Name: ANSWER KEY

Signature: ____________________

Math 5651 (V. Reiner) Final Exam
Tuesday, May 8, 2018

This is a 120 minute exam. No books, notes, calculators, cell phones, watches or other electronic devices are allowed. You can leave answers as fractions, with binomial or multinomial coefficients, and Gamma function values $\Gamma(\alpha)$ unevaluated.

There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Show all of your calculations on this test paper.

Problem	Score
1.	
2.	
3.	
4.	
5.	
6.	
Total:	

Reminders:

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbf{Pr}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$S = \sqcup_{i=1}^{n} B_i \Rightarrow \mathbf{Pr}(A) = \sum_{i=1}^{n} \mathbf{Pr}(A \cap B_i) = \sum_{i=1}^{n} \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i) \text{ and } \mathbf{Pr}(B_i|A) = \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)/\mathbf{Pr}(A)$$

$$\text{cdf } F(x) := \mathbf{Pr}(X \le x), \text{while pdf } f(x) = \frac{\partial}{\partial x}F(x), \text{ and } g_1(x|y) = f(x,y)/f_2(y) \text{ with } f_2(y) = \int_{x=-\infty}^{x=+\infty} f(x,y)dx$$

$$\text{When } \underline{Y} = \underline{r}(\underline{X}) \Leftrightarrow \underline{X} = \underline{s}(\underline{Y}), \text{ then } f(\underline{x}), g(\underline{y}) \text{ satisfy } g(\underline{y}) = f(\underline{s}(y)) \cdot |J| \text{ where } J := \det\left(\frac{\partial s_i}{\partial y_j}\right)$$

$$\mathbf{E}X = \int_{-\infty}^{+\infty} x f(x)dx, \text{ and } \mathrm{var}(X) = E(X - EX)^2 = E(X^2) - (EX)^2, \text{ with } \sigma(X) := +\sqrt{\mathrm{var}(X)}$$

$$\mathrm{cov}(X,Y) = E((X - EX)(Y - EY)) = E(XY) - EX \cdot EY = \sigma_X \sigma_Y \rho(X,Y)$$

$$\mathrm{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{var}(X_i) + 2 \sum_{1 \le i < j \le n} \mathrm{cov}(X_i, X_j)$$

$$\Gamma(\alpha) := \int_{x=0}^{x=\infty} x^{\alpha-1} e^{-x} dx \text{ for } \alpha > 0, \text{ and } \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \text{ for } \alpha > 1, \text{ with } \Gamma(1) = 1$$

Discrete Distributions

	Bernoulli with parameter p	Binomial with parameters n and p
p.f.	$f(x) = p^x(1-p)^{1-x}$, for $x = 0, 1$	$f(x) = \binom{n}{x} p^x(1-p)^{n-x}$, for $x = 0, \dots, n$
Mean	p	np
Variance	$p(1-p)$	$np(1-p)$
m.g.f.	$\psi(t) = pe^t + 1 - p$	$\psi(t) = (pe^t + 1 - p)^n$

	Uniform on the integers $a, \dots, b$	Hypergeometric with parameters A, B, and n
p.f.	$f(x) = \frac{1}{b-a+1}$, for $x = a, \dots, b$	$f(x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$, for $x = \max\{0, n-b\}, \dots, \min\{n, A\}$
Mean	$\frac{b+a}{2}$	$\frac{nA}{A+B}$
Variance	$\frac{(b-a)(b-a+2)}{12}$	$\frac{nAB}{(A+B)^2}\frac{A+B-n}{A+B-1}$
m.g.f.	$\psi(t) = \frac{e^{(b+1)t} - e^{at}}{(e^t-1)(b-a+1)}$	Nothing simpler than $\psi(t) = \sum_x f(x)e^{tx}$

