Name: Vic Reiner - ANSWER KEY

Signature: ______

Math 5651 Lecture 002 (V. Reiner) Midterm Exam I
Thursday, February 25, 2016

This is a 115 minute exam. No books, notes, calculators, cell phones, watches or other electronic devices are allowed. You can leave answers as fractions, with binomial or multinomial coefficients unevaluated.

There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	10/10
2.	15/15
3.	15/15
4.	15/15
5.	15/15
6.	15/15
7.	15/15
Total:	100/100

Reminders:

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbf{Pr}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$S = B_1 \sqcup \cdots \sqcup B_n \Rightarrow \mathbf{Pr}(A) = \sum_{i=1}^{n} \mathbf{Pr}(A \cap B_i) = \sum_{i=1}^{n} \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)$$

and Bayes Theorem $\quad \mathbf{Pr}(B_i|A) = \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)/\mathbf{Pr}(A)$

$$\mathbf{E}X = \sum_k k \cdot f(k) \quad \text{for a discrete random variable with p.f. } f(k)$$

$$X = \mathrm{Bin}(n,p) \text{ has p.f. } f(k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ for } k \in \{0,1,2,\ldots,n\}$$

$$X = \mathrm{Hypergeom}(A,B,n) \text{ has p.f. } f(k) = \binom{A}{k}\binom{B}{n-k}/\binom{A+B}{n} \text{ for } k \in \{0,1,2,\ldots,\min\{A,n\}\}$$

$$X = \mathrm{Poi}(\lambda) \text{ has p.f } f(k) = e^{-\lambda}\frac{\lambda^k}{k!} \text{ for } k \in \{0,1,2,\ldots\}$$

Problem 1. *(10 points)* When rolling a fair 6-sided dice having 1, 2, 3, 4, 5, 6 on its sides n times, consider the two events A, B where A is rolling a number 3 or less every time, and B is rolling the same number every time. Are A and B dependent or independent? You must support your answer with calculations to receive any credit.

(4/10 for ignoring n and doing only $n=1$)

(8/10 for picking some $n \neq 1$)

$$Pr(A) = \left(\frac{3}{6}\right)^n \left(= \left(\frac{1}{2}\right)^n\right) \quad \text{(3pts)}$$

$$Pr(B) = \sum_{i=1}^{6} Pr(\text{rolling } i \text{ every time}) = 6 \cdot \left(\frac{1}{6}\right)^n \quad \text{(3pts)}$$

$$\Rightarrow Pr(A)Pr(B) = 6 \cdot \left(\frac{1}{6}\right)^n \cdot \left(\frac{3}{6}\right)^n = \frac{6}{12^n}$$

$$Pr(A \cap B) = \sum_{i=1}^{3} Pr(\text{rolling } i \text{ every time}) = 3\left(\frac{1}{6}\right)^n = \frac{3}{6^n}$$

A, B Independent $\Leftrightarrow$ (4 pts) $Pr(A \cap B) = Pr(A)Pr(B) \Leftrightarrow \frac{3}{6^n} = \frac{6}{12^n} \Leftrightarrow \frac{3}{6} = \left(\frac{6}{12}\right)^n \Leftrightarrow \frac{1}{2} = \left(\frac{1}{2}\right)^n \Leftrightarrow n = 1$

(or $Pr(A|B) = Pr(A)$)

Problem 2. *(15 points)* If I choose a rearrangement of the 11 letters in the word "MISSISSIPPI" into a possibly nonsensical string of 11 letters, with all rearrangements equally likely, then what is the probablity that *at least* one (and possibly more than one) of the following three events occurs?:

- The four letters "SSSS" appear all adjacent. ← (call this event S)
- The four letters "IIII" appear all adjacent. ← (— " — I)
- The two letters "PP" appear adjacent. ← (— " — P)

