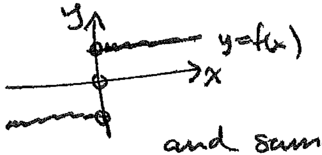
e.g. $f(x) = \frac{|x|}{x} = \left\{\begin{matrix} +1 \text{ if } x>0 \\ -1 \text{ if } x<0 \end{matrix}\right\}$ has same natural domain

$U = \mathbb{R}^1 - \{0\}$

$f: U \to \mathbb{R}^1$



and same questions make sense, but $\lim_{x\to 0}\frac{|x|}{x}$ does not exist,

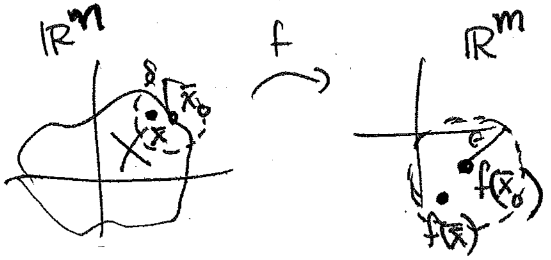
since $\lim_{x\to 0^+}\frac{|x|}{x} = +1$, $\lim_{x\to 0^-}\frac{|x|}{x} = -1$.

This issue is compounded in $\mathbb{R}^n$!

9/28/2016> The expected definitions and properties in $\mathbb{R}^n$...

DEF'N: For a subset $X \subset \mathbb{R}^n$ and function $\bar{f}: X \to \mathbb{R}^m$, and for any $\bar{x}_0 \in \overline{X}$ (=closure of X), say $\bar{f}$ has limit $\bar{a}$ at $\bar{x}_0$ (written $\lim_{\bar{x}\to\bar{x}_0} f(\bar{x}) = \bar{a}$)

if $\forall \epsilon > 0$ $\exists \delta > 0$ such that $\forall \bar{x} \in X$ $|\bar{x} - \bar{x}_0| < \delta \Rightarrow f(\bar{x}) - f(\bar{x}_0)| < \epsilon$.



PROP 1.5.21 (limits of functions are unique) If $\bar{a} = \lim_{\bar{x}\to\bar{x}_0} f(\bar{x})$, $\bar{b} = \lim_{\bar{x}\to\bar{x}_0} f(\bar{x})$ then $\bar{a} = \bar{b}$.

PROP 1.5.22 (limits of functions are componentwise) If $\bar{f}(\underset{\bar{x}\in\mathbb{R}^n}{\bar{x}}) = \begin{pmatrix} f_1(\bar{x}) \\ \vdots \\ f_n(\bar{x}) \end{pmatrix} \in \mathbb{R}^m$

then $\lim_{\bar{x}\to\bar{x}_0} \bar{f}(\bar{x}) = \underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \Leftrightarrow \left\{\begin{matrix} \lim_{\bar{x}\to\bar{x}_0} f_1(\bar{x}) = a_1 \\ \vdots \\ \lim_{\bar{x}\to\bar{x}_0} f_m(\bar{x}) = a_m \end{matrix}\right\}$

THM 1.5.23 (Limit laws) Let $X \subset \mathbb{R}^n$ with $\bar{f}, \bar{g} : \mathbb{R}^n \to \mathbb{R}^m$ (vector-valued functions)

$h : \mathbb{R}^n \to \underset{\mathbb{R}}{\overset{\parallel}{\mathbb{R}^1}}$ (a scalar-valued function)

and $\bar{x}_0 \in \bar{X}$ with

$$\lim_{\bar{x} \to \bar{x}_0} \bar{f}(\bar{x}) = \bar{a} \in \mathbb{R}^m$$

$$\lim_{\bar{x} \to \bar{x}_0} \bar{g}(\bar{x}) = \bar{b} \in \mathbb{R}^m$$

$$\lim_{\bar{x} \to \bar{x}_0} h(\bar{x}) = c \in \mathbb{R}$$

Then (1) $\displaystyle\lim_{\bar{x} \to \bar{x}_0} (\bar{f}(x) + \bar{g}(x)) = \bar{a} + \bar{b}$

(2) $\displaystyle\lim_{\bar{x} \to \bar{x}_0} h(\bar{x}) \bar{f}(\bar{x}) = c\,\bar{a}$

(3) If $c \neq 0$ then $\displaystyle\lim_{\bar{x} \to \bar{x}_0} \frac{f(\bar{x})}{h(\bar{x})} \left(= \lim_{\bar{x} \to \bar{x}_0} \frac{1}{h(\bar{x})} \begin{pmatrix} f_1(\bar{x}) \\ \vdots \\ f_m(\bar{x}) \end{pmatrix} \right) = \frac{1}{c} . \bar{a}$

(4) $\displaystyle\lim_{\bar{x} \to \bar{x}_0} \bar{f}(\bar{x}) \underset{\text{dot products}}{\bullet} \vec{g}(\bar{x}) = \bar{a} \underset{\text{dot products}}{\bullet} \bar{b}$