	Geometric with parameter p	Negative binomial with parameters r and p
p.f.	$f(x) = p(1-p)^x$, for $x = 0, 1, \dots$	$f(x) = \binom{r+x-1}{x} p^r(1-p)^x$, for $x = 0, 1, \dots$
Mean	$\frac{1-p}{p}$	$\frac{r(1-p)}{p}$
Variance	$\frac{1-p}{p^2}$	$\frac{r(1-p)}{p^2}$
m.g.f.	$\psi(t) = \frac{p}{1-(1-p)e^t}$, for $t < \log(1/[1-p])$	$\psi(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$, for $t < \log(1/[1-p])$

	Poisson with mean λ	Multinomial with parameters n and $(p_1, \dots, p_k)$
p.f.	$f(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, for $x = 0, 1, \dots$	$f(x_1, \dots, x_k) = \binom{n}{x_1, \dots, x_k} p_1^{x_1} \cdots p_k^{x_k}$, for $x_1 + \cdots + x_k = n$ and all $x_i \geq 0$
Mean	λ	$E(X_i) = np_i$, for $i = 1, \dots, k$
Variance	λ	$\text{Var}(X_i) = np_i(1-p_i)$, $\text{Cov}(X_i, X_j) = -np_ip_j$, for $i, j = 1, \dots, k$
m.g.f.	$\psi(t) = e^{\lambda(e^t-1)}$	Multivariate m.g.f. can be defined, but is not defined in this text.

Continuous Distributions

	Beta with parameters α and β	Uniform on the interval $[a, b]$
p.d.f.	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, for $0 < x < 1$	$f(x) = \frac{1}{b-a}$, for $a < x < b$
Mean	$\frac{\alpha}{\alpha+\beta}$	$\frac{a+b}{2}$
Variance	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$\frac{(b-a)^2}{12}$
m.g.f.	Not available in simple form	$\psi(t) = \frac{e^{-at} - e^{-bt}}{t(b-a)}$

	Exponential with parameter β	Gamma with parameters α and β
p.d.f.	$f(x) = \beta e^{-\beta x}$, for $x > 0$	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$, for $x > 0$
Mean	$\frac{1}{\beta}$	$\frac{\alpha}{\beta}$
Variance	$\frac{1}{\beta^2}$	$\frac{\alpha}{\beta^2}$
m.g.f.	$\psi(t) = \frac{\beta}{\beta-t}$, for $t < \beta$	$\psi(t) = \left(\frac{\beta}{\beta-t}\right)^\alpha$, for $t < \beta$

	Normal with mean μ and variance σ^2	Bivariate normal with means μ_1 and μ_2, variances σ_1^2 and σ_2^2, and correlation ρ
p.d.f.	$f(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	Formula is too large to print here. See Eq. (5.10.2) on page 338.
Mean	μ	$E(X_i) = \mu_i$, for $i = 1, 2$
Variance	σ^2	Covariance matrix: $\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$
m.g.f.	$\psi(t) = \exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$	Bivariate m.g.f. can be defined, but is not defined in this text.

Problem 1. *(20 points)* Assume that each of the functions $f(k)$ or $f(x)$ below defines a p.f. or p.d.f., and find the unknown constant c.

a. (5 points) $f(x) = \begin{cases} cx^{13}(1-x)^{1.5} & \text{for } x \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$

$X = \text{Beta}(\underset{\underset{14}{\parallel}}{\alpha}, \underset{\underset{2.5}{\parallel}}{\beta})$, so $c = \dfrac{\Gamma(14+2.5)}{\Gamma(14)\Gamma(2.5)}$

b. (5 points) $f(k) = c\left(\frac{2}{7}\right) \cdot \left(\frac{5}{7}\right)^k$ for $k = 0, 1, 2, 3, \ldots$.

