**Math 5707 Graph theory**
**Spring 2013, Vic Reiner**
**Midterm exam 2- Due Wednesday April 17, in class**

**Instructions:** This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

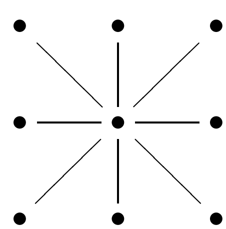
1. (10 points total) Prove that in the Gale-Shapley deferred acceptance algorithm used to match medical students and residency programs, if a particular medical student and a particular residency program each rank each other as their number one choice, then the two will be matched by the algorithm, no matter what any of the other preferences look like.

2. (15 points) Chapter III, Exercise 74 from Bollobás: For every pair of positive integers $k, \ell$ having $1 \leq k \leq \ell$, exhibit a simple graph $G_{k,\ell}$ simultaneously having

- vertex-connectivity $\kappa(G_{k,\ell}) = k$, and
- edge-connectivity $\lambda(G_{k,\ell}) = \ell$.

You must explain why it has both of these properties.

3. (15 points) Fix $d \geq 2$ a positive integer, and let $G = (V, E)$ be a $d$-regular simple graph. Show that one can disjointly decompose the edge set $E = E_1 \sqcup \cdots \sqcup E_t$ so that each subset $E_i$ forms the edges of a *star-shaped* complete bipartite graph $K_{1,d}$ if and only if $G$ is bipartite. E.g. for $d = 8$, here is the star-shaped bipartite graph $K_{1,8}$:



4. (20 points) Recall that in a (simple) graph $G = (V, E)$, one denotes by $\Delta(G)$ the maximum vertex degree, and denotes by $\alpha(G)$ the maximum size of an *independent* or *stable* set, that is, a subset $V' \subset V$ for which there are no edges $e = \{x, y\} \subset V'$ in $E$.

Prove that

$$\alpha(G) \geq \frac{|V|}{1 + \Delta(G)}.$$

5. (20 points) When a soccer ball is made up only of pentagons and hexagons, sewn together to form a cubic (3-regular) graph, how many pentagons will there be?
(Note: the number of hexagons can vary, but you will end up showing that the number of pentagons is fixed!)

6. (20 points) Fix positive integers $k, \ell$, and let $S$ be a set of size $k \cdot \ell$. Show that given any two disjoint decompositions of $S$ into $k$-element subsets $A_i$, $B_j$

$$S = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_\ell$$
$$S = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_\ell$$

there is at least one *common set of representatives* $\{s_1, \ldots, s_\ell\} \subset S$ for both decompositions, that is, a subset $\{s_1, \ldots, s_\ell\}$ that contains exactly one element from each of the $A_i$, and simultaneously contains exactly one element from each of the $B_j$.

E.g., if $k = 3, \ell = 4$, so $k \cdot \ell = 12$, and if one decomposes $S = \{1, 2, \ldots, 12\}$ as

$$\begin{aligned} S &= \{1, \underline{2}, 3\} \sqcup \{4, \underline{5}, 6\} \sqcup \{7, \underline{8}, 9\} \sqcup \{\underline{10}, 11, 12\} &&= A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4, \\ S &= \{1, 4, \underline{8}\} \sqcup \{\underline{2}, 3, 12\} \sqcup \{\underline{5}, 6, 9\} \sqcup \{7, \underline{10}, 11\} &&= B_1 \sqcup B_2 \sqcup B_3 \sqcup B_4 \end{aligned}$$

then $\{s_1, s_2, s_3, s_4\} = \{2, 5, 8, 10\}$ is a common set of representatives, underlined in both decompositions.