(5) If $c = 0$ and $\bar{f}$ is <u>bounded</u>, i.e. $\exists R \in \mathbb{R}$ with $|\bar{f}(\bar{x})| \leq R \ \forall \bar{x} \in X$, (<u>without</u> assuming $\lim_{\bar{x} \to \bar{x}_0} \bar{f}(\bar{x})$ exists!)

then $\displaystyle\lim_{\bar{x} \to \bar{x}_0} h(\bar{x}) \bar{f}(\bar{x}) = \bar{0}$

(6). If $\bar{a} = \bar{0}$ and h is <u>bounded</u>, i.e. $\exists R \in \mathbb{R}$ with $|h(\bar{x})| < R \ \forall \bar{x} \in X$, (without assuming $\lim_{\bar{x} \to \bar{x}_0} h(\bar{x})$ exists!)

then $\displaystyle\lim_{\bar{x} \to \bar{x}_0} h(\bar{x}) f(\bar{x}) = \bar{0}$

"<u>proofs</u>" : (1)(2)(5)(6) are pretty easy.

Read the book's proof of (4) on your own – it <u>is</u> instructive!

Let's try (3) ourselves....

By PROP 1.5.22, can work componentwise and just show

$$\lim_{\bar{x} \to \bar{x}_0} \frac{f_1(\bar{x})}{h(\bar{x})} = \frac{a_1}{c} \quad \text{if} \quad \lim_{\bar{x} \to \bar{x}_0} f_1(\bar{x}) = a_1 \quad \text{and} \quad \lim_{\bar{x} \to \bar{x}_0} h(\bar{x}) = c \ (\neq 0).$$

So given $\epsilon > 0$, we want to find $\delta > 0$ making $\left| \frac{f_1(\bar{x})}{h(\bar{x})} - \frac{a_1}{c} \right| < \epsilon$ if $|\bar{x} - \bar{x}_0| < \delta$.

Write

$$\left|\frac{f_1(\bar{x})}{h(\bar{x})} - \frac{a_1}{c}\right| = \left|\frac{cf_1(\bar{x}) - a_1 h(\bar{x})}{c\,h(\bar{x})}\right| = \left|\frac{cf_1(\bar{x}) \overbrace{- c_1a_1 + c_1a_1}^{\text{telescoping - a sometimes useful trick!}} - a_1 h(\bar{x})}{c h(\bar{x})}\right|$$

$$= \frac{1}{|h(\bar{x})|}\left|f_1(\bar{x}) - a_1 + a_1(c - h(\bar{x}))\right|$$

$$\leq \underbrace{\frac{1}{|h(\bar{x})|}}\Big(\underbrace{|f_1(\bar{x}) - a_1|}_{\substack{\text{can make}\\ \text{this } <\epsilon \\ \text{if } |\bar{x}-\bar{x}_0|<\delta_1 \\ \text{for some } \delta_1>0}} + |a_1|\underbrace{|c - h(\bar{x})|}_{\substack{\text{can make}\\ \text{this } <\epsilon \\ \text{if } |\bar{x}-\bar{x}_0|<\delta_2 \\ \text{for some } \delta_2>0}}\Big)$$

Can make $\underbrace{|h(\bar{x})| > |c| - \frac{|c|}{2}}> \frac{|c|}{2}$ if $|\bar{x}-\bar{x}_0| < \delta_3$ for some $\delta_3 > 0$

make $|h(\bar{x}) - c| < \frac{|c|}{2}$, so $|h(\bar{x})| - |c|$

Then for $|\bar{x} - \bar{x}_0| < \delta = \min\{\delta_1, \delta_2, \delta_3\}$, one has

$$\left|\frac{f_1(\bar{x})}{h(\bar{x})} - \frac{a_1}{c}\right| < \frac{1}{|c|/2}\left(\epsilon + |a_1|\cdot\epsilon\right) = \frac{2\epsilon(1+|a_1|)}{|c|} \longrightarrow 0 \text{ as } \epsilon \to 0,$$

so this is enough (read THM 1.5.14 ("Elegance is not required")!) ∎

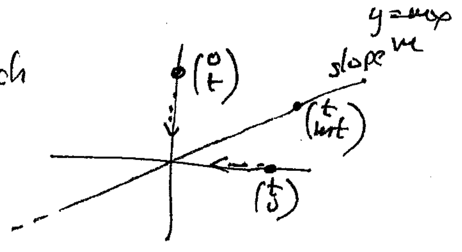
EXAMPLES:

