(14)
9/14/2016 Identities & inverse matrices

DEF'N: $I_n := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$ is the $(n\times n)$ identity matrix,

representing $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$
$\vec{v} \longmapsto \vec{v}$
the identity map

and $AI_n = A \quad \forall\ m\times n\ A$

$I_n B = B \quad \forall\ n\times p\ B$

DEF'N: If A is $m\times n$, B is $n\times m$ and $AB = I_m$ then A is called a left-inverse for B

and B is called a right-inverse for A

EXAMPLE: $\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

So A has B as a right-inverse, but A has no left-inverse i.e. a 3×2 C with $CA = I_3$ (why?)

B has A as a left-inverse and B has no right inverse

(They're also not unique! e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ x & y \end{bmatrix} = I_2$ for any $x, y \in \mathbb{R}$)

DEF'N: An ~~square~~ $m\times n$ matrix A is invertible if $\exists$ some $n\times m$ matrix B ~~~~

which is both a left and right inverse for A, i.e. $AB = I_m$, $BA = I_n$. Then we say $B \stackrel{?}{=} A^{-1}$.

(Worrisome!)
Q: Is it possible that $\overset{m\times n}{A}$ has a left-inverse $\overset{n\times m}{C}$ with $C \neq B$?
and a right-inverse $\underset{n\times m}{B}$ but $AB = I_m$, $CA = I_n$

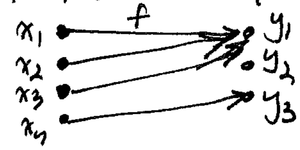
No, associativity prevents this: (PROP 1.2.4)
In this situation, calculate CAB two ways:

$$B = I_n B = (CA)B = CAB = C(AB) = CI_m = C$$

i.e. it forces $B = C$

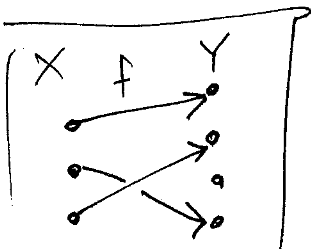
It's not clear yet whether A invertible forces A to be square (it does, we'll see later).

This is reminiscent of what happens for functions $f: X \longrightarrow Y$ between any sets X, Y



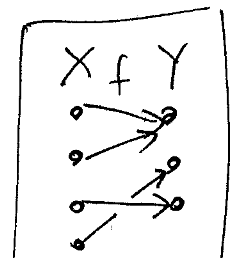
DEF'N: $f: X \to Y$ is one-to-one (injective) if $f(x_1)=f(x_2)$ implies $x_1=x_2$

$f: X \to Y$ is onto (surjective) if $\forall y \in Y \; \exists x \in X$ with $f(x)=y$

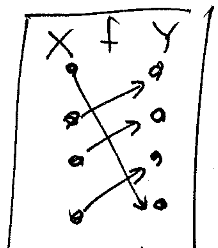


injective

$f: X \to Y$ is bijective if it's both injective & surjective



surjective



bijective

It's easy to see that $f: X \to Y$ is injective $\Leftrightarrow$ f has a left-inverse

i.e. $g: Y \to X$ such that $g \circ f = 1_X$ ← the identity map on X

i.e. $g(f(x)) = x \;\; \forall x \in X$

(But g is far from unique!)

$f: X \to Y$ is surjective $\Leftrightarrow$ f has a right-inverse

i.e. $g: Y \to X$ such that $f \circ g = 1_Y$

i.e. $f(g(y)) = y \;\; \forall y \in Y$

(But g is far from unique!)

and $\quad f: X \to Y$ is bijective $\Leftrightarrow$ f has a left and right inverse $g: Y \to X$ with $g \circ f = 1_X$, $f \circ g = 1_Y$

(and then $g = f^{-1}$ is unique!)

(16) PROP 1.3.14: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ that is bijective will have $T^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ also linear and $[T^{-1}]$, $[T]$ are inverse matrices

proof: For the 1st part we need to check $\forall \bar{v}, \bar{w} \in \mathbb{R}^m$ and $c \in \mathbb{R}$ that

$$T^{-1}(\bar{v}+\bar{w}) = T^{-1}(\bar{v}) + T^{-1}(\bar{w})$$
$$T^{-1}(c\bar{w}) = c\,T^{-1}(\bar{w})$$

But note ~~[illegible]~~

linearity of T

- $T(T^{-1}(\bar{v})+T^{-1}(\bar{w})) = T(T^{-1}(\bar{v})) + T(T^{-1}(\bar{w})) = \bar{v}+\bar{w} = T(T^{-1}(\bar{v}+\bar{w}))$
- $T(c\,T^{-1}(\bar{w})) = c\,T(T^{-1}(\bar{w})) = c\bar{w} = T(T^{-1}(c\bar{w}))$

so the fact that T is injective forces $T^{-1}(\bar{v})+T^{-1}(\bar{w}) = T^{-1}(\bar{v}+\bar{w})$
$c\,T^{-1}(\bar{w}) = T^{-1}(c\bar{w})$.

