Math 4707: Introduction to Combinatorics
Midterm 1

Each problem is worth 10 points and answers must be justified to receive full credit. This exam is open notes, book, and web, but you cannot interact with anyone (including online forums). The midterm is due on Monday, February 27th at the beginning of class and late exams will not be accepted (electronic submissions are welcomed). Good luck.

Problem 1. A *standard tableaux* of the partition $\{n,n\}$ is a filling of the $2 \times n$ rectangle such that is strictly increasing across the rows and down columns and the entries are in bijection with $[2n]$. For example, consider $n = 3$, then we have

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}\,.$$

How many standard tableaux of a $2 \times n$ rectangle are there?

Hint: Consider a map to words where the i-th value is N if i is in the upper row and E if i is in the lower row.

Problem 2. Let D_n denote the number of derangements of $[n]$. Show combinatorially (i.e., by using the definition of derangements) that

$$D_{n+1} = n(D_n + D_{n-1}).$$

Problem 3. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} j^k = (j+1)^n$$

for all $j \in \mathbb{Z}_{>0}$.

Problem 4. Consider the partition $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_\ell\}$, where $\lambda_i \in \mathbb{Z}_{>0}$, such that $\sum_{i=1}^{\ell} \lambda_i = n$. Let $P_i = \sum_{j=1}^{i} \lambda_j$ and $P_0 = 0$. Let S_λ denote the subset of $\sigma \in S_n$ such that $\sigma_{P_{i-1}+1} < \sigma_{P_{i-1}+2} < \cdots < \sigma_{P_{i-1}+\lambda_i}$ for all $1 \leq i \leq \ell$. Show that

$$|S_\lambda| = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_\ell!}.$$

Problem 5. The *Narayana number* $N(n,k)$ is defined as the number of Dyck words with exactly k *peaks*, a N step followed by an E step (if you turn the corresponding Dyck path such that the diagonal is the ground, these look like peaks of mountains). Show that

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}.$$

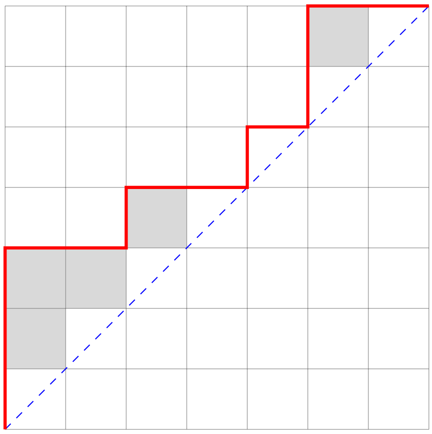
Hint: Recall that we can express $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$, where the right hand side can be thought of as a pair of subsets.

Problem 6. Problem 4.3.16.

Problem 7. (a) Determine a closed form for the number of binary strings of length $2n$ with an even number of 1's.

(b) Compute the number of binary strings of length $2n$ such that there are an even number of 1's and there are no consecutive 1's.

Problem 8. The *area* of a Dyck path is the number of full squares between the Dyck path and the main diagonal. For example:



has area 5, which are the boxes shaded in light gray. Let

$$C_n(q) = \sum_D q^{a(D)},$$

where we sum over all Dyck paths D in an $n \times n$ grid and $a(D)$ is the area of D. Show that

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_{k-1}(q) C_{n-k}(q).$$