Math 5705 Undergraduate enumerative combinatorics
Spring 2002, Vic Reiner
Midterm exam 3- Due Wednesday April 24, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are not allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. Let $B_n$ denote the set of permutations $\pi$ in the symmetric group $\mathfrak{S}_n$ on $n$ elements which satisfy $\pi^4 := \pi \circ \pi \circ \pi \circ \pi = 1$.
   (a) (10 points) Characterize the elements of $B_n$ in terms of their cycle structure, that is, give an equivalent definition of the elements of $B_n$ in terms of the sizes of their cycles.
   (b) (15 points) Let $b_n = |B_n|$, and express the exponential generating function $B(x) := \sum_{n \geq 0} b_n \frac{x^n}{n!}$ as an elementary function.

2. (20 points) Tucker, Section 8.2, Problem #34 on page 327.

3. (30 points) Tucker, Section 8.3, Problem #12 on page 338.
   (Each part is worth 15 points.)

4. (25 points) Use inclusion-exclusion to give a formula for the number of cyclic permutations $\pi$ in $\mathfrak{S}_n$ (that is, permutations of $[n] := \{1, 2, \ldots, n\}$ that have only one cycle of size $n$) with the property that $\pi(i) \neq i + 1 \mod n$ for all $i$. In other words,
   \[ \pi(1) \neq 2, \pi(2) \neq 3, \ldots, \pi(n - 1) \neq n, \text{ and } \pi(n) \neq 1. \]
   Your final answer should be in the form $\sum_k f(k)$ where $k$ ranges over some integers and $f(k)$ is some function of $k$. 