MISSISSIPPI has
4 S's
4 I's
2 P's
1 M's

Want $Pr(S \cup I \cup P) = Pr(S) + Pr(I) + Pr(P) - \left(Pr(S \cap I) + Pr(S \cap P) + Pr(I \cap P)\right) + Pr(S \cap I \cap P)$

$$= \frac{1}{\binom{11}{4\,4\,2\,1}} \left[\binom{8}{\underset{\text{SSSS as a single letter}}{1}\,4\,2\,1} + \binom{8}{4\,\underset{\text{IIII as a single letter}}{1}\,2\,1} + \binom{10}{4\,4\,\underset{\text{PP as a single letter}}{1}\,1} - \left(\binom{5}{\underset{\text{SSSS}}{1}\,\underset{\text{IIII}}{1}\,2\,1} + \binom{7}{\underset{\text{SSSS}}{1}\,4\,\underset{\text{PP}}{1}\,1} + \binom{7}{4\,\underset{\text{IIII}}{1}\,\underset{\text{PP}}{1}\,1} \right) + \binom{4}{\underset{\text{SSSS}}{1},\underset{\text{IIII}}{1},\underset{\text{PP}}{1},1} \right]$$

(5 pts) (5 pts) (5 pts)

8/15 for only calculating
$Pr(S)$
$Pr(I)$
$Pr(P)$
separately
or saying $Pr(S \cup I \cup P) = Pr(S) + Pr(I) + Pr(P)$

Problem 3. *(15 points)* Your friend Bill Shakespeare tells you that he has written a new play, either a comedy or a tragedy. However, you know that whenever he says this, it is *always* written by someone else, a ghost-writer which is either Chris Marlowe, Frank Bacon, or Benny Jonson, each with their own fixed frequencies for writing tragedies versus comedies:

	Marlowe	Bacon	Jonson
fraction of time used as ghost-writer	1/2	1/4	1/4
% chance of writing a tragedy	90%	50%	20%

a. (5 points) Before he shows you the play, what is the probability that it is a comedy?

$$Pr(\text{comedy}) = Pr(\text{comedy}|\text{Marlowe})Pr(\text{Marlowe}) + Pr(\text{comedy}|\text{Bacon})Pr(\text{Bacon}) + Pr(\text{comedy}|\text{Jonson}) \cdot Pr(\text{Jonson})$$

$$= (0.10)\cdot\left(\tfrac{1}{2}\right) + (0.50)\left(\tfrac{1}{4}\right) + (0.80)\left(\tfrac{1}{4}\right)$$

$$\left(= 0.375 = \tfrac{3}{8}\right)$$

b. (10 points) You read it, and it is a comedy. What is the probability that Jonson wrote it?

$$Pr(\text{Jonson}|\text{comedy}) = \frac{Pr(\text{comedy}|\text{Jonson})Pr(\text{Jonson})}{Pr(\text{comedy})}$$

$$= \frac{(0.80)\left(\frac{1}{4}\right)}{(0.10)\left(\frac{1}{2}\right) + (0.50)\left(\frac{1}{4}\right) + (0.80)\left(\frac{1}{4}\right)} \quad \left(= \tfrac{8}{15}\right)$$

($\frac{8}{10}$ for switching "comedy" to "tragedy")

Problem 4. *(15 points total)* A group of 14 women and 8 men people pair off to make 11 pairs of swimming buddies. If all possible pairings equally likely, what is the probability that all the pairs are single-sex, that is, woman-woman or man-man?

$S = \{\text{all pairings}\}$

$A = \{\text{all single-sex pairings}\}$

$$\Pr(A) = \frac{|A|}{|S|} = \frac{\overbrace{(13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1)}^{\#\text{ways to pair off the women}}\overbrace{(7\cdot 5\cdot 3\cdot 1)}^{\#\text{ways to pair off the men}}}{21\cdot 19\cdot 17\cdot 15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1} \quad \left(= \frac{7\cdot 5\cdot 3\cdot 1}{21\cdot 19\cdot 17\cdot 15}\right)$$

(5pts)