① $$\lim_{\binom{x}{y}\to\binom{2}{3}} \frac{x}{x+y} = \frac{\lim\limits_{\binom{x}{y}\to\binom{2}{3}} x}{\boxed{\lim\limits_{\binom{x}{y}\to\binom{2}{3}} x + \lim\limits_{\binom{x}{y}\to\binom{2}{3}} y}} = \frac{2}{\boxed{2+3}} = \frac{2}{5}$$

$$\lim_{\binom{x}{y}\to\binom{2}{3}} \frac{x}{x+y} = \frac{\lim\limits_{\binom{x}{y}\to\binom{2}{3}} x}{\boxed{\lim\limits_{\binom{x}{y}\to\binom{2}{3}} (x+y)}} = \frac{\lim\limits_{\binom{x}{y}\to\binom{2}{3}} x}{\lim\limits_{\binom{x}{y}\to\binom{2}{3}} x + \lim\limits_{\binom{x}{y}\to\binom{2}{3}} y}$$

needed this to exist, and be nonzero

② $\lim\limits_{\binom{x}{y}\to\binom{0}{0}} \frac{x}{x+y}$ doesn't exist; depends on angle of approach



$$\lim_{\binom{0}{t}\to\binom{0}{0}} \frac{0}{0+t} = 0$$

$$\lim_{\binom{t}{0}\to\binom{0}{0}} \frac{t}{t+0} = 1$$

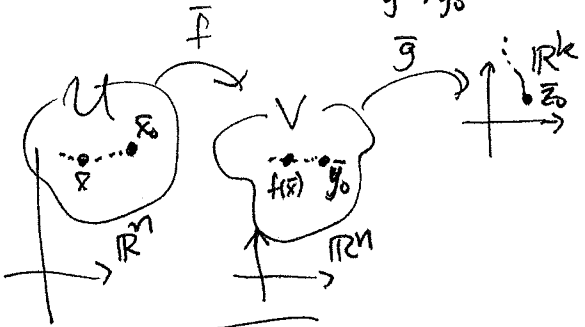
$$\lim_{\binom{t}{mt}\to\binom{0}{0}} \frac{t}{t+mt} = \frac{1}{m+1}$$

"$m=\infty$"

③ Read Example 1.5.25 for how nasty $\lim\limits_{\binom{x}{y}\to\binom{0}{0}} \frac{|y|e^{-\frac{|y|}{x^2}}}{x^2}$ is; along all straight lines, limit is 0; along parabolas $y=kx^2$, it is $|k|e^{-|k|}$!

THM 1.5.24 (limit of composition) If we have $\underset{\underset{\mathbb{R}^n}{\cap}}{\mathcal{U}} \xrightarrow{\bar f} \underset{\underset{\mathbb{R}^m}{\cap}}{V} \xrightarrow{\bar g} \mathbb{R}^k$

and both $\lim_{\bar x \to \bar x_0} f(\bar x) = \bar y_0$, $\lim_{\bar y \to \bar y_0} g(\bar y) = \bar z_0$ exist, then $\lim_{\bar x \to \bar x_0} (\bar g \circ \bar f)(\bar x) = \bar z_0$ exists too.



Proof: not hard; read it in book ∎

Continuity also proceeds as one might expect..

DEF'N: $\underset{\underset{\mathbb{R}^n}{\cap}}{X} \xrightarrow{\bar f} \mathbb{R}^m$ is continuous at $\bar x_0 \in X$ if $\lim_{\bar x \to \bar x_0} f(\bar x) = f(\bar x_0)$

i.e. $\forall \epsilon > 0$ $\exists \delta > 0$ such that $\forall \bar x \in X$ with $|\bar x - \bar x_0| < \delta$ one has $|f(\bar x) - f(\bar x_0)| < \epsilon$.

$\bar f$ is continuous on X if it is continuous at every $\bar x_0 \in X$.

9/30/2016

easily follow from the limit laws

THM 1.5.28 $\bar f, \bar g : \underset{\underset{\mathbb{R}^n}{\cap}}{\mathcal{U}} \to \mathbb{R}^m$, $h : \mathcal{U} \to \mathbb{R}$ all continuous at $\bar x_0$

$\Rightarrow$
1. $\bar f + \bar g$ cont. at $\bar x_0$
2. $h\bar f$ cont. at $\bar x_0$
3. $\frac{\bar f}{h}$ cont. at $\bar x_0$ if $h(\bar x_0) \neq 0$
4. $\bar f \cdot \bar g$ cont. at $\bar x_0$
5. (..some bounded statement...)

THM 1.5.29: $\underset{\underset{\mathbb{R}^n}{\cap}}{\mathcal{U}} \xrightarrow{f} \underset{\underset{\mathbb{R}^m}{\cap}}{V} \xrightarrow{g} \mathbb{R}^k$ with $\bar f$ cont. at $\bar x_0$, $\bar g$ cont. at $f(\bar x_0)$

then $\bar g \circ \bar f$ cont. at $\bar x_0$

COROLLARY 1.5.30: Polynomial functions $f : \mathbb{R}^n \to \mathbb{R}$ are continuous on $\mathbb{R}^n$, and rational functions $f(\bar x) = \frac{g(\bar x)}{h(\bar x)}$ (so g, h polynomial) are continuous at $\bar x_0 \in \mathbb{R}^n$ with $h(\bar x_0) \neq 0$.