Math 5705 Undergraduate enumerative combinatorics
Spring 2005, Vic Reiner
Final exam - Due Wednesday May 4, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult. Explain your reasoning- answers without justification or proof (where appropriate) will receive **no credit**.

1. (20 points total) Let π be a permutation of $[n] := \{1, 2, \ldots, n\}$ with the property that π^6 is the identity permutation.

(a) (10 points) What possible cycle sizes can occur in the cycle notation for such a π? Explain why.

(b) (10 points) Let a_n denote the number of such permutations π of $[n]$. Find an expression for the exponential generating function $f(x) := \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ as some simple transcendental function, involving no summations or products.

2. (15 points) Chapter 4, Supplementary Problem #8 on page 99. Your answer should be a simple explicit function of n that involves no summations. Also, I would ignore the comment about it being "Relevant to Appendix C"; it would seem more appropriate for it to have said "(Ordinary) generating functions are relevant".

3. (25 points total) For a positive integer m, Euler's φ-*function* $\varphi(m)$ is defined to be the number of integers in $[m] := \{1, 2, \ldots, m\}$ that are relatively prime to m, that is, which have no prime factors in common with m. For example, $\varphi(12) = 4$, since the subset of integers in $[12]$ relatively prime to 12 is $\{1, 5, 7, 11\}$.

(a) (5 points) Compute $\varphi(49, 000, 000)$ using inclusion-exclusion.
(Hint: Let $A := [49, 000, 000]$ and let A_1, A_2, and A_3 be the subsets of A consisting of those which are divisible by $2, 5$, and 7 respectively.)

(b) (15 points) Let $m = p_1^{e_1} \cdots p_n^{e_n}$ with p_i distinct primes and e_i positive integers. Find a formula for $\varphi(m)$ as a finite summation involving p_i, e_i (and possibly m itself).

(c) (5 points) Factor the sum from part (b) as a finite product.

4. (15 points total) Consider the following triangular array of 10 billiard balls

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & & 3 & & \\ & 4 & & 5 & & 6 & \\ 7 & & 8 & & 9 & & 10 \end{array}$$

which we imagine to be arranged in a perfect equilateral triangle. The arrangement then has six symmetries forming the *dihedral group* D_3: there are three rotational symmetries $\rho_0, \rho_{120}, \rho_{240}$ through $0, 120$, and 240 degrees, and reflections r_1, r_2, r_3 through three different lines of symmetry (each symmetry line being an altitude of the triangle).

(a) (5 points) For each of the six elements of D_3, write down the corresponding permutation of $[10]$ as it acts on the billiard balls, in *cycle notation*.

(b) (5 points) Consider the action of these symmetries permuting the colorings of the balls with n colors. For each of the six elements of D_3, write down the number of n-colorings of the balls that it fixes.

(c) (5 points) Write down a polynomial in n that expresses the number of orbits of n-colorings of the balls under the action of D_3.

5. (25 points total) Recall the Stirling number $S(k,n)$ of the second kind is the number of partitions of $[k]$ into exactly n non-empty blocks. Define two families of generating functions (one ordinary, one exponential) for $S(k,n)$ as follows:

$$f_n^{ord} := \sum_{k\geq 0} S(k,n)x^k$$

$$f_n^{exp} := \sum_{k\geq 0} S(k,n)\frac{x^k}{k!}.$$

In particular, our conventions for $S(k,0)$ imply that

$$f_0^{ord}(x) = S(0,0) + S(1,0)x + S(2,0)x^2 + \cdots = 1.$$

Recall the recurrence that we derived for $S(k,n)$:

$$S(k,n) = S(k-1,n-1) + nS(k-1,n) \text{ for } k,n \geq 1.$$

(a) (10 points) Use this to prove a simple relation that exhibits $f_n^{ord}(x)$ as something times $f_{n-1}^{ord}(x)$.

(b) (5 points) Derive a simple product expression for $f_n^{ord}(x)$ (involving no summations).

(c) (10 points) Show by any means that

$$f_n^{exp}(x) = \frac{(e^x-1)^n}{n!}.$$

Hint: Part(c) does not use parts (a) or (b). One possible approach might involve considering the number $\tilde{S}(k,n)$ of surjective functions $[k] \to [n]$, and proving the equivalent (why?) statement that

$$\sum_{k\geq 0} \tilde{S}(k,n)\frac{x^k}{k!} = (e^x-1)^n.$$