Math 4707 Intro to combinatorics and graph theory
Fall 2016, Vic Reiner
Midterm exam 2- Due Wednesday Nov. 23, in class

Instructions: There are 5 problems, worth 20 points each, totaling 100 points. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total) Recall that a forest is a graph containing no cycles, and a tree is a connected forest.

(a) (10 points) Prove that a tree having at least one vertex with degree d always has at least d distinct leaves.

(b) (5 points) Prove that a tree T with n vertices has

$$\sum_{v \in V} (\deg_T(v) - 1) = n - 2.$$

(c) (5 points) Given a forest with n vertices and c connected components, how many edges will it contain (as a function of n and c)?

2. (20 points total; 5 points each) For each $n = 1, 2, \ldots$, define a bipartite graph G_n on vertex $V = X \sqcup Y$ where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ with edge set

$$E := \{\{x_i, y_j\} : i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n, \text{ and } i \neq j\}$$

This means, for example, that G has $2n$ vertices and $n(n-1)$ edges.

Explain, with proof, exactly for which values of n in $\{1, 2, 3, \ldots\}$ does the graph G_n contain ...

(a) ... a spanning tree?

(b) ... a closed Euler tour/circuit?

(c) ... a perfect matching?

(d) ... a closed Hamiltonian tour/circuit?

3. (20 points) Your company has 6 employees $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and 6 tasks to perform $\{y_1, y_2, y_3, y_4, y_5, y_6\}$, but each employee has a different set of tasks that they are capable of doing:

employee	tasks they can do
x_1	$\{y_2, y_4, y_5\}$
x_2	$\{y_1, y_2, y_3, y_5, y_6\}$
x_3	$\{y_2, y_4, y_5\}$
x_4	$\{y_2, y_4\}$
x_5	$\{y_2, y_4, y_5\}$
x_6	$\{y_1, y_3, y_5, y_6\}$

Match each employee to at most one task, so that different employees end up doing different tasks, in such a way that the maximum number of tasks are performed. Prove that your answer attains the maximum.

4. (20 points) Prove that the number of *unlabeled* trees on n vertices (that is, isomorphism classes of trees on n vertices) is at most $\binom{2n-2}{n-1}$. (Hint: how did we already get an upper bound, in lecture or in the book, on the number of such unlabeled trees?)

5. (20 points) (Problem 7.2.11 on p. 134 of our text)
Prove that a graph G with *no multiple edges and no self-loops* having n vertices and strictly more than $\binom{n-1}{2}$ edges must be connected.