gives a linear transformation

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

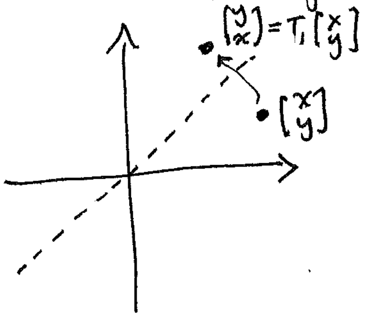
$$\bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \longmapsto T(\bar{v}) = A\bar{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ \vdots & & \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

(2) Every linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of this form, namely $T(\bar{v}) = [T]\bar{v}$ where $[T]$ is the $m \times n$ matrix whose j^{th} column is $T(\bar{e}_j)$, i.e.

$$[T] = \begin{bmatrix} | & | & & | \\ T(\bar{e}_1) & T(\bar{e}_2) & \cdots & T(\bar{e}_n) \\ | & | & & | \end{bmatrix}$$

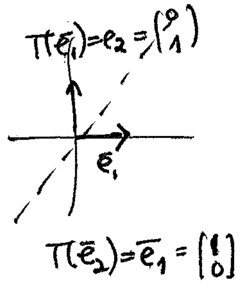
EXAMPLES of (2):

T_1 = reflection in $\mathbb{R}^2$ through symmetry line $y=x$



$$T_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

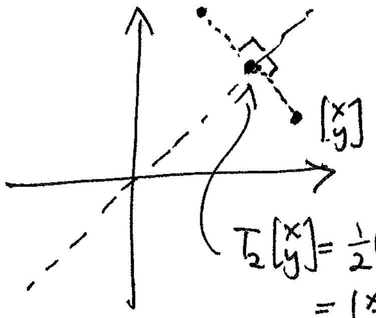
Images of $\bar{e}_1, \bar{e}_2$?



so $$[T_1] = \begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

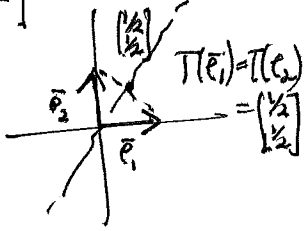
Check: $$[T_1]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \overset{\checkmark}{=} T_1\begin{bmatrix} x \\ y \end{bmatrix}$$

T_2 = projection in $\mathbb{R}^2$ orthogonally onto line $y=x$



$$T_2\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} y \\ x \end{bmatrix}\right) = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix}$$

Images of $\bar{e}_1, \bar{e}_2$?



so $$[T_2] = \begin{bmatrix} | & | \\ T(\bar{e}_1) & T(\bar{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Check: $$[T_2]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix} \overset{\checkmark}{=} T_2\begin{pmatrix} x \\ y \end{pmatrix}$$

Proof of THM 1.3.4:

(1) Given an $m \times n$ matrix $A = \begin{bmatrix} \overbrace{— \vec{a}_1^T —}^{n} \\ — \vec{a}_m^T — \end{bmatrix}$, note that

$$T(\bar{v}) := A\bar{v} = \begin{bmatrix} — \vec{a}_1^T — \\ \vdots \\ — \vec{a}_m^T — \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \bar{a}_1^T\bar{v} \\ \vdots \\ \bar{a}_m^T\bar{v} \end{bmatrix}$$

so to show T is linear, i.e. $T(c\bar{v}) = cT(\bar{v})$, $T(\bar{v}+\bar{w}) = T(\bar{v}) + T(\bar{w})$

it helps to deal with the $m=1$ row case first, i.e. $A = [— \bar{a}^T —]$

and then $T(\bar{v}) = \vec{a}^T\bar{v}$, with $T(c\bar{v}) = \vec{a}^T \cdot c\bar{v} = \sum_{i=1}^{n} a_i c v_i = c\sum_{i=1}^{n} a_i v_i = c\,\vec{a}^T\bar{v} = cT(\bar{v})$

$$T(\bar{v}+\bar{w}) = \vec{a}^T(\bar{v}+\bar{w}) = \sum_{i=1}^{n} a_i(v_i+w_i) = \vec{a}^T\bar{v} + \vec{a}^T\bar{w} = T(\bar{v}) + T(\bar{w})$$

$\Rightarrow$ Generally $T(\bar v)=A\bar v$ has $T(c\bar v)=\begin{bmatrix} \bar a_1^T c\bar v \\ \vdots \\ a_m^T c\bar v\end{bmatrix} = c\begin{bmatrix}\bar a_1^T v \\ \\ a_m^T v\end{bmatrix} = cT(\bar v)$

$$T(\bar v+\bar w)=\begin{bmatrix}\bar a_1^T(\bar v+\bar w) \\ \vdots \\ \bar a_m^T(\bar v+\bar w)\end{bmatrix} = \begin{bmatrix}\bar a_1^T\bar v \\ \vdots \\ a_m^T\bar v\end{bmatrix} + \begin{bmatrix}\bar a_1^T\bar w \\ \vdots \\ \bar a_m^T\bar w\end{bmatrix} = T(\bar v)+T(\bar w)$$