8/15 for $\dfrac{\binom{14}{7}\binom{8}{4}}{\binom{22}{11}}$ or $\dfrac{\binom{14}{2,2,2,2,2,2,2}\binom{8}{2,2,2,2}}{\binom{22}{2,2,2,2,2,2,2,2,2,2,2}}$

Problem 5. *(15 points total)* Prove that if the events A, B, C are (jointly/mutually, not just pairwise) independent, then the events A^c and $B \cup C$ are also independent.

$$\Pr(A^c \cap (B\cup C)) \overset{(3\text{pts})}{=} \Pr((A^c\cap B) \cup (A^c \cap C))$$

$$\overset{(3\text{pts})}{=} \Pr(A^c\cap B) + \Pr(A^c \cap C) - \Pr(\underbrace{(A^c\cap B)\cap(A^c\cap C)}_{=A^c\cap B\cap C})$$

$$\overset{(3\text{pts})}{=} \Pr(B) - \Pr(A\cap B) + \Pr(C) - \Pr(A\cap C) - (\Pr(A) - \Pr(A\cap B\cap C))$$

A, B, C independent (3pts)

$$= \Pr(B) - \Pr(A)\Pr(B) + \Pr(C) - \Pr(A)\Pr(C) - \Pr(A) + \Pr(A)\Pr(B)\Pr(C)$$

$\cancel{= \Pr(A)(1 - \Pr(B) + 1 - \Pr(C) - 1 + \Pr(B)\Pr(C))}$

$\cancel{= \Pr(A)(1 - \Pr(B) - \Pr(C) + \Pr(B)\Pr(C))}$

$\cancel{= \Pr(A)(1 - \Pr(B\cup C))}$

SAME!

(3pts) versus

$$\Pr(A^c)\cdot\Pr(B\cup C) = (1 - \Pr(A))(\Pr(B) + \Pr(C) - \underbrace{\Pr(B\cap C)}_{=\Pr(B)\Pr(C)})$$

$$= \Pr(B) + \Pr(C) - \Pr(B)\Pr(C) - \Pr(A)\Pr(B) - \Pr(A)\Pr(C) + \Pr(A)\Pr(B)\Pr(C)$$

Hence $\Pr(A^c \cap (B\cup C)) = \Pr(A^c)\Pr(B\cup C)$

Problem 6. *(15 points total)* You have two coins in your pocket, one *fair* coin having probability of heads 1/2, and one *unfair* coin in which the probability of heads is 2/3. You pull one of the two out of your pocket, with either coin equally probable, and start flipping it, generating a sequence of heads and tails.

(0/5 for using $7/12 = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{2}{3}$ as head probability)

(4/5 for changing "at most" to "exactly")

(1/5 for recognizing $B(n,p)$ is relevant)

a. (5 points) What is the probability that in a total of 10 flips you get at most 8 heads?

$$Pr(\leq 8 \text{ heads}) \overset{(2pts)}{=} 1 - Pr(\geq 9 \text{ heads})$$

$$\overset{(1pts)}{=} 1 - \left[Pr(\geq 9 \text{ heads} \mid \text{fair coin}) \cdot Pr(\text{fair coin}) + Pr(\geq 9 \text{ heads} \mid \text{unfair coin}) Pr(\text{unfair coin})\right]$$

$$\overset{(2pts)}{=} 1 - \left[\left(\binom{10}{9}\left(\tfrac{1}{2}\right)^{10} + \binom{10}{10}\left(\tfrac{1}{2}\right)^{10}\right)\left(\tfrac{1}{2}\right) + \left(\binom{10}{9}\left(\tfrac{2}{3}\right)^{9}\left(\tfrac{1}{3}\right)^{1} + \binom{10}{10}\left(\tfrac{2}{3}\right)^{10}\right)\left(\tfrac{1}{2}\right)\right]$$

$$\left(= 1 - \tfrac{1}{2}\left(10\left(\tfrac{1}{2^{10}}\right) + 1\cdot\left(\tfrac{1}{2^{10}}\right) + 10\left(\tfrac{2}{3}\right)^{9}\left(\tfrac{1}{3}\right) + \left(\tfrac{2}{3}\right)^{10}\right)\right)$$

b. (5 points) What is the probability that the first tails occurs exactly on the 5^{th} flip?