$X = \text{Geom}(\underset{\underset{2/7}{\parallel}}{p})$, so $c = 1$

c. (5 points) $f(x) = \begin{cases} cx^4 e^{-3x-6} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$

$X = \text{Gamma}(\underset{\underset{5}{\parallel}}{\alpha}, \underset{\underset{3}{\parallel}}{\beta})$, so $ce^{-6} = \dfrac{3^5}{\Gamma(5)}$

$c = \dfrac{3^5 \cdot e^6}{\Gamma(5)}$

d. (5 points) $f(x) = ce^{-3x^2-6}$ for $x \in \mathbb{R}$.

$f(x) = ce^{-6} \cdot e^{-3x^2} = ce^{-6} \cdot e^{-\frac{1}{2}(6x^2)} = ce^{-6} \cdot e^{-\frac{1}{2}\left(\frac{x-0}{1/\sqrt{6}}\right)^2}$

$\Rightarrow X = N(\underset{\underset{0}{\parallel}}{\mu}, \underset{\underset{\frac{1}{\sqrt{6}}}{\parallel}}{\sigma})$, so $ce^{-6} = \dfrac{1}{\sigma\sqrt{2\pi}} = \dfrac{\sqrt{6}}{\sqrt{2\pi}}$

$c = \sqrt{\dfrac{3}{\pi}}\, e^6$

Problem 2. *(20 points total)*
True or False? Some explanation required for each answer.

Assume that $X = \text{Binom}(n_1, p_1)$ and $Y = \text{Binom}(n_2, p_2)$ are any pair of *independent* binomial random variables. Let $Z = X + Y$ be their sum.

a. (5 points) $\mathbf{E}(Z) = n_1p_1 + n_2p_2$.

TRUE, since $\mathbb{E}(Z) = \mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y$
$= n_1p_1 + n_2p_2$

b. (5 points) $Z = \text{Binom}(n_1 + n_2, p_1 + p_2)$.

FALSE, e.g. if $p_1, p_2 > \frac{1}{2}$ then $p_1 + p_2 > \frac{1}{2}$, so $\text{Binom}(n_1+n_2, p_1+p_2)$ doesn't even make sense.

c. (5 points) $\text{var}(Z) = n_1p_1(1 - p_1) + n_2p_2(1 - p_2)$.

TRUE, since $\text{var}(Z) = \text{var}(X+Y) \overset{X,Y\ \underline{\text{independent}}}{=} \text{var}(X) + \text{var}(Y)$
$= n_1p_1(1-p_1) + n_2p_2(1-p_2)$

d. (5 points) The correlation $\rho(X, Y) = 0$.

TRUE, since X, Y <u>independent</u>.

Problem 3. *(20 points)* Let X have pdf $f(x) = \begin{cases} 5e^{-5x} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$

a. (5 points) Compute an explicit formula for the cdf $F(x)$ of X.

$$F(x) = Pr(X \le x) = \int_{t=-\infty}^{t=x} f(t)\,dt = \begin{cases} \int_{t=0}^{t=x} 5e^{-5t}\,dt = \left[-e^{-5t}\right]_{t=0}^{t=x} = 1 - e^{-5x} & \text{if } x \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

b. (5 points) What choice of an estimate m_0 for X will minimize the mean square error $\mathbf{E}((X - m_0)^2)$?

$m_0 = \mathbb{E}X = \frac{1}{5}$ since $X = Exp(\underset{5}{\overset{}{\beta}})$

c. (5 points) What choice of an estimate m_1 for X will minimize the mean absolute error $\mathbf{E}(|X - m_1|)$? m_1 = any median for X,

i.e. $\frac{1}{2} = F(m_1) = 1 - e^{-5m_1} \Rightarrow e^{-5m_1} = \frac{1}{2}$

$-5m_1 = \log(\frac{1}{2})$

$m_1 = \frac{\log(2)}{5}$

d. (5 points) Let $Y = \frac{X_1+X_2+X_3+X_4+X_5}{5}$ be the sample mean for five *independent* samples from the random variable X, that is, X_1, X_2, X_3, X_4, X_5 are i.i.d., with the same distribution as X. Compute an explicit formula for the moment generating function $\psi_Y(t)$ of Y.