(2) It helps here to note $A\bar v = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \\ a_{m1} & \cdots & a_{mn}\end{bmatrix}\begin{bmatrix}v_1 \\ v_2 \\ \vdots \\ v_n\end{bmatrix} = \begin{bmatrix} v_1a_{11}+\cdots+v_na_{1n} \\ \vdots \\ v_1a_{m1}+\cdots+v_na_{mn}\end{bmatrix}$

$$\boxed{\text{i.e. } A\bar v = v_1\underbrace{\begin{bmatrix}a_{11}\\ \vdots \\ a_{m1}\end{bmatrix}}_{\substack{1^{st}\text{ col}\\ \text{of } A}} + v_2\underbrace{\begin{bmatrix}a_{12}\\ \vdots \\ a_{m2}\end{bmatrix}}_{\substack{2^{nd}\text{ col}\\ \text{of } A}} + \cdots + v_n\underbrace{\begin{bmatrix}a_{1n}\\ \vdots \\ a_{mn}\end{bmatrix}}_{\substack{n^{th}\text{ col}\\ \text{of } A}}}$$

So given <u>any</u> linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$, applying the boxed fact to $A=[T] := \begin{bmatrix} | & | & & | \\ T(\bar e_1) & T(\bar e_2) & \cdots & T(\bar e_n) \\ | & | & & | \end{bmatrix}$

one gets $[T]\bar v = \begin{bmatrix} | & & | \\ T(\bar e_1) & \cdots & T(\bar e_n) \\ | & & | \end{bmatrix}\begin{bmatrix}v_1\\ \vdots \\ v_n\end{bmatrix} = v_1T(\bar e_1)+\cdots+v_nT(\bar e_n)$

$$\overset{\text{by linearity of } T}{=} T(v_1\bar e_1+\cdots+v_n\bar e_n)$$

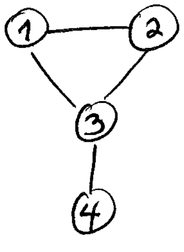
$$= T\left(\begin{bmatrix}v_1\\ v_2\\ \vdots \\ v_n\end{bmatrix}\right) = T(\bar v) \quad ∎$$

THM 1.3.10 ~~COROLLARY~~: Composing linear transformations $T_1:\mathbb{R}^n\to\mathbb{R}^m$, $T_2:\mathbb{R}^m\to\mathbb{R}^p$ gives a linear transformation $T_2\circ T_1:\mathbb{R}^n\to\mathbb{R}^p$ with matrix $[T_2\circ T_1]=[T_2][T_1]$

9/16/2016 > Another way matrix multiplication arises naturally...

<u>Counting Walks in graphs</u>

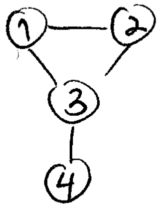
<u>DEF'N</u>: Given a <u>graph</u> G with <u>nodes</u>/<u>vertices</u> labeled $1,2,\ldots,m$ and <u>edges</u> between some pairs $\{i,j\}$ of vertices,

e.g. $G=$ 

one can form its <u>adjacency</u> <u>matrix</u> $m\times n$

e.g. $A=\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$ having $a_{ij}=\begin{cases}1 & \text{if } \{i,j\} \text{ is an edge of } G \\ 0 & \text{if not.}\end{cases}$

Then powers $A, A^2, A^3, \ldots$ have entries that count walks along edges of G

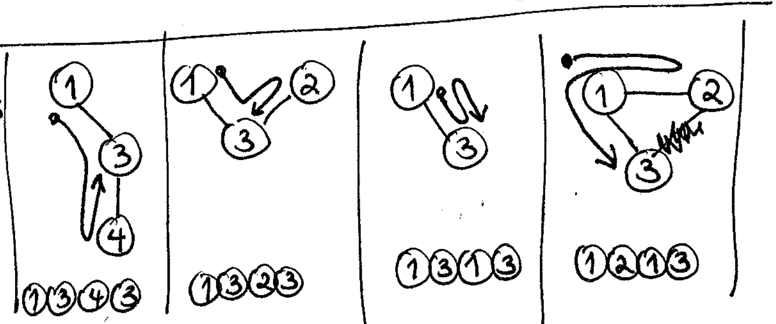


$$A^2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 3 & 1 & 1 & 3 & 0 \\ 4 & 1 & 1 & 0 & 1 \end{array} \qquad A^3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & 4 & 1 \\ 2 & 3 & 2 & 4 & 1 \\ 3 & 4 & 4 & 2 & 3 \\ 4 & 1 & 1 & 3 & 0 \end{array}$$

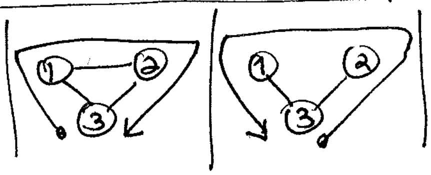
PROPOSITON 1.2.23: $(A^n)_{i,j}$ = # of walks along edges of G from i to j taking exactly n steps

for $n \geq 1$

e.g. $(A^3)_{1,3} = 4$ counts



$(A^3)_{3,3} = 2$ counts



proof of PROP 1.2.23:

Induction on n, with base case n=1 true by definition of adjacency matrix A.

In the inductive step, assume the assertion of the PROP is true for n, and we'll show it for n+1:

$$(A^{n+1})_{i,j} = (A^n \cdot A)_{i,j} = \sum_{k=1}^{m} (A^n)_{ik} \, a_{kj}$$

0 or 1 depending on whether $\{j,k\}$ is an edge of G

$$= \sum_{\substack{k \text{ with} \\ \text{an edge } \{j,k\} \\ \text{in } G}} (A^n)_{ik}$$

# of walks with n steps from i to k in G

= # of walks with n+1 steps from i to k in G ∎