$$Pr(1^{st} \text{ tails on } 5^{th} \text{ flip}) \overset{(2pts)}{=} Pr(1^{st} \text{ tails on } 5^{th} \text{ flip} \mid \text{fair coin}) Pr(\text{fair coin}) + Pr(1^{st} \text{ tails on } 5^{th} \text{ flip} \mid \text{unfair coin}) Pr(\text{unfair coin})$$

$$= Pr((H,H,H,H,T) \mid \text{fair coin}) Pr(\text{fair coin}) + Pr((H,H,H,H,T) \mid \text{unfair coin}) Pr(\text{unfair coin})$$

$$= \left(\tfrac{1}{2^5}\right)\left(\tfrac{1}{2}\right) + \left(\left(\tfrac{2}{3}\right)^{4}\left(\tfrac{1}{3}\right)^{1}\right)\cdot\left(\tfrac{1}{2}\right) \quad \left(= \tfrac{1}{2^6} + \tfrac{2^3}{3^5}\right)$$

(3pts)

c. (5 points) After you pull it out of your pocket, you do one test flip, and get tails. What is the probability that you pulled the fair coin out of your pocket?

$$Pr(\text{fair coin} \mid \text{tails}) \overset{(3pts)}{=} \frac{Pr(\text{tails} \mid \text{fair coin}) Pr(\text{fair coin})}{Pr(\text{tails} \mid \text{fair coin}) Pr(\text{fair coin}) + Pr(\text{tails} \mid \text{unfair coin}) Pr(\text{unfair coin})}$$

$$= \frac{\left(\tfrac{1}{2}\right)\cdot\left(\tfrac{1}{2}\right)}{\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3}\right)\cdot\left(\tfrac{1}{2}\right)} \quad \left(= \frac{\tfrac{1}{4}}{5/12} = \frac{3}{5}\right)$$

Problem 7. *(15 points)* Assume $X = Poi(\lambda)$ is a Poisson random variable with mean λ, and let $Y = X^2$, so that Y only takes on the values k^2 for $k = 0, 1, 2, \ldots$, and

$$\mathbf{Pr}(Y = k^2) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

What is the expected value $\mathbf{E}(Y)$ of Y?

(Hint: I think it helps to rewrite $k^2 = k(k-1) + k$.)

$$E(Y) = \sum_{k=0}^{\infty} k^2 \cdot f(k^2) \qquad \text{where } f(k^2) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$$= \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda}\frac{\lambda^k}{k!}$$

$$= \sum_{k=0}^{\infty} \big(k(k-1) + k\big) e^{-\lambda}\frac{\lambda^k}{k!}$$

(5pts)

$$= e^{-\lambda}\left[\sum_{k=0}^{\infty} k(k-1)\cdot\frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} k\cdot\frac{\lambda^k}{k!}\right]$$

$k(k-1) = 0$ when $k = 0, 1$ and $k = 0$ when $k = 0$ (!)

$$= e^{-\lambda}\left[\sum_{k=2}^{\infty} k(k-1)\cdot\frac{\lambda^k}{k!} + \sum_{k=1}^{\infty} k\cdot\frac{\lambda^k}{k!}\right]$$

(5pts)

$$= e^{-\lambda}\left[\lambda^2\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!} + \lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}\right]$$

$$= e^{-\lambda}\left[\lambda^2\cdot e^{\lambda} + \lambda\cdot e^{\lambda}\right]$$

(5pts)

$$= \lambda^2 + \lambda$$

$\left(\frac{1}{15}\right.$ for "$E(Y) = \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda}\frac{\lambda^{k^2}}{(k^2)!}$" which is ~~false~~. $\left.\right)$