$$\psi_Y(t) = \psi_{\frac{1}{5}(X_1+\dots+X_5)}(t)$$

$$= \psi_{X_1+\dots+X_5}\left(\frac{t}{5}\right)$$

independence

$$\overset{\downarrow}{=} \psi_{X_1}\left(\frac{t}{5}\right)\psi_{X_2}\left(\frac{t}{5}\right)\psi_{X_3}\left(\frac{t}{5}\right)\psi_{X_4}\left(\frac{t}{5}\right)\psi_{X_5}\left(\frac{t}{5}\right)$$

$$= \psi_X\left(\frac{t}{5}\right)^5 = \left(\frac{5}{5-\frac{t}{5}}\right)^5 = \left(\frac{25}{25-t}\right)^5$$

$X = Exp(5)$

Problem 4. *(15 points total)* A group of n restaurant patrons whose names are Person 1, Person 2, ..., Person n each give their hat to the hat-check attendant. Later, the attendant gives them each back a hat, uniformly at random, that is, with all distributions equally likely.

a. (5 points) What is the probability (as a function of n) that Persons $1, 2, 3$ end up with a *three-cycle* of each other's hats, that is, either

- 1 gets 2's hat, and 2 gets 3's hat, and 3 gets 1's hat, or
- the reverse, that is, 2 gets 1's hat, and 3 gets 2's hat, and 1 gets 3's hat?

Let $A_{\{i,j,k\}}$ be the event that $\{i,j,k\}$ get their hats' three-cycled.

Then $$\Pr(A_{\{1,2,3\}}) = \frac{|A_{\{1,2,3\}}|}{|S|} = \frac{(n-3)! \cdot 2}{n!} = \frac{2}{n(n-1)(n-2)}$$

distribute hats to Persons $4,5,6,\ldots,n$ arbitrarily

pick $1 \to 2 \to 3 \to 1$ versus $1 \to 3 \to 2 \to 1$

b. (10 points) Let X denote the random variable which counts the number of unordered triples $\{i, j, k\}$ of $\{1, 2, \ldots, n\}$ for which Persons i, j, k end up with a three-cycle of each other's hats. Compute the expected value $\mathbb{E}X$.

Note $$X = \sum_{\substack{\text{triples} \\ \{i,j,k\} \subset \{1,2,\ldots,n\}}} X_{\{i,j,k\}} \quad \text{where } X_{\{i,j,k\}} = \begin{cases} 1 & \text{if } A_{\{i,j,k\}} \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\underset{\substack{\text{linearity} \\ \text{of} \\ \text{expectation}}}{\Longrightarrow} \quad \mathbb{E}X = \sum_{\{i,j,k\}} \mathbb{E}(X_{\{i,j,k\}})$$

$$= \sum_{\{i,j,k\}} \Pr(A_{\{i,j,k\}}) = \binom{n}{3} \cdot \Pr(A_{\{1,2,3\}})$$

$$= \binom{n}{3} \cdot \frac{2}{n(n-1)(n-2)}$$

$$= \frac{n(n-1)(n-2)}{3!} \cdot \frac{2}{n(n-1)(n-2)} = \frac{1}{3}$$

Problem 5. *(10 points)* Among dog breeds, assume that the weight of the average beagle is 20 pounds, with standard deviation 3 pounds, approximately following a normal distribution, and that the average pug weighs 10 pounds, with standard deviation 2 pounds, also approximately following a normal distribution.

I pick 5 beagles at random (independently) and compute the average of their weights, then independently pick 7 pugs at random (also independently), and average their weights.

Express the probability that the average weight for the 5 beagles is higher than the average weight for the 7 pugs, in terms of the cdf $\Phi(z)$ for a standard normal random variable $Z = \mathcal{N}(0,1)$.

