Math 4707 Intro to combinatorics and graph theory
Fall 2016, Vic Reiner
Final exam- Due Wednesday Dec. 14, in class

Instructions: There are 5 problems, worth 20 points each, totaling 100 points. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total; 10 points each part) DNA strings can be thought of as ordered sequences of 4 possible nucleotides:

- A (=adenine)
- C (=cytosine)
- G (=guanine)
- T (=thymine).

For example, $CCGATAAGGGCTCA$ is such a sequence.

(a) How many such sequences are there of length 1000?

(b) How many such sequence are there of length 1000, with equally many A's as C's, and equally many G's as T's, but three times as many G's as A's.

2. (20 points total; 10 points each part)
(a) How many paths are there in the plane $\mathbb{R}^2$ going from $(0,0)$ to $(40000, 44000)$ taking unit length steps in either the north or east direction at each step?

(b) How many such paths are there which avoid passing through any of these three 3 "bad" points

$$(10,11),\ (200,220),\ (3000,3300)?$$

3. (20 points; 10 points each part) Given a graph $G = (V, E)$ with no loops, recall from class that an *acyclic orientation* of G is an assignment of one of the two possible orientations to every edge e in E, so as to make G a directed graph, but so as to create no directed cycles in this digraph.

Recall also that the *chromatic polynomial* $\chi(G, k)$ was defined to be the polynomial in k that gives as its value for positive integers k the number of proper vertex-colorings of G with k colors.

(a) Show that the number of acyclic orientations for a tree T on n vertices does not depend upon the structure of the tree itself, by writing (with proof) this number as a function of n.

(b) Show that the chromatic polynomial $\chi(T, k)$ for a tree T on n vertices does not depend upon the structure of the tree itself, by writing (with proof) the $\chi(T, k)$, a polynomial in k, as a function of n.

4. (20 points total; 10 points each part) Let $G = (V, E)$ be a bipartite graph with vertices partitioned $V = X \sqcup Y$, and assume

- every x in X has the same degree $d_X \geq 1$, and
- every y in Y has the same degree $d_Y \geq 1$.

(a) Prove that $\frac{d_X}{d_Y} = \frac{|Y|}{|X|}$.

(b) Assuming without loss of generality that $d_X \geq d_Y$, show that there exists at least one matching $M \subset E$ with number of edges $|M| = |X|$.

5. (20 points) Standard soccer balls have 12 pentagons and some number of hexagons, sewn together with 3 polygons meeting at each vertex.

Your boss wants you to design a *new* soccer ball, again with exactly 3 polygons meeting at each vertex, but this time made up of only *squares* and hexagons. The boss insists that there should be at least 12 squares used in the ball.

Convince the boss that this is impossible, by deriving the exact number of squares that must be used in any such hypothetical soccer ball.