Hence T^{-1} is linear. But then $T^{-1} \circ T = 1_{\mathbb{R}^n}$, $T \circ T^{-1} = 1_{\mathbb{R}^m}$

$$\Rightarrow [T^{-1}][T] = [1_{\mathbb{R}^n}] = I_n, \quad [T][T^{-1}] = [1_{\mathbb{R}^m}] = I_m$$

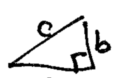
that is, $[T], [T^{-1}]$ are inverse matrices $\blacksquare$

§1.4 Geometry of $\mathbb{R}^n$

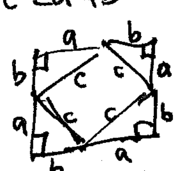
Dot products, cross products, determinants, lengths, etc...

- they help us to understand distances, angles, orthogonality in easy ways.

Recall 2 basic facts:

Pythagorean Theorem: 

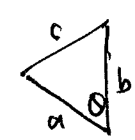
has $c^2 = a^2 + b^2$

proof: 

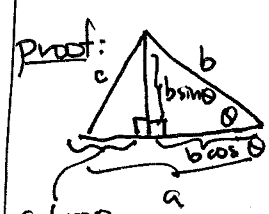
$(a+b)^2 =$ area of big square $= c^2 + 4 \cdot \frac{1}{2}ab$

$\Rightarrow a^2 + 2ab + b^2 = c^2 + 2ab$

$a^2 + b^2 = c^2$ $\blacksquare$

Law of Cosines: More generally 

has $c^2 = a^2 + b^2 - 2ab\cos\theta$

Proof: 

Pythagoras

$c^2 = (a - b\cos\theta)^2 + (b\sin\theta)^2$

$= a^2 - 2ab\cos\theta + b^2\cos^2\theta + b^2\sin^2\theta$

$= a^2 + b^2 - 2ab\cos\theta$ $\blacksquare$

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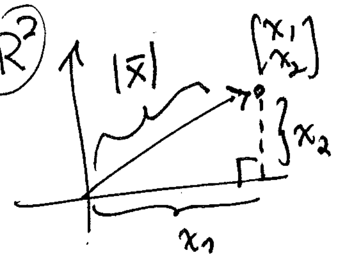
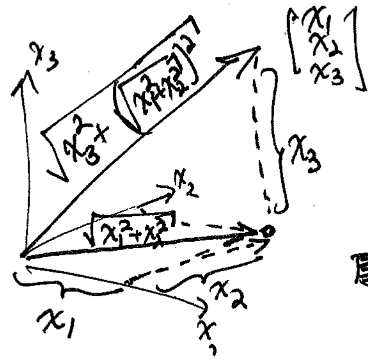
<u>COR:</u> In $\mathbb{R}^1$ and $\mathbb{R}^2$, $\mathbb{R}^3$, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ has <u>length squared</u> $= \bar{x} \bullet \bar{x} =: |\bar{x}|^2$
(and as <u>definition</u> in $\mathbb{R}^n$)

$$= \sum_{i=1}^{n} x_i^2$$

<u>proof:</u> $\mathbb{R}^2$



$\mathbb{R}^3$

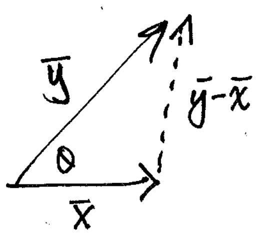


■

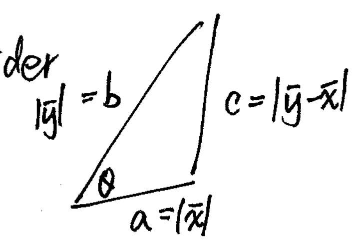
<u>Thus</u> $|\bar{x}| :=$ length of $\bar{x}$ $= \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\bar{x} \cdot \bar{x}}$

<u>COR</u> (PROP 1.43) More generally, ~~for~~ any $\bar{x}, \bar{y} \in \mathbb{R}^n$, their dot product $\bar{x} \bullet \bar{y} = |\bar{x}| \cdot |\bar{y}| \cos\theta$
$\left(:= \sum_{i=1}^{n} x_i y_i\right)$

if θ is the angle between them:



<u>proof:</u> Consider



and law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$\Downarrow$$

$$\underbrace{(\bar{y}-\bar{x}) \bullet (\bar{y}-\bar{x})} = \cancel{\bar{x} \cdot \bar{x}} + \cancel{\bar{y} \cdot \bar{y}} - 2|\bar{x}||\bar{y}|\cos\theta$$

distributivity of dot product (= matrix multiplication)

$$= \cancel{\bar{y} \cdot \bar{y}} - \bar{x} \cdot \bar{y} - \bar{y} \cdot \bar{x} + \cancel{\bar{x} \cdot \bar{x}}$$

$$\Rightarrow \quad -2\bar{x} \cdot \bar{y} = -2|\bar{x}||\bar{y}|\cos\theta$$

$$\bar{x} \cdot \bar{y} = |\bar{x}||\bar{y}|\cos\theta \quad ■$$

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So $\bar{x} \cdot \bar{y} = \begin{cases} 0 \Leftrightarrow \bar{x} \perp \bar{y} & \text{perpendicular/\underline{orthogonal}} \ (\cos\theta = 0, \ \theta = \pi/2) \\ > 0 \Leftrightarrow \bar{x}, \bar{y} \text{ acute} & (\text{so } \cos\theta > 0) \\ < 0 \Leftrightarrow \bar{x}, \bar{y} \text{ obtuse} & (\text{so } \cos\theta < 0) \end{cases}$