Let B_i = weight of i^{th} beagle, so $B_i \approx N(\overset{\mu}{\overset{\|}{20}}, \overset{\sigma}{\overset{\|}{3}})$ for $i=1,2,3,4,5$

P_i = weight of i^{th} pug, so $P_i \approx N(10, 2)$ for $i=1,2,3,4,5,6,7$

Then $\Pr\left(\frac{1}{5}(B_1+\dots+B_5) > \frac{1}{7}(P_1+\dots+P_7)\right)$

$$= \Pr\Big(\underbrace{\tfrac{1}{5}(B_1+\dots+B_5) - \tfrac{1}{7}(P_1+\dots+P_7)}_{\text{call this } X;} > 0\Big) = \Pr(X>0)$$

$X \approx N(\mu,\sigma)$ where $\mu = \mathbb{E}X = \frac{1}{5}(\underbrace{20+\dots+20}_{5 \text{ times}}) - \frac{1}{7}(\underbrace{10+\dots+10}_{7 \text{ times}})$

$= 20 - 10 = 10$

and $\sigma^2 = var(X) = var\left(\frac{1}{5}\sum_{i=1}^{5} B_i - \frac{1}{7}\sum_{i=1}^{7} P_i\right)$

$= \frac{1}{5^2}\left(\sum_{i=1}^{5} 3^2\right) + \frac{1}{7^2}\left(\sum_{i=1}^{7} 2^2\right)$

$= \frac{9}{5} + \frac{4}{7} = \frac{83}{35}$

so $\sigma = \sqrt{\frac{83}{35}}$

So $\Pr(X>0) = \Pr\Big(\underbrace{\frac{X-10}{\sqrt{\frac{83}{35}}}}_{\substack{Z \approx N(0,1) \\ \text{standard normal}}} > \frac{-10}{\sqrt{\frac{83}{35}}}\Big) \approx 1 - \Pr\left(Z \le \frac{-10}{\sqrt{\frac{83}{35}}}\right) = 1 - \Phi\left(\frac{-10}{\sqrt{\frac{83}{35}}}\right)$

Problem 6. *(15 points)* Let $X = \text{Exp}(\beta_1)$ and $Y = \text{Exp}(\beta_2)$ be two *independent* exponential random variables. Prove that, for any positive real constant c, one has $\Pr(Y \leq cX) = \dfrac{c\beta_2}{\beta_1 + c\beta_2}$.

(X,Y) have joint pdf $f(x,y) \underset{\text{independence}}{=} f_1(x) f_2(y) = \begin{cases} \beta_1 e^{-\beta_1 x} \cdot \beta_2 e^{-\beta_2 y} & \text{if } x,y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

and hence

$$\Pr(X \leq cY) = \int_{x=0}^{x=+\infty} \int_{y=0}^{y=cx} \beta_1 \beta_2 e^{-\beta_1 x} e^{-\beta_2 y} \, dy \, dx$$

$$= \beta_1 \int_{x=0}^{x=+\infty} e^{-\beta_1 x} \left[-e^{-\beta_2 y} \right]_{y=0}^{y=cx} dx$$

$$= \beta_1 \int_{x=0}^{x=+\infty} e^{-\beta_1 x} \left(1 - e^{-c\beta_2 x}\right) dx$$

$$= \beta_1 \int_{x=0}^{x=+\infty} \left(e^{-\beta_1 x} - e^{-(\beta_1 + c\beta_2)x}\right) dx$$

$$= \beta_1 \left(\left[\frac{-1}{\beta_1} e^{-\beta_1 x} \right]_0^\infty - \left[\frac{-1}{\beta_1 + c\beta_2} e^{-(\beta_1 + c\beta_2)x} \right]_0^\infty \right)$$

$$= \beta_1 \left(-\frac{1}{\beta_1} [0 - 1] + \frac{1}{\beta_1 + c\beta_2} [0 - 1] \right)$$

$$= 1 - \frac{\beta_1}{\beta_1 + c\beta_2}$$

$$= \frac{c\beta_2}{\beta_1 + c\beta_2}$$