Name: ______________________________

Signature: ______________________________

Math 5651 Lecture 001 (V. Reiner) Midterm Exam I
Thursday, February 22, 2018

This is a 115 minute exam. No books, notes, calculators, cell phones, watches or other electronic devices are allowed. You can leave answers as fractions, with binomial or multinomial coefficients unevaluated.

There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	________
2.	________
3.	________
4.	________
5.	________
6.	________
Total:	________

Reminders:

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbf{Pr}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$S = B_1 \sqcup \cdots \sqcup B_n \Rightarrow \mathbf{Pr}(A) = \sum_{i=1}^{n} \mathbf{Pr}(A \cap B_i) = \sum_{i=1}^{n} \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i),$$

and Bayes' Theorem: $\mathbf{Pr}(B_i|A) = \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)/\mathbf{Pr}(A)$

$$X = \mathrm{Bin}(n, p) \text{ has p.f. } f(k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ for } k \in \{0, 1, 2, \ldots, n\}$$

$$X = \mathrm{Hypergeom}(A, B, n) \text{ has p.f. } f(k) = \binom{A}{k}\binom{B}{n-k} \Big/ \binom{A+B}{n} \text{ for } k \in \{0, 1, 2, \ldots, \min\{A, n\}\}$$

$$X = \mathrm{Poi}(\lambda) \text{ has p.f } f(k) = e^{-\lambda} \frac{\lambda^k}{k!} \text{ for } k \in \{0, 1, 2, \ldots\}$$

Problem 1. *(10 points)*

a. (3 points) You flip a fair coin 4 times. What is the probability that you never see two of the same flip consecutively, that is, no "heads, heads" and no "tails, tails"?

$$\Pr(\text{HTHT or THTH}) = \frac{1}{2^4} + \frac{1}{2^4} = \frac{1}{2^3}$$

b. (7 points) Same question, except that this time you flip the coin n times, where $n \geq 2$. (Your answer for the probability should be a function of n.)

$$\Pr(\text{HTHT}\cdots \text{ or THTH}\cdots) = \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}}$$

Problem 2. *(15 points)* You know that the number of lightning strikes in a certain area during a month is a Poisson random variable with parameter λ.

a. (5 points) What is the probability that next month there are exactly 3 lightning strikes? (The parameter λ will appear in your answer.)

$$X = \#\text{ house fires} = \text{Poi}(\lambda) \Rightarrow \Pr(X=3) = e^{-\lambda} \cdot \frac{\lambda^3}{3!}$$

b. (10 points) If you know that last month there was *at least one* lightning strike, then what is the probability that there were between 2-4 lightning strikes?

$$\Pr(X \in \{2,3,4\} \mid X \geq 1) = \frac{\Pr(X \in \{2,3,4\} \text{ and } X \geq 1)}{\Pr(X \geq 1)}$$

$$= \frac{\Pr(X \in \{2,3,4\})}{1 - \Pr(X=0)} = \frac{e^{-\lambda}\left(\frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!}\right)}{1 - e^{-\lambda}}$$

Problem 3. *(15 points)* Your favorite radio station chooses its morning disc jockey randomly among four possibilities: Charlotte, Emily, Anne, who each work 2 mornings per week on average, and Bramwell who works 1 morning per week on average. Charlotte, Emily and Anne all like to play classical music $\frac{1}{3}$ of the time, and non-classical music the other $\frac{2}{3}$, while Bramwell plays *only* classical music.

$Ch = \{\text{Charlotte is DJ}\}$
$E = \{\text{Emily is DJ}\}$
$A = \{\text{Anne is DJ}\}$
$B = \{\text{Bramwell is DJ}\}$
$Cl = \{\text{Classical playing}\}$

a. (5 points) What is the *a priori* chance that, when you wake up tomorrow morning, Charlotte will be the disc jockey?

$$Pr(Ch) = \frac{2}{7} \left(= \frac{|Ch|}{|S|} \right)$$

b. (10 points) If you wake up and hear classical music playing, what is the chance that Bramwell is the disc jockey that morning?

$$Pr(B|Cl) \overset{\text{Bayes}}{=} \frac{Pr(Cl|B)\,Pr(B)}{Pr(Cl|B)Pr(B) + Pr(Cl|A)Pr(A) + Pr(Cl|E)Pr(E) + Pr(Cl|Ch)Pr(Ch)}$$

$$= \frac{1 \cdot \frac{1}{7}}{1 \cdot \frac{1}{7} + \frac{1}{3} \cdot \frac{2}{7} + \frac{1}{3} \cdot \frac{2}{7} + \frac{1}{3} \cdot \frac{2}{7}} = \frac{1}{3}$$

Problem 4. *(20 points total)*
True or False? Each of your answers **must be justified by calculation**.

a. (5 points) Given events A, B with $\mathbf{Pr}(A), \mathbf{Pr}(B) > 0$, if $A \cap B = \varnothing$, then A, B are independent.

