**Math 5705 Undergraduate enumerative combinatorics**
**Fall 2002, Vic Reiner**
**Final exam - Due Friday December 13, in class**

**Instructions:** This is an open book, open library, open notes, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (14 points) Chapter 4, Supplementary problem 11 on page 102.

2. (14 points) Appendix C, Supplementary problem 6 on page 174.

3. For a positive integer $n$, the *Euler $\phi$-function* $\phi(n)$ is defined to be the number of integers in $\{0, 1, 2, \ldots, n-1\}$ which are *relatively prime* to $n$, that is, the number of them which share no common factors with $n$. For example, $\phi(12) = |\{1, 5, 7, 11\}| = 4$.

Note that 0 is never relatively prime to $n$, as it is divisible by every factor of $n$, but 1 is always relatively prime to $n$. For those of you who have seen modular arithmetic, $\phi(n)$ is the number of elements in the integers $\mathbb{Z}/n\mathbb{Z}$ modulo $n$ that have a multiplicative inverse.

(a) (5 points) Use inclusion-exclusion to compute $\phi(n)$ for

$$n = 63,000,000 = 2^6 3^2 5^6 7^1.$$

(Note: An answer that does not use inclusion-exclusion will not get any points. Hint: For each of the primes $p = 2, 3, 5, 7$ define the set $A_p$ to be the subset of numbers in $\{0, 1, 2, \ldots, n-1\}$ which are divisible by $p$).

(b) (5 points) Given the prime factorization of the number $n$ as $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ in which the $p_i$'s are distinct primes and the $e_i$'s are positive integers, write down an expression for $\phi(n)$ in terms of the $p_i$'s and $e_i$'s via inclusion-exclusion.

(c) (4 points) Prove that if $n$ has prime factorization as given in part (b), then

$$\phi(n) = \prod_{i=1}^{r} \left( p_i^{e_i} - p_i^{e_i - 1} \right).$$

4. (a) (10 points) Find the ordinary generating function $f(x) := \sum_{n \geq 0} a_n x^n$ for the sequence $a_0, a_1, \ldots$ defined by the recurrence and

initial conditions

$$a_n = 6a_{n-1} - 9a_{n-2}, \qquad a_0 = 0, a_1 = 1.$$

(b) (5 points) Find a simple explicit formula for $a_n$.

5. Let $S(n,k)$ be the Stirling number of the second kind, counting partitions of an $n$-element set into $k$ blocks. Fixing $k$, form the ordinary generating function $F_k(x) := \sum_{n\geq 0} S(n,k)x^n$.

(a) (4 points) Write down a simple explicit formula (with no summations) for $F_1(x)$.

(b) (6 points) Use the recurrence

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

to find a simple relation between $F_k(x)$ and $F_{k-1}(x)$.

(c) (4 points) Use part (b) to find a simple explicit formula (with no summations) for $F_k(x)$.

6. (14 points) Say that a *tennis club configuration* (not standard terminology) on $[n] := \{1, 2, \ldots, n\}$ consists of a choice of some subset $P$ of $[n]$ having evenly many elements along with a perfect pairing of these elements (think of these as the players who are currently paired off and playing singles tennis matches), together with a choice of a linear ordering on the remaining elements $[n] - P$ (think of these as the players currently standing in line at the drinking fountain).

Let $a_n$ denote the number of tennis club configurations on $[n]$. As a test of your understanding of the definition, you might want to check that the first few values are

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 3, \quad a_3 = 9, \quad a_4 = 39.$$

Find a simple expression for the exponential generating function $\sum_{n\geq 0} \frac{a_n x^n}{n!}$ in terms of elementary functions.

7. (a) (10 points) Prove the following identity by any means:

$$\sum_{k=0}^{n} \binom{n}{k} k(k-1) = n(n-1)\cdot 2^{n-2} \text{ for } n \geq 2.$$

(b) (5 points) Prove it again via a different method.
(Note: I reserve the right to be the final arbiter of what constitutes a "different" method.)