Math 4707 Intro to combinatorics and graph theory
Spring 2017, Vic Reiner
Final exam- Due Wednesday May 3, in class or in my mailbox by 4pm

Instructions: There are 5 problems, worth 20 points each, totaling 100 points. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total) In each part of this problem, a numerical answer with no justification will receive no credit.

(a) (5 points) How many rearrangements are there of the letters in PREPARATION? For example, AAEIONPPRRT is one of them, as is PREPARATION itself, as is TRRPPNOIEAA.

(b) (5 points) How many of the rearrangements in part (a) will have no occurrence of the two letters PP adjacent?

(c) (10 points) How many of the rearrangements in part (a) will have no occurrence of any pair of equal letters adjacent, that is no AA, no PP, and no RR? For example, PREPARATION itself is such a rearrangement.

2. (20 points total) For $n \geq 2$, define a bipartite graph $G_n = (X \sqcup Y, E)$ in which the vertex sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ both have size n, and the edge set

$$E = \{\{x_i, y_j\} : 1 \leq i \leq n \text{ and } 1 \leq j \leq n, \ \textbf{but } i \neq j\},$$

which has size $n^2 - n$.

Describe, with proof, the set of all values $n \geq 2$ for which G_n has ...

(a) (5 points) ... a spanning tree.

(b) (5 points) ... an Eulerian tour (that is, a closed Eulerian walk in our book's terminology).

(c) (5 points) ... a Hamiltonian cycle.

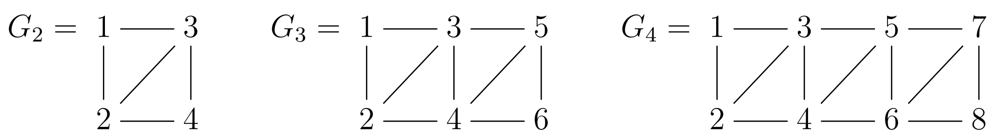
(d) (5 points) ... a **perfect** matching.

3. (20 points) Your friend shows you a 3-dimensional polyhedron whose polygonal faces are all either triangles or hexagons, and with exactly three polygonal faces meeting at each vertex.

Exactly how many triangular faces will it have?

(Hint: find various equations that relate the numbers v, e, f, t, h of vertices, edges, faces, triangular faces, hexagonal faces, respectively.)

4. (20 points) Consider a family of graphs $G_2, G_3, G_4, \ldots$, where G_n has vertices $V = \{1, 2, \ldots, 2n-1, 2n\}$, and edges as shown below



Compute explicitly the *chromatic polynomial* $\chi(G_n, k)$ for ...

(a) (5 points) ... the case $n = 2$, that is G_2.

(b) (5 points) ... the case $n = 3$, that is G_3.

(c) (5 points) ... the general case of G_n for $n \geq 2$.

(d) (5 points) How many acyclic orientations are there for G_n with $n \geq 2$, as a function of n?

5. (20 points) Recall for an undirected (multi-)graph $G = (V, E)$, an *orientation* ω of the edges of G is any of the $2^{\#E}$ different ways to assign to each undirected edge $e = \{v, w\}$ in E one of the two possible orientations $v \to w$ or $w \to$ to make it a directed arc.

Given an orientation ω for G show that the following two properties are equivalent, that is (a) occurs if and only if (b) occurs:

(a) Every directed arc $v_0 \to v_1$ in ω lies in at least one directed cycle $v_0 \to v_1 \to v_2 \to v_3 \to \cdots \to v_{m-1} \to v_m = v_0$ in ω.

(b) For every pair of vertices $\{v, w\}$ in V lying in the same connected component of G (so there exists some undirected path from v to w in G), there exist **both** directed paths $v \to \cdots \to w$ and $w \to \cdots \to v$ in ω.