FALSE : $0 = Pr(\underbrace{A \cap B}_{\varnothing}) \neq \underbrace{Pr(A)}_{>0}\underbrace{Pr(B)}_{>0} > 0$

b. (5 points) Given an event A with $0 < \mathbf{Pr}(A) < 1$, and if we let $B = A$, then A, B are dependent.

TRUE : $Pr(A \cap B) \neq Pr(A)\,Pr(B)$

$Pr(A \cap B) = Pr(A)$, $Pr(A)\,Pr(B) = Pr(A)^2$ ($\neq Pr(A)$ since $0 < Pr(A) < 1$)

c. (5 points) When sampling **without replacement** $r > 0$ red and $w > 0$ white balls from a box, let R_i, W_i, respectively, be the events that the i^{th} ball sampled is red, white, respectively.

Then $\mathbf{Pr}(R_2|W_1) > \mathbf{Pr}(R_2) > \mathbf{Pr}(R_2|R_1)$.

TRUE: $\frac{r}{r+w-1} > \frac{r}{r+w} > \frac{r-1}{r+w-1}$

Did this in homework & in lecture:

$$Pr(R_2) = Pr(R_2|R_1)Pr(R_1) + Pr(R_2|W_1)Pr(W_1) = \frac{r-1}{r+w-1}\cdot\frac{r}{r+w} + \frac{r}{r+w-1}\cdot\frac{w}{r+w} = \frac{r}{r+w}$$

d. (5 points) With notation as in part (c), events W_1 and R_2 are independent.

FALSE: $Pr(R_2|W_1) \neq Pr(R_2)$

$Pr(R_2|W_1) = \frac{r}{r+w-1}$, $Pr(R_2) = \frac{r}{r+w}$

Problem 5. *(20 points)* Let A, B, C be events with $\mathbf{Pr}(C) > 0$. Prove that $\mathbf{Pr}(A \cup B|C) = \mathbf{Pr}(A|C) + \mathbf{Pr}(B|C) - \mathbf{Pr}(A \cap B|C)$.

definition of conditional probability

$$\Pr(A \cup B|C) = \frac{\Pr((A \cup B) \cap C)}{\Pr(C)}$$

$(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$

$$= \frac{\Pr((A \cap C) \cup (B \cap C))}{\Pr(C)}$$

2-set inclusion-exclusion

$$= \frac{\Pr(A \cap C) + \Pr(B \cap C) - \Pr((A \cap C) \cap (B \cap C))}{\Pr(C)}$$

$$= \frac{\Pr(A \cap C)}{\Pr(C)} + \frac{\Pr(B \cap C)}{\Pr(C)} - \frac{\Pr(A \cap B \cap C)}{\Pr(C)}$$

definition of conditional probability

$$= \Pr(A|C) + \Pr(B|C) - \Pr(A \cap B|C)$$

Problem 6. *(20 points total)* You have two coins in your pocket, one *fair* coin having probability of heads 1/2, and one *unfair* coin in which the probability of heads is 2/3. You pull one of the two out of your pocket, with either coin equally probable, and start flipping it, generating a sequence of heads and tails.

$F = \{\text{fair coin pulled out}\}$

$U = \{\text{unfair coin pulled out}\}$

$S = \{\text{1st tails on 7th flip}\}$

a. (10 points) What is the probability that the *first* tails is on the 7^{th} flip?

Law of total probability

$$\Pr(S) = \Pr(S|F)\Pr(F) + \Pr(S|U)\Pr(U)$$

$$= \underbrace{\left(\tfrac{1}{2}\right)^6\left(\tfrac{1}{2}\right)}_{HHHHHHT}\cdot\left(\tfrac{1}{2}\right) + \underbrace{\left(\tfrac{2}{3}\right)^6\left(\tfrac{1}{3}\right)}_{HHHHHHT}\cdot\left(\tfrac{1}{2}\right)$$

$$= \frac{1}{2^8} + \frac{1}{6}\cdot\left(\frac{2}{3}\right)^6$$

b. (10 points) After you pull it out of your pocket, you do one test flip, and get heads. What is the probability that you pulled the fair coin out?

$H = \{\text{heads in test flip}\}$

$$\Pr(F|H) \overset{\text{Bayes}}{=} \frac{\Pr(H|F)\Pr(F)}{\Pr(H|F)\Pr(F) + \Pr(H|U)\Pr(U)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{2}} = \frac{3}{7}$$