Math 4707 Intro to combinatorics and graph theory
Fall 2011, Vic Reiner
Midterm exam 1- Due Wednesday Oct 19, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (15 points) Recall that the Fibonacci numbers are defined by a recurrence $F_{n+1} = F_n + F_{n-1}$, with initial conditions $F_0 = 0, F_1 = 1$.

Without using the recurrence to compute $F_{1,000,000,000}$ explicitly, predict how many decimal digits it will contain, up to an error of 2 digits. Explain why your answer is correct to within 2 digits.
(Hint: recall that we know an exact formula for F_n).

2. (20 points total) Prove that every odd positive integer n will have $n^2 - 1$ divisible by 8.

3. (20 points) (Exercise 3.8.5 on p. 62 of our text) Find the value of k that maximizes $k\binom{99}{k}$
(**Warning:** A solution that calculates all values $k\binom{99}{k}$ for $k = 0, 1, 2, \ldots, 99$ will be given no credit, but is fine to check your answer!)

4. (20 points total; 10 points each) Recall that $a \equiv b \bmod m$, or a is congruent to b modulo m, means that a, b have the same remainder upon division by m, or that $a - b$ is a multiple of m.

(a) Fill in the blanks that make the following conjecture correct:
Conjecture: *The Fibonacci numbers (defined as in Problem 1) have*

$$F_n \equiv \begin{cases} 0 \bmod 3 & \text{if } n \equiv \underline{?} \text{ or } \underline{?} \bmod 8 \\ 1 \bmod 3 & \text{if } n \equiv \underline{?} \text{ or } \underline{?} \text{ or } \underline{?} \bmod 8 \\ 2 \bmod 3 & \text{if } n \equiv \underline{?} \text{ or } \underline{?} \text{ or } \underline{?} \bmod 8 \end{cases}$$

(b) Prove your conjecture.

5. (25 points total; 5 points each part) For parts (a)-(d) below, your answer should be a simple function of n that is allowed to contain binomial coefficients or factorials, but no summation symbols (Σ) nor dots (+....+).

(a) (5 points) Find a_n, the number of paths in the plane $\mathbb{R}^2$ going from $(0, 0)$ to $(3n, 3n)$ taking unit length steps in either the north or east direction at each step.

(b) (5 points) Find b_n, the number of paths as in part (a) that touch neither of the bad points $(n, n), (2n, 2n)$.

(c) (5 points) Find c_n, the number of paths as in part (a) that touch neither of the bad points $(2n, n), (n, 2n)$.

(d) (5 points) What is the probability that, a path as in part (a) chosen randomly, with all paths equally likely, is actually a path as in part (c)?

(e) (5 points) Which one is larger, b_n or c_n, asymptotically for large n? Justify your answer.