Math 5707 Graph theory
Spring 2013, Vic Reiner
Midterm exam 1- Due Wednesday Feb. 27, in class

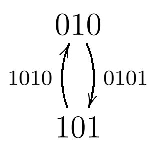
Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1.(15 points total) Define a directed graph $G(k, n) = (V, A)$ whose vertex set V consists of all words $(a_1, a_2, \ldots, a_n)$ of length n from an alphabet of k letters in which $a_i \neq a_{i+1}$ for $i = 1, 2, \ldots, n-1$, and the arc set A has all arcs of this form:

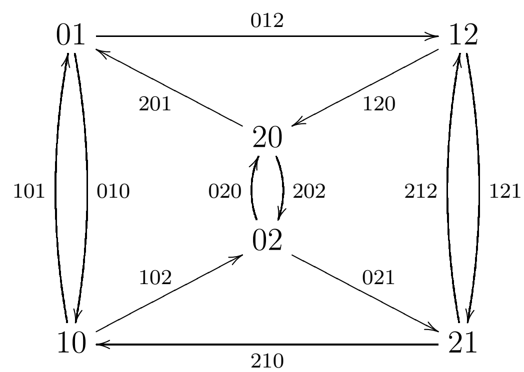
$$\begin{array}{c}(a_0, a_1, \ldots, a_{n-1}) \\ \Big\downarrow {\scriptstyle (a_0, a_1, \ldots, a_{n-1}, a_n)} \\ (a_1, \ldots, a_{n-1}, a_n)\end{array}$$

For example,

$G(2, 3) =$



and $G(3, 2) =$



(a) (5 points) Show that $|V| = k(k-1)^{n-1}$ and $|A| = k(k-1)^n$.

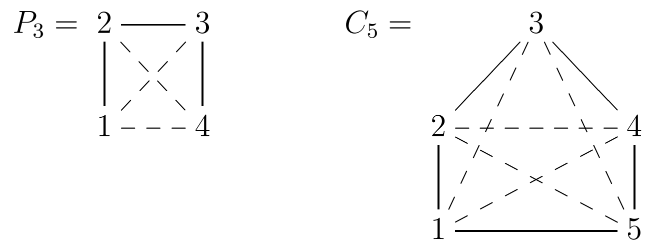
(b) (10 points) Prove for all k, n that $G(k, n)$ has a directed Euler tour.

2.(15 points total) Let T be a minimum cost spanning tree in a graph $G = (V, E)$ with respect to some edge-cost function $c : E \to \mathbb{R}$.

Prove or disprove: For every pair of vertices x, y in V, the unique path from x to y within T will achieve the minimum cost among all paths from x to y within G.

3. (20 points total) Recall that for a simple graph $G = (V, E)$, its *complement graph* $\bar{G}$ is the simple graph on the same vertex set V, but with the complementary set of edges. That is, $\{x, y\}$ is an edge of $\bar{G}$ if and only if $\{x, y\} \notin E$.

Say G is *self-complementary* if $\bar{G}$ is isomorphic to G. For example, the path P_3 with 3 edges is self-complementary, as is the 5-cycle C_5 (edges of $\bar{G}$ are shown dashed):



(a) (10 points) Prove that a self-complementary graph must have $n = |V|$ either congruent to 0 or 1 modulo 4, that is, either n is divisible by 4 or n has remainder 1 on division by 4.

(b) (5 points) Prove that for any n divisible by 4 there exists a self-complementary graph having n vertices.
(Hint: break the vertex set into 4 equal size groups and then use P_3 as a guideline for how to connect between groups).

(c) (5 points) Prove that for any n having remainder 1 on division by 4 there exists a self-complementary graph having n vertices.
(Hint: Figure out how to add one more vertex to your construction from part (b).)

4. (15 points total) Show that a tree that has no vertices of degree two will have more leaf vertices (that is, degree one vertices) than non-leaf vertices, and in fact, at least *two more* leaves than non-leaves.

5. (15 points) Chapter I, Exercise 80 from Bollobás: Show that every connected (simple, undirected) graph $G = (V, E)$ with $m = |E|$ **even** has an orientation of its edges making a digraph $D = (V, A)$ in which every vertex has **even** outdegree.

6. (20 points total) Let $G = (X \sqcup Y, E)$ be a bipartite graph for which there exist positive integers d_X, d_Y such that every x in X has the same degree $d_G(x) = d_X$ and every y in Y has the same degree $d_G(y) = d_Y$.

(a) (10 points) Prove that $d_X/d_Y = |Y|/|X|$.

(b) (10 points) Prove that if $d_X \geq d_Y$ then there exists a matching $M \subseteq E$ that matches every x in X.